

Rayleigh waves in an incompressible orthotropic half-space coated by a thin elastic layer

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THE PRESENT PAPER IS CONCERNED with the propagation of Rayleigh waves in an orthotropic elastic half-space coated with a thin orthotropic elastic layer. The half-space and the layer are both incompressible and they are in welded contact to each other. The main purpose of the paper is to establish an approximate secular equation of the wave. By using the effective boundary condition method an approximate secular equation of third-order in terms of the dimensionless thickness of the layer is derived. It is shown that this approximate secular equation has high accuracy. From it an approximate formula of third-order for the velocity of Rayleigh waves is obtained and it is a good approximation. The obtained approximate secular equation and formula for the velocity will be useful in practical applications.

Key words: Rayleigh waves, incompressible orthotropic elastic half-space, thin incompressible orthotropic elastic layer, approximate secular equation, approximate formula for the velocity.

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1. Introduction

THE STRUCTURES OF A THIN FILM ATTACHED TO SOLIDS, modeled as half-spaces coated with a thin layer, are widely applied in modern technology [1], measurements of mechanical properties of thin supported films play an important role in understanding the behaviors of these structures in applications, see, e.g., [2] and references therein. Among various measurement methods, the surface/guided wave method [3], is used most extensively, and for which the guided Rayleigh wave is a convenient and versatile tool [1, 4]. When using the Rayleigh wave tool, the explicit dispersion relations of Rayleigh waves are employed as theoretical bases for extracting the mechanical properties of the thin films from experimental data. They are therefore the main purpose of the investigations of Rayleigh waves propagating in half-spaces covered with a thin layer. Taking

the assumption of thin layer, explicit dispersion relations can be derived by replacing (approximately) the entire effect of the thin layer on the half-space by the so-called effective boundary conditions which relate the displacements and the stresses of the half-space at its surface. For deriving the effective boundary conditions, ACHENBACH and KESHEVA [5], TIERSTEN [6] replaced the thin layer by a plate modeled by different theories: Mindlin's plate theory and the plate theory of low-frequency extension and flexure, while BOVIK [7] expanded the stresses at the top surface of the layer into Taylor series in its thickness. The Taylor expansion approach was then developed by NIKLASSON [8], ROKHLIN and HUANG [9], BENVENISTE [10], STEIGMANN and OGDEN [11], TING [12], VINH and LINH [13, 14], VINH and ANH [15] and VINH *et. al.* [16] to establish the effective boundary conditions. ACHENBACH and KESHEVA [5], TIERSTEN [6], BOVIK [7] and TUAN [17] assumed that the layer and the substrate are both isotropic and the authors derived approximate secular equations of second-order. STEIGMANN and OGDEN [11] considered a transversely isotropic layer with residual stress overlying an isotropic half-space and the authors derived an approximate second-order secular equation. WANG *et al.* [18] considered an isotropic half-space covered with a thin electrode layer and they obtained an approximate secular equation of first-order. In VINH and LINH [13] the layer and the half-space are both assumed to be orthotropic and compressible, and an approximate secular equation of third-order was obtained. In VINH and LINH [14], the layer and the half-space are both subjected to homogeneous pre-strains and an approximate secular equation of third-order was established that is valid for any pre-strain and for a general strain energy function. In [15, 16] the contact between the layer and the half-space is assumed to be smooth, and approximate secular equations of third-order [15] and fourth-order [16] were established.

The main purpose of this paper is to establish an approximate secular equation of Rayleigh waves propagating in an incompressible orthotropic elastic half-space coated by a thin incompressible orthotropic elastic layer. By using the effective boundary condition method, an approximate secular equation of third-order in terms of the dimensionless thickness of the layer is derived. A numerical investigation shows that this approximate secular equation has high accuracy. Based on the obtained approximate dispersion relation, an approximate formula of third-order for the velocity of Rayleigh waves is derived and it is a good approximation. The obtained approximate secular equation and the approximate velocity formula are good tools for evaluating the mechanical properties of thin films deposited on half-spaces. It should be noted that due to the presence of the hydrostatic pressure associated with the incompressibility constraint, the derivation of the effective boundary conditions becomes more complicated than the one for the compressible case.

2. Effective boundary conditions of third-order

Consider an elastic half-space $x_2 \geq 0$ coated by a thin elastic layer $-h \leq x_2 \leq 0$. Both the layer and half-space are assumed to be orthotropic and they are in welded contact with each other. Note that same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer. We are interested in the plain strain so that

$$(2.1) \quad u_i = u_i(x_1, x_2, t), \quad \bar{u}_i = \bar{u}_i(x_1, x_2, t), \quad i = 1, 2, \quad u_3 \equiv \bar{u}_3 \equiv 0,$$

where t is the time. Suppose that the material of the layer is incompressible. Then, the strain-stress relations are [19]

$$(2.2) \quad \begin{aligned} \bar{\sigma}_{11} &= -\bar{p} + \bar{c}_{11}\bar{u}_{1,1} + \bar{c}_{12}\bar{u}_{2,2}, \\ \bar{\sigma}_{22} &= -\bar{p} + \bar{c}_{12}\bar{u}_{1,1} + \bar{c}_{22}\bar{u}_{2,2}, \\ \bar{\sigma}_{12} &= \bar{c}_{66}(\bar{u}_{1,2} + \bar{u}_{2,1}), \end{aligned}$$

where $\bar{\sigma}_{ij}$, \bar{p} and \bar{c}_{ij} are respectively the stress, the hydrostatic pressure associated with the incompressibility constraint and the material constants, commas indicate differentiation with respect to the spatial variables x_k . In the absence of body forces, the equations of motion are

$$(2.3) \quad \begin{aligned} \bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} &= \bar{\rho}\ddot{\bar{u}}_1, \\ \bar{\sigma}_{12,1} + \bar{\sigma}_{22,2} &= \bar{\rho}\ddot{\bar{u}}_2, \end{aligned}$$

where $\bar{\rho}$ is the mass density, a dot signifies differentiation with respect to the time t . The incompressibility gives

$$(2.4) \quad \bar{u}_{1,1} + \bar{u}_{2,2} = 0.$$

Taking into account (2.1), Eqs. (2.2)-(2.4) are written in matrix form as

$$(2.5) \quad \begin{bmatrix} \bar{\mathbf{U}}' \\ \bar{\mathbf{T}}' \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{T}} \end{bmatrix}$$

where $\bar{\mathbf{U}} = [\bar{u}_1 \ \bar{u}_2]^T$, $\bar{\mathbf{T}} = [\bar{\sigma}_{12} \ \bar{\sigma}_{22}]^T$, the symbol “ T ” indicates the transpose of a matrix, the prime signifies differentiation with respect to x_2 and

$$(2.6) \quad \begin{aligned} \mathbf{M}_1 &= \begin{bmatrix} 0 & -\partial_1 \\ -\partial_1 & 0 \end{bmatrix}, & \mathbf{M}_2 &= \begin{bmatrix} 1/\bar{c}_{66} & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{M}_3 &= \begin{bmatrix} -\bar{\delta}\partial_1^2 + \bar{\rho}\partial_t^2 & 0 \\ 0 & \bar{\rho}\partial_t^2 \end{bmatrix}, & \mathbf{M}_4 &= \mathbf{M}_1, \end{aligned}$$

where $\bar{\delta} = \bar{c}_{11} + \bar{c}_{22} - 2\bar{c}_{12}$ and we use the notations $\partial_1^2 = \partial^2/\partial x_1^2$, $\partial_t^2 = \partial^2/\partial t^2$, $\partial_1 = \partial/\partial x_1$. It follows from (2.5) that

$$(2.7) \quad \begin{bmatrix} \bar{\mathbf{U}}^{(n)} \\ \bar{\mathbf{T}}^{(n)} \end{bmatrix} = \mathbf{M}^n \begin{bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{T}} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix}, \quad n = 1, 2, 3, \dots, \quad x_2 \in [-h, 0].$$

Let h be small (i.e., the layer is thin), by expanding into Taylor series $\mathbf{T}(-h)$ at $x_2 = 0$ up to the third-order of h we have

$$(2.8) \quad \bar{\mathbf{T}}(-h) = \bar{\mathbf{T}}(0) - h\bar{\mathbf{T}}'(0) + \frac{h^2}{2}\bar{\mathbf{T}}''(0) - \frac{h^3}{6}\bar{\mathbf{T}}'''(0).$$

Suppose that surface $x_2 = -h$ is free of traction, i.e., $\bar{\mathbf{T}}(-h) = 0$, using (2.7) at $x_2 = 0$ for $n = 1, 2, 3$ into (2.8) yields

$$(2.9) \quad \begin{aligned} & \left\{ \mathbf{I} - h\mathbf{M}_4 + \frac{h^2}{2}(\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2) \right. \\ & \quad \left. - \frac{h^3}{6}[(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3)\mathbf{M}_2 + (\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2)\mathbf{M}_4] \right\} \bar{\mathbf{T}}(0) \\ & + \left\{ -h\mathbf{M}_3 + \frac{h^2}{2}(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3) \right. \\ & \quad \left. - \frac{h^3}{6}[(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3)\mathbf{M}_1 + (\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2)\mathbf{M}_3] \right\} \bar{\mathbf{U}}(0) = 0. \end{aligned}$$

Since the half-space and the layer are in welded contact with each other at the interface $x_2 = 0$, it follows: $\mathbf{U}(0) = \bar{\mathbf{U}}(0)$ and $\mathbf{T}(0) = \bar{\mathbf{T}}(0)$. Thus, from (2.9)

$$(2.10) \quad \begin{aligned} & \left\{ \mathbf{I} - h\mathbf{M}_4 + \frac{h^2}{2}(\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2) \right. \\ & \quad \left. - \frac{h^3}{6}[(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3)\mathbf{M}_2 + (\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2)\mathbf{M}_4] \right\} \mathbf{T}(0) \\ & + \left\{ -h\mathbf{M}_3 + \frac{h^2}{2}(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3) \right. \\ & \quad \left. - \frac{h^3}{6}[(\mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_4\mathbf{M}_3)\mathbf{M}_1 + (\mathbf{M}_3\mathbf{M}_2 + \mathbf{M}_4^2)\mathbf{M}_3] \right\} \mathbf{U}(0) = 0. \end{aligned}$$

The relation (2.10) is called the approximate effective boundary condition of third-order in matrix form that replaces (approximately) the entire effect of the thin layer on the substrate. Introducing the expressions of the matrices \mathbf{M}_k given by (2.6) into Eq. (2.10) yields the effective boundary conditions in component form, namely

$$(2.11) \quad \sigma_{12} + h(\sigma_{22,1} + \bar{\delta}u_{1,11} - \bar{\rho}\ddot{u}_1) + \frac{h^2}{2} \left(r_1\sigma_{12,11} + \frac{\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{12} + \bar{\delta}u_{2,111} - 2\bar{\rho}\ddot{u}_{2,1} \right) \\ + \frac{h^3}{6} \left(r_1\sigma_{22,111} + \frac{\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{22,1} - r_2u_{1,1111} - \bar{\rho}r_3\ddot{u}_{1,11} - \frac{\bar{\rho}^2}{\bar{c}_{66}}\ddot{u}_{1,tt} \right) = 0 \quad \text{at } x_2 = 0,$$

$$(2.12) \quad \sigma_{22} + h(\sigma_{12,1} - \bar{\rho}\ddot{u}_2) + \frac{h^2}{2} (\sigma_{22,11} + \bar{\delta}u_{1,111} - 2\bar{\rho}\ddot{u}_{1,1}) \\ + \frac{h^3}{6} \left(r_1\sigma_{12,111} + \frac{2\bar{\rho}}{\bar{c}_{66}}\ddot{\sigma}_{12,1} + \bar{\delta}u_{2,1111} - 3\bar{\rho}\ddot{u}_{2,11} \right) = 0 \quad \text{at } x_2 = 0,$$

where $r_1 = 1 - \bar{\delta}/\bar{c}_{66}$, $r_2 = \bar{\delta}(\bar{\delta}/\bar{c}_{66} - 2)$, $r_3 = 2r_1 + 1$.

3. Approximate secular equation of third-order

Suppose that the elastic half-space is also incompressible. Then, the unknown vectors $\mathbf{U} = [u_1 u_2]^T$, $\mathbf{T} = [\sigma_{12} \sigma_{22}]^T$ are satisfied by Eq. (2.5) without bars. In addition to this equation there are required the effective boundary conditions (2.11) and (2.12) and the decay condition at $x_2 = +\infty$ is as follows

$$(3.1) \quad \mathbf{U} = \mathbf{T} = \mathbf{0} \quad \text{at } x_2 = +\infty.$$

Now, we consider a Rayleigh wave travelling in the x_1 -direction with velocity c , wave number k and decaying in the x_2 -direction. According to OGDEN and VINH [19] the displacement components of the Rayleigh wave are given by

$$(3.2) \quad u_1 = -k(b_1 B_1 e^{-kb_1 x_2} + b_2 B_2 e^{-kb_2 x_2}) e^{ik(x_1 - ct)}, \\ u_2 = -ik(B_1 e^{-kb_1 x_2} + B_2 e^{-kb_2 x_2}) e^{ik(x_1 - ct)},$$

where B_1, B_2 are constants to be determined from the effective boundary conditions (2.11) and (2.12), b_1, b_2 are roots of the characteristic equation

$$(3.3) \quad \gamma b^4 - (2\beta - X)b^2 + (\gamma - X) = 0$$

whose real parts are positive to ensure the decay condition (13), $X = \rho c^2$, and

$$(3.4) \quad \gamma = c_{66}, \quad \beta = (\delta - 2\gamma)/2, \quad \delta = c_{11} + c_{22} - 2c_{12}.$$

From Eq. (3.3) it follows

$$(3.5) \quad b_1^2 + b_2^2 = \frac{(2\beta - X)}{\gamma} = S, \quad b_1^2 \cdot b_2^2 = \frac{\gamma - X}{\gamma} = P.$$

It is not difficult to verify that if the Rayleigh wave exists ($\rightarrow b_1, b_2$ having positive real parts), then

$$(3.6) \quad 0 < X < c_{66}$$

and

$$(3.7) \quad b_1 \cdot b_2 = \sqrt{P}, \quad b_1 + b_2 = \sqrt{S + 2\sqrt{P}}.$$

Substituting (3.2) into Eqs. (2.2) corresponding to the half-space and taking into account (2.3) yield

$$(3.8) \quad \begin{aligned} \sigma_{12} &= k^2 \{ \beta_1 B_1 e^{-kb_1 x_2} + \beta_2 B_2 e^{-kb_2 x_2} \} e^{ik(x_1 - ct)}, \\ \sigma_{22,1} &= k^3 \{ \gamma_1 B_1 e^{-kb_1 x_2} + \gamma_2 B_2 e^{-kb_2 x_2} \} e^{ik(x_1 - ct)}, \end{aligned}$$

in which $\beta_n = c_{66}(b_n^2 + 1)$, $\gamma_n = (X - \delta + \beta_n)b_n$, $n = 1, 2$.

Introducing (3.2) and (3.8) into the effective boundary conditions (2.11) and (2.12) leads to two equations for B_1 , B_2 , namely

$$(3.9) \quad \begin{aligned} f(b_1)B_1 + f(b_2)B_2 &= 0, \\ F(b_1)B_1 + F(b_2)B_2 &= 0, \end{aligned}$$

where

$$(3.10) \quad \begin{aligned} f(b_n) &= \beta_n + \varepsilon \{ \gamma_n - (\bar{X} - \bar{\delta})b_n \} + \frac{\varepsilon^2}{2} \left\{ 2\bar{X} - \bar{\delta} - \left(r_1 + \frac{\bar{X}}{\bar{c}_{66}} \right) \beta_n \right\} \\ &\quad + \frac{\varepsilon^3}{6} \left\{ - \left(r_1 + \frac{\bar{X}}{\bar{c}_{66}} \right) \gamma_n + \left(r_2 + \bar{X} r_3 + \frac{\bar{X}^2}{\bar{c}_{66}} \right) b_n \right\}, \\ F(b_n) &= \gamma_n + \varepsilon \{ \bar{X} - \beta_n \} + \frac{\varepsilon^2}{2} \{ -\gamma_n + b_n(2\bar{X} - \bar{\delta}) \} \\ &\quad + \frac{\varepsilon^3}{6} \left\{ \beta_n \left(r_1 + 2\frac{\bar{X}}{\bar{c}_{66}} \right) + \bar{\delta} - 3\bar{X} \right\}, \quad n = 1, 2, \quad \bar{X} = \bar{\rho}c^2. \end{aligned}$$

Due to $B_1^2 + B_2^2 \neq 0$, the determinant of coefficients of the homogeneous system (3.9) must vanish. This gives

$$(3.11) \quad f(b_1)F(b_2) - f(b_2)F(b_1) = 0.$$

Substituting (3.10) into (3.11) and taking into account (3.5) and (3.7), after lengthy calculations whose details are omitted we arrive at

$$(3.12) \quad A_0 + A_1\varepsilon + \frac{A_2}{2}\varepsilon^2 + \frac{A_3}{6}\varepsilon^3 + O(\varepsilon^4) = 0,$$

where $\varepsilon = kh$ called the dimensionless thickness of the layer, and

$$\begin{aligned}
 A_0 &= c_{66} \left\{ (X - \delta)(b_1 b_2 - 1) - c_{66}(b_1^2 + 1)(b_2^2 + 1) \right\}, \\
 A_1 &= c_{66}(b_1 + b_2)[\bar{X} + b_1 b_2(\bar{X} - \bar{\delta})], \\
 (3.13) \quad A_2 &= - \left(\frac{\bar{X}}{\bar{c}_{66}} - \frac{\bar{\delta}}{\bar{c}_{66}} \right) A_0 - 2\bar{X}(\bar{X} - \bar{\delta}) + \bar{\delta} [X - \delta + c_{66}(b_1 + b_2)^2], \\
 A_3 &= c_{66}(b_1 + b_2) \left\{ 3\bar{X} \left(1 - r_1 - \frac{\bar{X}}{\bar{c}_{66}} \right) - 2\bar{\delta} - b_1 b_2 \left[r_2 + \bar{X} \left(r_3 - 3 + \frac{\bar{X}}{\bar{c}_{66}} \right) \right] \right\},
 \end{aligned}$$

in which $b_1 b_2$ and $b_1 + b_2$ are given by (3.5) and (3.7). Equation (3.12) is the desired approximate secular equation of third-order that is totally explicit. In the dimensionless form the equation (3.12) becomes

$$(3.14) \quad D_0 + D_1 \varepsilon + \frac{D_2}{2} \varepsilon^2 + \frac{D_3}{6} \varepsilon^3 + O(\varepsilon^4) = 0,$$

where

$$\begin{aligned}
 D_0 &= (x - e_\delta) \sqrt{P} + x, \\
 D_1 &= r_\mu [r_v^2 x + (x r_v^2 - \bar{e}_\delta) \sqrt{P}] \sqrt{S + 2\sqrt{P}}, \\
 (3.15) \quad D_2 &= -(x r_v^2 - \bar{e}_\delta) D_0 - 2r_\mu^2 r_v^2 x (x r_v^2 - \bar{e}_\delta) + r_\mu \bar{e}_\delta (x - e_\delta + S + 2\sqrt{P}), \\
 D_3 &= -r_\mu \sqrt{S + 2\sqrt{P}} \{ -3x r_v^2 (\bar{e}_\delta - x r_v^2) + 2\bar{e}_\delta \\
 &\quad + \sqrt{P} [\bar{e}_\delta (\bar{e}_\delta - 2) + x r_v^2 (x r_v^2 - 2\bar{e}_\delta)] \}, \\
 P &= 1 - x, \quad S = e_\delta - 2 - x,
 \end{aligned}$$

and

$$\begin{aligned}
 x &= \frac{X}{c_{66}}, \quad e_\delta = \frac{\delta}{c_{66}}, \quad \bar{e}_\delta = \frac{\bar{\delta}}{\bar{c}_{66}}, \quad r_\mu = \frac{\bar{c}_{66}}{c_{66}}, \quad r_v = \frac{c_2}{\bar{c}_2}, \\
 c_2 &= \sqrt{\frac{c_{66}}{\rho}}, \quad \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}.
 \end{aligned}$$

It is clear from (3.14) and (3.15) that the squared dimensionless Rayleigh wave velocity $x = c^2/c_2^2$ depends on five dimensionless parameters: e_δ , \bar{e}_δ , r_μ , r_v and ε . Note that $e_\delta > 0$, $\bar{e}_\delta > 0$ because $c_{ii} > 0$, \bar{c}_{ii} ($i = 1, 2, 6$), $c_{11} + c_{22} - 2c_{12} > 0$ and $\bar{c}_{11} + \bar{c}_{22} - 2\bar{c}_{12} > 0$ (see OGDEN and VINH [19]).

When the layer is absent, i.e., $\varepsilon = 0$, Eq. (3.14) becomes

$$D_0 = (x - e_\delta) \sqrt{1 - x} + x = 0$$

that coincides with the secular equation of Rayleigh waves in an incompressible orthotropic elastic half-space, see [19].

When the layer and the half-space are both transversely isotropic (with the isotropic axis being the x_3 -axis): $c_{11} = c_{22}$, $\bar{c}_{11} = \bar{c}_{22}$, $c_{11} - c_{12} = 2c_{66}$, $\bar{c}_{11} - \bar{c}_{12} = 2\bar{c}_{66}$, then

$$(3.16) \quad e_\delta = \bar{e}_\delta = 4, \quad S = 2 - x.$$

From (3.15) and (3.16), D_0, D_1, D_2, D_3 are expressed by:

$$(3.17) \quad \begin{aligned} D_0 &= (x - 4)\sqrt{1 - x} + x, \\ D_1 &= r_\mu(1 + \sqrt{1 - x}) [(r_v^2 x - 4)\sqrt{1 - x} + r_v^2 x], \\ D_2 &= -(r_v^2 x - 4)D_0 - 2r_\mu^2 r_v^2 x(r_v^2 x - 4) + 8r_\mu(\sqrt{1 - x} - 1), \\ D_3 &= -r_\mu(1 + \sqrt{1 - x}) \\ &\quad \times [-12r_v^2 x + 8 + 3r_v^4 x^2 + (8 - 8r_v^2 x + r_v^4 x^2)\sqrt{1 - x}]. \end{aligned}$$

When the layer and the half-space are both isotropic, D_0, D_1, D_2, D_3 are also given by (3.17), but in which $x = \rho c^2 / \mu$, μ is the shear modulus.

Figure 1 presents the dependence on ε of the squared dimensionless Rayleigh wave velocity $x = c^2 / c_2^2$ that is calculated by the exact dispersion relation (3.9)

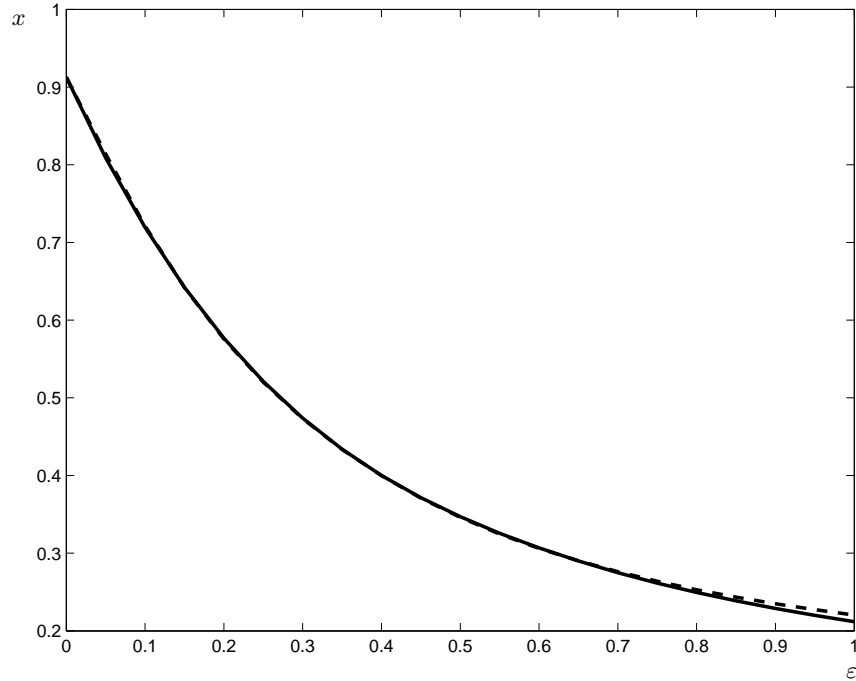


FIG. 1. The Rayleigh wave velocity curves drawn by solving the exact dispersion relation (3.9) in [17] (solid line) and by solving the approximate secular equation (3.14) (dashed line) with $e_\delta = \bar{e}_\delta = 4$, $r_\mu = 1$, $r_v = 3$.

in Tuan [17] (solid line), by the approximate secular equation (3.14) (dashed line) with $e_\delta = \bar{e}_\delta = 4$, $r_\mu = 1$, $r_v = 3$. Figure 1 shows that the approximate and exact velocity curves are very close to each other. This says that the obtained third-order approximate secular equations have high accuracy.

4. Third-order approximate formula for the velocity

In this section, we establish an approximate formula of third-order for the squared dimensionless Rayleigh wave velocity $x(\varepsilon)$ that is of the form

$$(4.1) \quad x(\varepsilon) = x(0) + x'(0)\varepsilon + \frac{1}{2}x''(0)\varepsilon^2 + \frac{1}{6}x'''(0)\varepsilon^3 + O(\varepsilon^4),$$

where $x(0)$ is the squared dimensionless velocity of Rayleigh waves propagating in an incompressible orthotropic elastic half-space that, according to [19], is given by

$$(4.2) \quad x(0) = 1 - \frac{1}{9} \left[-1 + \sqrt[3]{9e_\delta + 16 + 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2 \right. \\ \left. + \sqrt[3]{9e_\delta + 16 - 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2 \right]^2,$$

in which the roots are understood as real roots. In view of the relation

$$(4.3) \quad \sqrt[3]{9e_\delta + 16 - 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2 \\ = \frac{4 - 3e_\delta}{\sqrt[3]{9e_\delta + 16 + 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2},$$

$x(0)$ is given by the formula

$$(4.4) \quad x(0) = 1 - \frac{1}{9} \left[-1 + \sqrt[3]{9e_\delta + 16 + 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2 \right. \\ \left. + \frac{4 - 3e_\delta}{\sqrt[3]{9e_\delta + 16 + 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)}}/2} \right]^2,$$

that is more convenient to use because

$$9e_\delta + 16 + 3\sqrt{3}\sqrt{e_\delta(4e_\delta^2 - 13e_\delta + 32)} > 0$$

for all positive values of e_δ . From (3.14) it follows that

$$\begin{aligned}
 (4.5) \quad & x'(0) = -\frac{D_1}{D_{0x}} \Big|_{x=x(0)}, \\
 & x''(0) = -\frac{D_2 D_{0x}^2 - 2D_{0x} D_1 D_{1x} + D_1^2 D_{0xx}}{D_{0x}^3} \Big|_{x=x(0)}, \\
 & x'''(0) = -\frac{D_3 + 3D_{2x}x'(0) + 3D_{1xx}x'^2(0) + 3D_{1xx}x''(0) + 3D_{0xxx}x'(0)x''(0) + D_{0xxx}x'^3(0)}{D_{0x}} \Big|_{x=x(0)},
 \end{aligned}$$

where D_1, D_2, D_3 are given by (3.15) and

$$\begin{aligned}
 (4.6) \quad & D_{0x} = \frac{2 - 3x + e_\delta}{2\sqrt{1-x}} + 1, \\
 & D_{0xx} = \frac{e_\delta - 4 + 3x}{4\sqrt{(1-x)^3}}, \\
 & D_{0xxx} = \frac{3(e_\delta - 2 + x)}{8\sqrt{(1-x)^5}}, \\
 & D_{1x} = r_\mu \left[\left(r_\nu^2 + \frac{2r_\nu^2 + \bar{e}_\delta - 3r_\nu^2 x}{2\sqrt{1-x}} \right) \sqrt{S + 2\sqrt{P}} \right. \\
 & \quad \left. - [r_\nu^2 x + (r_\nu^2 x - \bar{e}_\delta)\sqrt{P}] \frac{\sqrt{P} + 1}{2\sqrt{P}\sqrt{S + 2\sqrt{P}}} \right], \\
 & D_{1xx} = r_\mu \left[\frac{\bar{e}_\delta - 4r_\nu^2 + 3r_\nu^2 x}{4\sqrt{P^3}} \sqrt{S + 2\sqrt{P}} \right. \\
 & \quad \left. - \frac{\sqrt{P} + 1}{\sqrt{P}\sqrt{S + 2\sqrt{P}}} \left(r_\nu^2 + \frac{2r_\nu^2 + \bar{e}_\delta - 3r_\nu^2 x}{2\sqrt{1-x}} \right) \right] \\
 & \quad - \frac{r_\mu [r_\nu^2 x + (r_\nu^2 x - \bar{e}_\delta)\sqrt{P}]}{4\sqrt{P^3}\sqrt{(S + 2\sqrt{P})^3}} [S + 2\sqrt{P} + \sqrt{P}(\sqrt{P} + 1)^2], \\
 & D_{2x} = -r_\nu^2 D_0 - (r_\nu^2 x - \bar{e}_\delta) D_{0x} - 2r_\mu^2 r_\nu^2 (2r_\nu^2 x - \bar{e}_\delta) - \frac{r_\mu \bar{e}_\delta}{\sqrt{1-x}}.
 \end{aligned}$$

Figure 2 presents the dependence on ε of the Rayleigh wave velocity $x = c^2/c_2^2$ that is calculated by the exact dispersion relation (3.9) in [17] (solid line) and by the approximate formula (4.1) (dashed line) with $e_\delta = \bar{e}_\delta = 4$, $r_\mu = 0.8$, $r_\nu = 1.2$. It shows that the approximate formula (4.1) is a good approximation for the Rayleigh wave velocity.

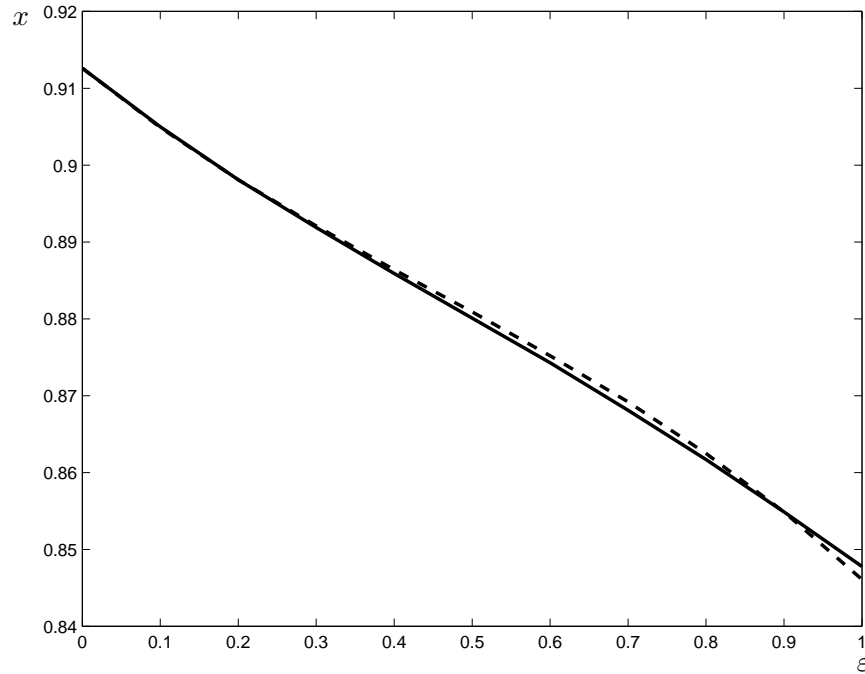


FIG. 2. The Rayleigh wave velocity curves drawn by solving the exact dispersion relation (3.9) in [17] (solid line) and by using the approximate formula (4.1) (dashed line) with $e_\delta = \bar{e}_\delta = 4$, $r_\mu = 0.8$, $r_v = 1.2$.

5. Conclusions

In this paper, the propagation of Rayleigh waves in an incompressible orthotropic elastic half-space coated by a thin incompressible orthotropic elastic layer with the welded contact is investigated. First, an approximate effective boundary condition of third-order in matrix form is established that replaces the entire effect of the layer on the half-space. Then, by using it, an approximate secular equation of third-order is obtained. Based on this secular equation an approximate formula of third-order for the Rayleigh wave velocity is derived. It is shown that the obtained approximate secular equation and the approximate formula for the velocity are good approximations. They will be useful in practical applications.

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