

Fractional continua for linear elasticity

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FRACTIONAL CONTINUA IS A GENERALISATION of the classical continuum body. This new concept shows the application of fractional calculus in continuum mechanics. The advantage is that the obtained description is non-local. This natural non-locality is inherently a consequence of fractional derivative definition which is based on the interval, thus variates from the classical approach where the definition is given in a point. In the paper, the application of fractional continua to one-dimensional problem of linear elasticity under small deformation assumption is presented.

Key words: non-local models, fractional calculus.

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1. Introduction

SINCE ITS INVENTION IN 1695 [1], fractional calculus became an individual branch of pure mathematics with many successful applications, e.g., in fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics, and others, cf. [2–8] and cited therein. This interest in fractional calculus (theory of differential equation of arbitrary order) for applications in physics can be understand as looking for better models describing the reality [9]. The reason is that fractional differential operators introduce non-local effects as consequence of their definition which is based on the interval.

It is commonly accepted that non-local formulations play essential role in the physical models while locality assumption limits the processes to be covered by a specific model. Considering as an example the description of the material deformation including length scale, we are able to cover the phenomena such as scale effects or strain softening where classical approach is no longer valid [10, 11].

First articles in this field were written in the 1960s predicting phenomena such as stress concentration at holes, crack-tip stresses, bending stiffness of thin beams or stresses at free surfaces (cf. [12] and cited therein). Currently, we single out in general two common ways to classify the introduction of the length scale, i.e., explicit [13–15] (e.g., via classical strain gradients) or implicit [16–20] (i.e., via relaxation time in Perzyna’s type viscoplasticity). Nevertheless, it is still desirable to discover new concepts of non-local formulations in continuum mechanics.

Non-locality resulting from fractional calculus application can operate in different spaces. For example, one can introduce non-locality (memory) in the time space when generalising the time derivative (cf. [6] in viscoelasticity), in the stress space when generalising the directions of plastic strain (cf. [21]) or in the spatial space when generalising the spatial derivative (cf. [22–29]). The last case constitutes the main subject of this paper.

The formulation presented herein has some crucial advantages in contrast to previous investigations, where fractional derivative operates in spatial space. We have the following features [29]: (i) the proposed new formulation has clear physical interpretation and is developed by analogy to general framework of classical continuum mechanics; (ii) we deal with finite deformations (in contrast to [26, 27] where small deformations are considered only); (iii) contrary to previous works, e.g., [26–28] the generalised fractional measures of the deformation, e.g., fractional deformation gradients or fractional strains have the same physical dimensions as classical one (thus, their classical interpretation remain unchanged); (iv) characteristic length scale of the particular material is defined explicitly (an in classical non-local models); (v) objectivity requirements are proved; (vi) and finally the discussed concept is based on the fractional material and spatial line elements in contrast to [28] where the whole formulation is based on fractional motion (which can be important because in more general formulations displacement field may not exist cf. [30], Box 3.1, pp. 57). Finally, some important similarities with [22, 26, 27], when considering special case of the presented formulation, namely small deformations, will be clarified in Section 4.1.3.

The paper is divided into four main parts. In Section 2, fundamental concepts of fractional calculus are presented to clarify assumptions imposed during non-local fractional continua definition. The non-local fractional kinematics of a continuum body in Euclidean space, defined in general setup, is presented in Section 3. Section 4 deals with the concept of small strains for fractional continua and governing equations for boundary value problem (BVP) are stated. In Section 5, detailed discussion on numerical implementation as well as illustrative examples, showing the dependency of solution of BVP vs. order of applied fractional derivative and characteristic material length scale, are presented.

2. Remarks on fractional calculus

The theory of derivatives of non-integer order dates back to 30 September 1695 when Leibniz had concerned in letter to L'Hospital derivative of order one half [1]. The genius sentence by Leibniz was: *“It will lead to a paradox, from which one day useful consequences will be drawn”*.

There are many definitions of fractional derivatives with the most popular (called after their inventors) given by Grünwald–Letnikov (GL), Riemann–Liouville (RL), and Caputo (C) [2–4]. Nevertheless, they all share one common attribute, i.e., they are all defined on an interval contrary to integer order differential operators defined in a point. It should be emphasised that derivatives mentioned can be equivalent under specific assumptions and capture classical one when the order of derivative becomes integer one.

Let us more precisely characterise the Caputo (C) type derivative – the one used in this paper. Therefore, consider the n -fold integration of a function f which is given by

$$(2.1) \quad I^n f(t) = \frac{1}{\Gamma(n)} \int_a^t (t - \tau)^{n-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{N},$$

where Γ is the Euler gamma function. If we replace in Eq. (2.1) n on an arbitrary $\alpha > 0$ we obtain (left) fractional integral operator in Riemann-Liouville (RL) sense

$$(2.2) \quad {}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{R}^+.$$

Based on relation Eq. (2.2) one can define the Caputo (C) derivative (left sided) as

$$(2.3) \quad {}_a^C D_t^\alpha f(t) = {}_a I_t^{m-\alpha} (D^m f)(t),$$

where $m = [\alpha] + 1$ and ${}_a^C D_t^\alpha$ stands for Caputo (C) differential operator. It is clear for $\alpha = n \in \mathbb{N} \setminus \{0\}$ (then $m = \alpha$) classical derivative is captured and for $\alpha = 0$ we have ${}_a^C D_t^0 f(t) = f(t)$. It should be emphasised, that the specific for Caputo's derivative is that for a constant function is equal zero and requires standard (like in the classical differential equations) initial and/or boundary conditions. For other types of fractional derivatives such conditions can be violated.

Finally, the explicit definitions of left- and right-sided Caputo's derivatives are:

– left-sided Caputo’s derivative for $t > a$ and $n = [\alpha] + 1$

$$(2.4) \quad {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau;$$

– right-sided Caputo’s derivative for $t < b$ and $n = [\alpha] + 1$

$$(2.5) \quad {}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau - t)^{\alpha - n + 1}} d\tau,$$

where $\alpha > 0$ denotes the real order of the derivative, D denotes ‘derivative’ and a, t, b are so called terminals. Both definitions include integration over the interval, namely (a, t) or (t, b) , respectively. The terminals a and b can be chosen arbitrarily.

At the end of this short introduction to fractional calculus, let us consider the Caputo’s type derivative for $t \in (a, b)$. We call such derivative a Riesz–Caputo (RC) derivative cf. [31] – this type of fractional derivative is crucial for further definition of fractional continua. Since any linear combination of derivatives Eqs. (2.4) and (2.5) defines new one [2], hence we define it for $t \in (a, b) \subseteq \mathbb{R}$ and $0 < \alpha < 1$ such as (when α is an integer, the usual definition of a derivative is used cf. [31, 32])

$$(2.6) \quad {}^{RC} D_b^\alpha f(t) = \frac{1}{2} \frac{\Gamma(2 - \alpha)}{\Gamma(2)} ({}_a^C D_t^\alpha f(t) + (-1)^n {}_t^C D_b^\alpha f(t)).$$

The factor $\Gamma(2 - \alpha)/\Gamma(2)$ will be clarified in Section 3.3 where it appears for objectivity reasons.

In the remaining part of this paper the RC derivative is shortly denoted as D^α with the possibility of writing variable under the D in case of partial differentiation of multivariate functions. For example $D_{X_1}^\alpha f$ represents partial fractional derivative of f with respect to the variable X_1 over the interval which should be explicitly defined before $X_1 \in (a, b)$. It is important that for $\alpha = 1$ we have

$$(2.7) \quad {}^{RC} D_b^1 f(t) = \frac{d}{dt} f(t).$$

3. Fractional continua

3.1. Fractional deformation gradients

The description is given in the Euclidean space in Cartesian coordinates (leaving more general setup as a future task). We refer to \mathcal{B} as the reference

configuration of the continuum body while \mathcal{S} denotes its current configuration. Points in \mathcal{B} are denoted by \mathbf{X} and in \mathcal{S} by \mathbf{x} . Coordinate system for \mathcal{B} is denoted by $\{X_A\}$ with base \mathbf{E}_A and for \mathcal{S} we have $\{x_a\}$ with base \mathbf{e}_a .

The regular motion of the material body \mathcal{B} can be written as

$$(3.1) \quad \mathbf{x} = \phi(\mathbf{X}, t),$$

and its inverse as

$$(3.2) \quad \mathbf{X} = \varphi(\mathbf{x}, t),$$

thus $\phi_t : \mathcal{B} \rightarrow \mathcal{S}$ is a C^1 actual configuration of \mathcal{B} in \mathcal{S} , at time t .

Gradient of ϕ defines two-point tensor field \mathbf{F} , called deformation gradient, which describes all local deformation properties and is the primary measure of deformation [33, 34]. Thus we have:

$$(3.3) \quad \mathbf{F}(\mathbf{X}, t) = \frac{\partial \phi(\mathbf{X}, t)}{\partial \mathbf{X}} \quad \text{or} \quad F_{aA} = \frac{\partial \phi_a}{\partial X_A} \mathbf{e}_a \otimes \mathbf{E}_A.$$

It is clear that \mathbf{F} is a linear transformation for each $\mathbf{X} \in \mathcal{B}$ and $t \in I \subset \mathbb{R}^1$, therefore

$$(3.4) \quad d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad \text{or} \quad dx_a = F_{aA}dX_A \mathbf{e}_a,$$

and inverse transformation

$$(3.5) \quad d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \quad \text{or} \quad dX_A = F_{Aa}^{-1}dx_a \mathbf{E}_A,$$

where

$$(3.6) \quad \mathbf{F}^{-1}(\mathbf{x}, t) = \frac{\partial \varphi(\mathbf{x}, t)}{\partial \mathbf{x}} \quad \text{or} \quad F_{Aa}^{-1} = \frac{\partial \varphi_A}{\partial x_a} \mathbf{E}_A \otimes \mathbf{e}_a.$$

We can generalise the deformation gradient and its inverse as follows:

$$(3.7) \quad \tilde{\mathbf{F}}_X(\mathbf{X}, t) = \ell_X^{\alpha-1} D_X^\alpha \phi(\mathbf{X}, t) \quad \text{or} \quad \tilde{F}_{aA} = \ell_A^{\alpha-1} D_{X_A}^\alpha \phi_a \mathbf{e}_a \otimes \mathbf{E}_A,$$

and

$$(3.8) \quad \tilde{\mathbf{F}}_x(\mathbf{x}, t) = \ell_x^{\alpha-1} D_x^\alpha \varphi(\mathbf{x}, t) \quad \text{or} \quad \tilde{F}_{Aa} = \ell_a^{\alpha-1} D_{x_a}^\alpha \varphi_A \mathbf{E}_A \otimes \mathbf{e}_a,$$

where ℓ_X and ℓ_x are length scales in \mathcal{B} and \mathcal{S} , respectively. We assume additionally that $\ell = \ell_X = \ell_x$. In general we have the following relations:

$$(3.9) \quad \tilde{\mathbf{F}}_x \tilde{\mathbf{F}}_X \neq \mathbf{I} = \delta_{AB} \mathbf{E}_A \otimes \mathbf{E}_B,$$

and

$$(3.10) \quad \tilde{\mathbf{F}}_{Xx}^{\alpha} \neq \mathbf{i} = \delta_{ab} \mathbf{e}_a \otimes \mathbf{e}_b,$$

where δ denotes the Kronecker delta. It should be emphasised that fractional deformation gradients $\tilde{\mathbf{F}}_X$ and $\tilde{\mathbf{F}}_x$ are non-local due to the definition of RC fractional differential operator which is based on an interval defined dependently on material being described.

Let us now discuss the role of the length-scale parameters appearing in definitions Eqs (3.7) and (3.8) which is twofold.

Thus firstly, without ℓ the unit of the fractional deformation gradients would be $m^{1-\alpha}$; hence, introduction of the length, similarly like in the classical non-local gradient methods, allows to finally obtain dimensionless quantity. In this way, we can compare the lengths of line elements $d\mathbf{X}$ and $d\mathbf{x}$ with their fractional counterparts $d\tilde{\mathbf{X}}$ and $d\tilde{\mathbf{x}}$ what would be crucial concerning possible strain definitions and clear geometrical interpretation.

Secondly, it is necessary to introduce the length scale in order to fulfil the rigid body motion requirements, i.e., there should be no deformation for such a type of motion. This problem is clarified in the next Section 3.3 where illustrative example is discussed. Notice that from purely mathematical point of view, those parameters could be omitted, however this is not the case in physical theory.

After such introduction the fractional counterparts of the material and the spatial line elements can be introduced, namely

$$(3.11) \quad d\tilde{\mathbf{x}} = \tilde{\mathbf{F}}_X d\mathbf{X} \quad \text{or} \quad d\tilde{x}_a = \tilde{F}_{aA} dX_A \mathbf{e}_a,$$

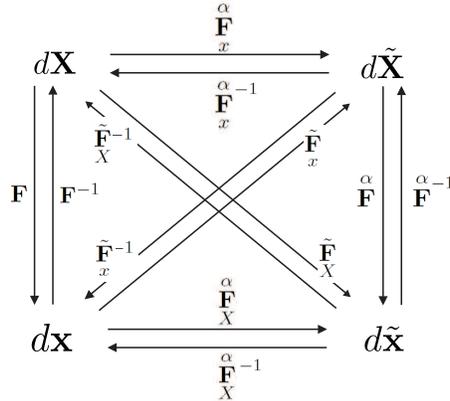


FIG. 1. The relations between material and spatial-line elements with their fractional counterparts.

and

$$(3.12) \quad d\tilde{\mathbf{X}} = \tilde{\mathbf{F}}_x d\mathbf{x} \quad \text{or} \quad d\tilde{X}_A = \tilde{F}_{x Aa} dx_a \mathbf{E}_A.$$

Using Eqs. (3.4), (3.5), (3.11) and (3.12) we have (cf. Fig. 1):

$$(3.13) \quad d\tilde{\mathbf{x}} = \overset{\alpha}{\mathbf{F}}_x d\tilde{\mathbf{X}} \quad \text{or} \quad d\tilde{x}_a = \overset{\alpha}{F}_{aA} d\tilde{X}_A \mathbf{e}_a,$$

$$(3.14) \quad d\tilde{\mathbf{X}} = \overset{\alpha}{\mathbf{F}}_X d\mathbf{X} \quad \text{or} \quad d\tilde{X}_B = \overset{\alpha}{F}_{xB} dX_A \mathbf{E}_B,$$

$$(3.15) \quad d\tilde{\mathbf{x}} = \overset{\alpha}{\mathbf{F}}_X d\mathbf{x} \quad \text{or} \quad d\tilde{x}_b = \overset{\alpha}{F}_{Xba} dx_a \mathbf{e}_b,$$

where $\overset{\alpha}{\mathbf{F}}_X = \tilde{\mathbf{F}}_X \mathbf{F}^{-1} \tilde{\mathbf{F}}_x^{-1}$, $\overset{\alpha}{\mathbf{F}}_x = \tilde{\mathbf{F}}_x \mathbf{F}$ and $\overset{\alpha}{\mathbf{F}}_X = \tilde{\mathbf{F}}_X \mathbf{F}^{-1}$. It is clear that $\overset{\alpha}{\mathbf{F}}_x$ and $\overset{\alpha}{\mathbf{F}}_X$ are not two point tensors while $\tilde{\mathbf{F}}_X$, $\tilde{\mathbf{F}}_x$ and \mathbf{F} are. Based on the properties of motion Eq. (3.1) we have shown that the inverse of $\tilde{\mathbf{F}}_X$ and $\tilde{\mathbf{F}}_x$ exists.

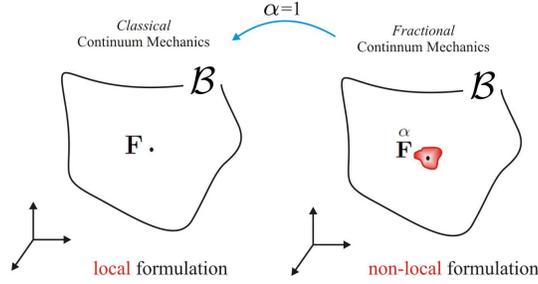


FIG. 2. The equivalence of fractional continua and classical continua for $\alpha = 1$.

As mentioned for $\alpha = 1$ RC fractional derivative becomes classical derivative and hence we recover classical local continuum mechanics where $\ell^{\alpha-1} = \ell^{1-1} = \ell^0 = 1$ do not influence the results, so (Fig. 2):

$$(3.16) \quad \mathbf{F} = \tilde{\mathbf{F}}_X = \tilde{\mathbf{F}}_x^{-1} = \overset{\alpha}{\mathbf{F}}_x,$$

$$(3.17) \quad \mathbf{F}^{-1} = \tilde{\mathbf{F}}_X^{-1} = \tilde{\mathbf{F}}_x = \overset{\alpha}{\mathbf{F}}_X^{-1},$$

$$(3.18) \quad \overset{\alpha}{\mathbf{F}}_x = \mathbf{I},$$

$$(3.19) \quad \overset{\alpha}{\mathbf{F}}_X = \mathbf{i},$$

$$(3.20) \quad d\mathbf{x} = d\tilde{\mathbf{x}},$$

$$(3.21) \quad d\mathbf{X} = d\tilde{\mathbf{X}}.$$

3.2. Fractional strains

We define the strains by analogy to the classical continuum mechanics based on the difference in scalar products in actual and reference configurations. The introduced line elements (Eqs. (3.4) and (3.5)) as well as their fractional counterparts (Eqs. (3.11) and (3.12)) allow to define four concepts of strains:

1. Classical formulation

$$(3.22) \quad dx dx - d\mathbf{X} d\mathbf{X} \equiv d\mathbf{X}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) d\mathbf{X} \equiv d\mathbf{x}(\mathbf{i} - \mathbf{F}^{-T} \mathbf{F}^{-1}) d\mathbf{x},$$

so

$$(3.23) \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}), \quad \mathbf{e} = \frac{1}{2}(\mathbf{i} - \mathbf{F}^{-T} \mathbf{F}^{-1}).$$

2. Formulation based on the fractional spatial line element ($d\tilde{\mathbf{x}}$) and classical material line element ($d\mathbf{X}$)

$$(3.24) \quad d\tilde{\mathbf{x}} d\tilde{\mathbf{x}} - d\mathbf{X} d\mathbf{X} \equiv d\mathbf{X}(\tilde{\mathbf{F}}_X^T \tilde{\mathbf{F}}_X - \mathbf{I}) d\mathbf{X} \equiv d\tilde{\mathbf{x}}(\mathbf{i} - \tilde{\mathbf{F}}_X^{-T} \tilde{\mathbf{F}}_X^{-1}) d\tilde{\mathbf{x}},$$

so

$$(3.25) \quad \tilde{\mathbf{E}}_X = \frac{1}{2}(\tilde{\mathbf{F}}_X^T \tilde{\mathbf{F}}_X - \mathbf{I}), \quad \tilde{\mathbf{e}}_X = \frac{1}{2}(\mathbf{i} - \tilde{\mathbf{F}}_X^{-T} \tilde{\mathbf{F}}_X^{-1}).$$

3. Formulation based on the classical spatial line element ($d\mathbf{x}$) and fractional material line element ($d\tilde{\mathbf{X}}$)

$$(3.26) \quad dx dx - d\tilde{\mathbf{X}} d\tilde{\mathbf{X}} \equiv d\tilde{\mathbf{X}}(\tilde{\mathbf{F}}_x^{-T} \tilde{\mathbf{F}}_x^{-1} - \mathbf{I}) d\tilde{\mathbf{X}} \equiv d\mathbf{x}(\mathbf{i} - \tilde{\mathbf{F}}_x^T \tilde{\mathbf{F}}_x) d\mathbf{x},$$

so

$$(3.27) \quad \tilde{\mathbf{E}}_x = \frac{1}{2}(\tilde{\mathbf{F}}_x^{-T} \tilde{\mathbf{F}}_x^{-1} - \mathbf{I}), \quad \tilde{\mathbf{e}}_x = \frac{1}{2}(\mathbf{i} - \tilde{\mathbf{F}}_x^T \tilde{\mathbf{F}}_x).$$

4. Formulation based on the fractional spatial line element ($d\tilde{\mathbf{x}}$) and fractional material line element ($d\tilde{\mathbf{X}}$)

$$(3.28) \quad d\tilde{\mathbf{x}} d\tilde{\mathbf{x}} - d\tilde{\mathbf{X}} d\tilde{\mathbf{X}} \equiv d\tilde{\mathbf{X}}(\tilde{\mathbf{F}}^{\alpha T} \tilde{\mathbf{F}}^{\alpha} - \mathbf{I}) d\tilde{\mathbf{X}} \equiv d\tilde{\mathbf{x}}(\mathbf{i} - \tilde{\mathbf{F}}^{-\alpha T} \tilde{\mathbf{F}}^{\alpha-1}) d\tilde{\mathbf{x}},$$

so

$$(3.29) \quad \tilde{\mathbf{E}}^{\alpha} = \frac{1}{2}(\tilde{\mathbf{F}}^{\alpha T} \tilde{\mathbf{F}}^{\alpha} - \mathbf{I}), \quad \tilde{\mathbf{e}}^{\alpha} = \frac{1}{2}(\mathbf{i} - \tilde{\mathbf{F}}^{-\alpha T} \tilde{\mathbf{F}}^{\alpha-1}).$$

Based on the above relations it is clear that the generalisation of classical continuum mechanics can be formulated just by exchanging classical deformation gradient with the one of its fractional counterparts. Thus, one can define:

$$(3.30) \quad \mathbf{E} = \frac{1}{2}(\overset{\diamond}{\mathbf{F}}^T \overset{\diamond}{\mathbf{F}} - \mathbf{I}) \quad \text{or} \quad E_{AB} = \frac{1}{2}(\overset{\diamond}{F}_{Aa}^T \overset{\diamond}{F}_{aB} - I_{AB}) \mathbf{E}_A \otimes \mathbf{E}_B,$$

$$(3.31) \quad \mathbf{e} = \frac{1}{2}(\mathbf{i} - \overset{\diamond}{\mathbf{F}}^{-\alpha T} \overset{\diamond}{\mathbf{F}}^{\alpha-1}) \quad \text{or} \quad e_{ab} = \frac{1}{2}(i_{ab} - \overset{\diamond}{F}_{aA}^{-\alpha T} \overset{\diamond}{F}_{Ab}^{\alpha-1}) \mathbf{e}_a \otimes \mathbf{e}_b,$$

where generalised pull-back transformation of \mathbf{e} gives

$$(3.32) \quad \mathbf{E} = \mathbf{F}^{\diamond T} \mathbf{e} \mathbf{F}^{\diamond},$$

while generalised push-forward of \mathbf{E} gives

$$(3.33) \quad \mathbf{e} = \mathbf{F}^{-\diamond T} \mathbf{E} \mathbf{F}^{-\diamond},$$

and

$$(3.34) \quad \mathbf{C} = \mathbf{F}^{\diamond T} \mathbf{F}^{\diamond} \quad \text{or} \quad C_{AB} = F_{Aa}^{\diamond T} F_{aB}^{\diamond} \mathbf{E}_A \otimes \mathbf{E}_B,$$

$$(3.35) \quad \mathbf{c} = \mathbf{F}^{-\diamond T} \mathbf{F}^{-\diamond} = \mathbf{b}^{-1} \quad \text{or} \quad c_{ab} = F_{aA}^{-\diamond T} F_{Ab}^{-\diamond} \mathbf{e}_a \otimes \mathbf{e}_b,$$

finally using the theorem of polar decomposition of non-singular second-order tensor we have

$$(3.36) \quad \mathbf{F}^{\diamond} = \mathbf{R} \mathbf{U} = \mathbf{v} \mathbf{R} \quad \text{or} \quad F_{aA}^{\diamond} = R_{aB} U_{BA} \mathbf{e}_a \otimes \mathbf{E}_B = v_{ab} R_{bA} \mathbf{e}_a \otimes \mathbf{E}_B,$$

and as a consequence

$$(3.37) \quad \mathbf{C} = \mathbf{U} \mathbf{U} \quad \text{and} \quad \mathbf{b} = \mathbf{v} \mathbf{v}.$$

In the above expression, depending on the formulation, \mathbf{F}^{\diamond} can be replaced with \mathbf{F} or $\tilde{\mathbf{F}}_X$ or $\tilde{\mathbf{F}}_x$ or $\tilde{\mathbf{F}}^{\alpha}$. According to the chosen \mathbf{F}^{\diamond} the associated others variables denote: \mathbf{E} is the classical Green–Lagrange strain tensor or its fractional counterpart (symmetric); \mathbf{e} is the classical Euler–Almansi strain tensor or its fractional counterpart (symmetric); \mathbf{C} is the classical right Cauchy–Green tensor or its fractional counterpart (symmetric and positive definite); $\mathbf{c} = \mathbf{b}^{-1}$ is the classical left Cauchy–Green tensor/Finger deformation tensor or its fractional counterpart (symmetric and positive definite); \mathbf{R} is orthogonal tensor; \mathbf{U} is the classical right stretch tensor (symmetric and positive definite) or its fractional counterpart, \mathbf{v} is the classical left stretch tensor (symmetric and positive definite) or its fractional counterpart.

We have also analogous definitions for the volume ratio and surface element mapping. Based on the possible linear transformations defined by Eqs. (3.4), (3.5), (3.11) and (3.12), and by analogy to the classical results we have

$$(3.38) \quad dv = \det(\mathbf{F}^{\diamond}) dV,$$

and

$$(3.39) \quad ds = \det(\mathbf{F}^{\diamond}) \mathbf{F}^{-\diamond T} d\mathbf{S},$$

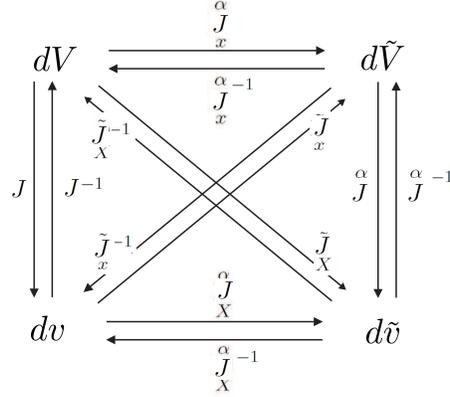


FIG. 3. The relations between material and spatial volumes with their fractional counterparts.

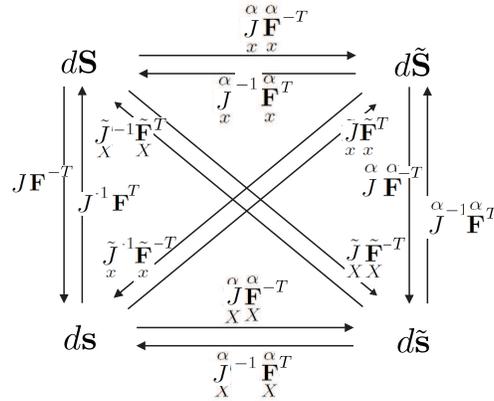


FIG. 4. The relations between material and spatial surface elements with their fractional counterparts.

where according to chosen \diamond \mathbf{F} the following variables denote: dv is the spatial volume element or its fractional counterpart, dV is the material volume element or its fractional counterpart, ds is the spatial surface element or its fractional counterpart, $d\mathbf{S}$ is the material surface element or its fractional counterpart – cf. Figs. 3 and 4 (by analogy to Fig. 1). In Figs. 3 and 4 we have denoted: $J = \det(\mathbf{F})$, $\tilde{J} = \det(\tilde{\mathbf{F}})$, $\tilde{J}_x = \det(\tilde{\mathbf{F}}_x)$, $\tilde{J}_X = \det(\tilde{\mathbf{F}}_X)$, $\overset{\alpha}{J} = \det(\overset{\alpha}{\mathbf{F}})$, $\overset{\alpha}{J}_x = \det(\overset{\alpha}{\mathbf{F}}_x)$, $\overset{\alpha}{J}_X = \det(\overset{\alpha}{\mathbf{F}}_X)$.

3.3. Remarks on objectivity

This new concept of fractional continua should not of course violate the objectivity requirements. It is clear that under the change of the observer the dis-

tances between arbitrary pairs of points in the space and time intervals between events should be preserved. As common, the change of the observer may equivalently be viewed as a certain rigid-body motions superimposed on the current configuration. Thoroughly we will use this concept to prove that the proposed fractional kinematics leads to the same results (in the sense of objectivity) as the classical ones. It should be emphasised that it is crucial to observe how fractional deformation gradients transform under isomorphism (superimposed rigid-body motions).

We denote the superimposed rigid-body motion on the current configuration \mathbf{x} as

$$(3.40) \quad \mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t),$$

where $\mathbf{Q}(t)$ is assumed to be the proper orthogonal tensor and $\mathbf{c}(t)$ denotes its translation. In classical sense the rigid body motion means that there is no deformation, i.e., $\mathbf{E} = \mathbf{e} = 0$. It is equivalent to say that for such a type of deformation, the deformation gradient is an orthogonal tensor, i.e., $\mathbf{F}^T\mathbf{F} = \mathbf{I}$, thus $\mathbf{F} = \mathbf{R}$ or in other words motion ϕ is a linear function on material coordinates X .

The described situation is not so obvious considering fractional deformation. In order to fully satisfy the requirements of the new formulation additional assumptions concerning ℓ and the type of linear combination in RC definition are needed (namely, the necessity of the term $\Gamma(2 - \alpha)/\Gamma(2)$ in Eq. (2.6)). This can be thought of as putting the physical constraints on the obtained fractional generalisation of classical kinematics. In order to understand the problem we will follow this logic.

Let us calculate $\tilde{\mathbf{F}}_X$ and $\tilde{\mathbf{F}}_x$ for the rigid-body motion under the assumption that in RC fractional derivative the terminals are $a = X_A - L/2$ and $b = L/2 + X_A$ (thus, we calculate the RC derivative on the interval with length L , and X_A is the point of interest). Hence we have

$$(3.41) \quad \tilde{\mathbf{F}}_X = \ell^{\alpha-1} \left(\frac{L}{2}\right)^{1-\alpha} \mathbf{R},$$

and

$$(3.42) \quad \tilde{\mathbf{F}}_x = \ell^{\alpha-1} \left(\frac{L}{2}\right)^{1-\alpha} \mathbf{R}^{-1}.$$

So, from Eqs. (3.41) and (3.42) we see that only for

$$(3.43) \quad \ell = \frac{L}{2},$$

the pure rotation is obtained and that is the reason this relation is chosen as a definition of ℓ .

Knowing the explicit definition of ℓ the transformations of deformation gradients can be expressed as

$$(3.44) \quad \mathbf{F}^* = \mathbf{Q}\mathbf{F},$$

$$(3.45) \quad \tilde{\mathbf{F}}_X^* = \mathbf{Q}\tilde{\mathbf{F}}_X,$$

$$(3.46) \quad \tilde{\mathbf{F}}_x^* = \tilde{\mathbf{F}}_x \mathbf{Q}^{-1},$$

so

$$(3.47) \quad \mathbf{F}_X^* = \mathbf{Q}\tilde{\mathbf{F}}_X \mathbf{F}_X^{-1} \mathbf{Q}^{-1} (\tilde{\mathbf{F}}_x \mathbf{Q}^{-1})^{-1} = \mathbf{Q}\tilde{\mathbf{F}}_X \mathbf{F}_X^{-1} \tilde{\mathbf{F}}_x = \mathbf{Q}\mathbf{F}_X^{\alpha},$$

$$(3.48) \quad \mathbf{F}_X^* = \mathbf{Q}\tilde{\mathbf{F}}_X \mathbf{F}_X^{-1} \mathbf{Q}^{-1} = \mathbf{Q}\mathbf{F}_X^{\alpha} \mathbf{Q}^T,$$

$$(3.49) \quad \mathbf{F}_x^* = \tilde{\mathbf{F}}_x \mathbf{Q}^{-1} \mathbf{Q}\mathbf{F}_x = \mathbf{F}_x^{\alpha}.$$

From Eqs. (3.44)-(3.49) it appears that all fractional counterparts of classical measures of deformation keep the same objectivity relations. Similarly to the classical approach also in this approach any material field (like e.g., \mathbf{C}, \mathbf{E}) is unaffected by a rigid-body motion superimposed on current configuration.

4. Fractional linear elasticity – isotropy

4.1. Infinitesimal fractional strains

4.1.1. Finite fractional strains in terms of fractional displacement gradient. Let us consider the relation between fractional displacement gradient tensor and fractional strains. As in the classical continuum mechanics one can define the relation between strains and displacement gradient tensor utilising introduced fractional gradient tensors $\tilde{\mathbf{F}}_X$ and $\tilde{\mathbf{F}}_x$.

The displacements in the material description \mathbf{U} are defined as (\mathbf{U} and should not be confused with the right stretch tensor defined previously):

$$(4.1) \quad \mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X},$$

and its fractional gradient

$$(4.2) \quad \text{Grad } \tilde{\mathbf{U}}_X = \tilde{\mathbf{F}}_X - \mathbf{I} \quad \text{or} \quad \ell^{\alpha-1} D_{X_A}^{\alpha} U_a = (\tilde{F}_{X_A}^a - I_{aA}) \mathbf{e}_a \otimes \mathbf{E}_A,$$

thus we have

$$(4.3) \quad \tilde{\mathbf{F}}_X = \text{Grad } \tilde{\mathbf{U}}_X + \mathbf{I}.$$

Similarly, the displacements in spatial description \mathbf{u} are defined as

$$(4.4) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t),$$

and its fractional gradient

$$(4.5) \quad \text{grad}_x \tilde{\mathbf{u}} = \mathbf{i} - \tilde{\mathbf{F}}, \quad \text{or} \quad \ell^{\alpha-1} D_{x_a}^\alpha u_A = (i_{Aa} - \tilde{F}_{Aa}) \mathbf{E}_A \otimes \mathbf{e}_a,$$

thus we have

$$(4.6) \quad \tilde{\mathbf{F}} = \mathbf{i} - \text{grad}_x \tilde{\mathbf{u}}.$$

By applying Eqs. (4.3) and (4.6) into the fractional strain definitions Eqs. (3.30) and (3.31) we obtain their dependence on the fractional displacement gradients. Thus one obtains:

$$(4.7) \quad \tilde{\mathbf{E}}_X = \frac{1}{2} (\text{Grad}_X \tilde{\mathbf{U}} + \text{Grad}_X \tilde{\mathbf{U}}^T + \text{Grad}_X \tilde{\mathbf{U}}^T \text{Grad}_X \tilde{\mathbf{U}}),$$

$$(4.8) \quad \tilde{\mathbf{e}}_X = \frac{1}{2} (-\text{Grad}_X \tilde{\mathbf{U}}^{-1} - \text{Grad}_X \tilde{\mathbf{U}}^{-T} - \text{Grad}_X \tilde{\mathbf{U}}^{-T} \text{Grad}_X \tilde{\mathbf{U}}^{-1}),$$

and

$$(4.9) \quad \tilde{\mathbf{E}}_x = \frac{1}{2} (-\text{grad}_x \tilde{\mathbf{u}}^{-1} - \text{grad}_x \tilde{\mathbf{u}}^{-T} + \text{grad}_x \tilde{\mathbf{u}}^{-T} \text{grad}_x \tilde{\mathbf{u}}^{-1}),$$

$$(4.10) \quad \tilde{\mathbf{e}}_x = \frac{1}{2} (\text{grad}_x \tilde{\mathbf{u}} + \text{grad}_x \tilde{\mathbf{u}}^T - \text{grad}_x \tilde{\mathbf{u}}^T \text{grad}_x \tilde{\mathbf{u}}),$$

and

$$(4.11) \quad \tilde{\mathbf{E}}^\alpha = \frac{1}{2} \left[(\text{Grad}_X \tilde{\mathbf{U}} + \text{Grad}_X \tilde{\mathbf{U}}^T) - (\text{grad}_x \tilde{\mathbf{u}}^{-1} + \text{grad}_x \tilde{\mathbf{u}}^{-T}) - (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + (\dots) \right],$$

$$(4.12) \quad \tilde{\mathbf{e}}^\alpha = \frac{1}{2} \left[-(\text{Grad}_X \tilde{\mathbf{U}}^{-1} + \text{Grad}_X \tilde{\mathbf{U}}^{-T}) + (\text{grad}_x \tilde{\mathbf{u}} + \text{grad}_x \tilde{\mathbf{u}}^T) + (\nabla \mathbf{u}^{-1} + \nabla \mathbf{u}^{-T}) + (\dots) \right].$$

For definitions $\tilde{\mathbf{E}}^\alpha$ and $\tilde{\mathbf{e}}^\alpha$ we have omitted second- and third-order terms denoting them (...) for clarity.

Of course for $\alpha = 1$ we have classical solution ($\text{Grad}_X \tilde{\mathbf{U}}^{-1} = -\nabla \mathbf{u}$ and $\text{grad}_x \tilde{\mathbf{u}}^{-1} = -\nabla \mathbf{U}$ like in classical continuum mechanics where $\nabla \mathbf{u} = -\nabla \mathbf{U}^{-1}$ and consequently $\nabla \mathbf{U} = -\nabla \mathbf{u}^{-1}$), so

$$(4.13) \quad \mathbf{E} = \frac{1}{2} (\nabla \mathbf{U} + \nabla \mathbf{U}^T + \nabla \mathbf{U}^T \nabla \mathbf{U}) = \frac{1}{2} (-\nabla \mathbf{u}^{-1} - \nabla \mathbf{u}^{-T} + \nabla \mathbf{u}^{-T} \nabla \mathbf{u}^{-1}),$$

$$(4.14) \quad \mathbf{e} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T - \nabla \mathbf{u}^T \nabla \mathbf{u}) = \frac{1}{2} (-\nabla \mathbf{U}^{-1} - \nabla \mathbf{U}^{-T} - \nabla \mathbf{U}^{-T} \nabla \mathbf{U}^{-1}),$$

where ∇ stands for classical gradient.

4.1.2. Small fractional strains. If we now take into account small deformation assumption, understood as omitting higher order terms in above strain definitions, we obtain infinitesimal fractional Cauchy's strain tensor

$$(4.15) \quad \overset{\diamond}{\varepsilon} = \frac{1}{2} \left[\text{Grad}_X \tilde{\mathbf{U}} + \text{Grad}_X \tilde{\mathbf{U}}^T \right] = \frac{1}{2} \left[\text{grad}_x \tilde{\mathbf{u}} + \text{grad}_x \tilde{\mathbf{u}}^T \right].$$

It is clear that for a rigid body motion, when displacement field is independent from a spatial variable [26], we obtain from Eq. (4.15) $\overset{\diamond}{\varepsilon} = 0$ (cf. Sec. 3.3, and RC derivative properties discussed in Sec. 2).

And again for $\alpha = 1$ classical Cauchy's strain tensor is recovered

$$(4.16) \quad \varepsilon = \overset{\diamond}{\varepsilon} = \frac{1}{2} \left[\nabla \mathbf{U} + \nabla \mathbf{U}^T \right] = \frac{1}{2} \left[\nabla \mathbf{u} + \nabla \mathbf{u}^T \right].$$

4.1.3. Some comments on one-dimensional case. For one-dimensional case Eq. (4.15) reduces to the following relation:

$$(4.17) \quad \overset{\diamond}{\varepsilon} = \text{Grad}_X \tilde{U} = \frac{1}{2} \ell^{\alpha-1} D_X^\alpha U,$$

or more explicitly when applying RC definition (cf. Eq. (2.6)) we have

$$(4.18) \quad \overset{\diamond}{\varepsilon} = \ell^{\alpha-1} \frac{\Gamma(2-\alpha)}{2} \left({}^C_{a_1} D_X^\alpha U - {}^C_X D_{a_2}^\alpha U \right).$$

The result given by Eq. (4.18) allows to observe crucial differences with one obtained in [22] (Eq. 9), [26] (Eq. 2.7) and [27] (Eq. 20). Thus first, as mentioned, Eq. (4.18) is obtained as a special case of general fractional finite strains. Second, in contrast to the previous results, in Eq. (4.18) the fractional derivative operates on the finite interval. Finally, the length scale ℓ is given explicitly, and is in the relation with the interval over which the fractional differential operator acts.

As a concluding remark of this section, it is important, that by analogy to [27], Eq. (4.18) can be expressed in terms of the so-called Marchaud or Riesz fractional derivatives.

4.2. Boundary value problem

Let us consider the problem of static deformation under the assumption that material is linear elastic and small fractional deformation holds. The governing equations are

$$(4.19) \quad \begin{cases} \sigma_{ij,j} + b_i = 0, \\ \overset{\diamond}{\varepsilon}_{ij} = \frac{1}{2}\ell^{\alpha-1} (D_{X_j}^\alpha U_i + D_{X_i}^\alpha U_j), \\ \sigma_{ij} = \mathcal{L}_{ijkl}^e \overset{\diamond}{\varepsilon}_{ij}, \\ U_i = \check{U}_i, \quad \mathbf{X} \in \Gamma_U, \\ \sigma_{ij} n_j = \check{t}_i, \quad \mathbf{X} \in \Gamma_\sigma, \\ \Gamma_U \cap \Gamma_\sigma = \emptyset \quad \text{and} \quad \Gamma_U \cup \Gamma_\sigma = \Gamma. \end{cases}$$

In above we have denoted: $\boldsymbol{\sigma}$ is the Cauchy stress tensor, \mathbf{b} is the body force, \mathcal{L}^e is the stiffness tensor, \mathbf{n} is the outward unit normal vector, \mathbf{t} is the Cauchy traction vector, Γ_U and Γ_σ are parts of boundary Γ where the displacements and the tractions are applied, respectively.

5. Numerical example

5.1. Description of the problem

To show the main features of the proposed formulation let us consider one-dimensional tension of fractional continuum body (Fig. 5). Total length of the body is $l = 1$, and $\frac{b}{E} = 0.1$. The body is fixed at the left end ($U(X = 0) = 0$) while on the right end the displacement ($U(X = l) = \check{U} = 0.01l$) is applied. The computations are carried out for the following set of crucial parameters in the presented description:

- the order of fractional derivative $\alpha \in \{0.2, 0.5, 0.9, 1.0\}$, and
- the length scale $\ell \in \{1\%l, 10\%l, 30\%l\}$.

5.2. Numerical scheme

According to Sec.4.2, the analysed problem of one-dimensional tension of the fractional continua under linear elasticity with Dirichlet's boundary conditions is defined by

$$(5.1) \quad \begin{cases} \frac{\partial}{\partial X} (\text{Grad}_X \tilde{U}) + \frac{b}{E} = 0, \\ U(X = 0) = 0, \\ U(X = l) = \check{U} = 0.01l. \end{cases}$$

Taking into account the result given by Eq. (4.18), it is clear that to solve Eq. (5.1) we need to calculate adequate left and right Caputo's derivatives of U . Discrete form of Eq. (5.1) is obtained through the following logic.

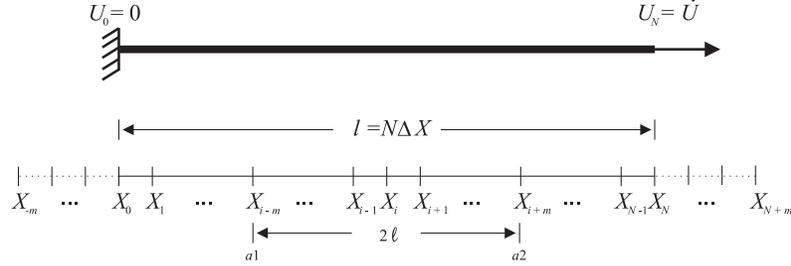


FIG. 5. Spatial discretization for one-dimensional fractional continuum body.

We introduce the homogeneous grid of nodes (see Fig. 5). We see, that additional fictitious nodes X_{-m}, \dots, X_{-1} and X_{N+1}, \dots, X_{N+m} placed outside the domain $[X_0, X_N]$ are introduced. We denote a value of the displacement at the node X_i as $U(X_i) = U_i$. Next, by analogy as in [35], we assume that for all fictitious nodes on the left the displacements are $U_{-m} = U_{-m+1} = \dots = U_{-1} = U_0$ and on the right the displacements are $U_{N+m} = U_{N+m-1} = \dots = U_{N+1} = U_N$.

For calculation of the displacements U_i for $i = 1, 2, \dots, N - 1$ we use the following approximation of first-order derivative (backward difference):

$$\begin{aligned}
 (5.2) \quad \frac{\partial}{\partial X} (\text{Grad } \tilde{U}_X) \Big|_{X=X_i} &\cong \frac{\text{Grad } \tilde{U}_X \Big|_{X=X_i} - \text{Grad } \tilde{U}_X \Big|_{X=X_{i-1}}}{\Delta X} \\
 &= \ell^{\alpha-1} \frac{\Gamma(2-\alpha)}{2\Delta X} \left({}^C_{X_{i-m}} D_X^\alpha U \Big|_{X=X_i} - {}^C_X D_{X_{i+m}}^\alpha U \Big|_{X=X_i} \right. \\
 &\quad \left. - {}^C_{X_{i-m-1}} D_X^\alpha U \Big|_{X=X_{i-1}} + {}^C_X D_{X_{i+m-1}}^\alpha U \Big|_{X=X_{i-1}} \right).
 \end{aligned}$$

Now we determine discrete form of the fractional operators occurring in (5.2). There are many numerical schemes for the fractional equations containing the left and right fractional derivatives (see [35, 36]). These schemes are based on the modified trapezoidal rule [7, 37, 38]. For $\alpha \in (0, 1)$ the left Caputo derivative is approximated by

$$\begin{aligned}
 (5.3) \quad {}^C_{X_{i-m}} D_X^\alpha U \Big|_{X=X_i} &= \frac{1}{\Gamma(1-\alpha)} \int_{X_{i-m}}^{X_i} \frac{U'(\tau) d\tau}{(X_i - \tau)^\alpha} \\
 &\cong \frac{(\Delta X)^{1-\alpha}}{\Gamma(3-\alpha)} \left\{ [(m-1)^{2-\alpha} - (m+\alpha-2)m^{1-\alpha}] U'(X_{i-m}) + U'(X_i) \right. \\
 &\quad \left. + \sum_{j=i-m+1}^{i-1} [(i-j-1)^{2-\alpha} - 2(i-j)^{2-\alpha} + (i-j+1)^{2-\alpha}] U'(X_j) \right\}.
 \end{aligned}$$

Using the forward difference formula for derivatives occurring in (5.3) we finally obtain the following discrete form of the left Caputo derivative:

$$(5.4) \quad {}^C_{X_{i-m}} D_X^\alpha U|_{X=X_i} \cong \frac{(\Delta X)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=i-m}^{i+1} U_j v_1(i, j),$$

where

$$(5.5) \quad v_1(i, j) = \begin{cases} -(m-1)^{2-\alpha} + (m+\alpha-2)m^{1-\alpha} & \text{for } j = i-m, \\ -(m-2)^{2-\alpha} + 3(m-1)^{2-\alpha} \\ \quad -(m+\alpha-2)m^{1-\alpha} - m^{2-\alpha} & \text{for } j = i-m+1, \\ -(i-j-1)^{2-\alpha} + 3(i-j)^{2-\alpha} \\ \quad -3(i-j+1)^{2-\alpha} + (i-j+2)^{2-\alpha} & \text{for } j = i-m+2, \dots, i-1 \wedge j \neq i, \\ 2^{2-\alpha} - 3 & \text{for } j = i, \\ 1 & \text{for } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for the right Caputo derivative we obtain the following formula:

$$(5.6) \quad {}^C_{X} D_{X_{i+m}}^\alpha U|_{X=X_i} = \frac{-1}{\Gamma(n-\alpha)} \int_{X_i}^{X_{i+m}} \frac{U'(\tau) d\tau}{(\tau - X_i)^\alpha}$$

$$\cong \frac{-(\Delta X)^{1-\alpha}}{\Gamma(3-\alpha)} \{ [(m-1)^{2-\alpha} - (m+\alpha-2)m^{1-\alpha}] U'(X_{i+m}) + U'(X_i) \\ + \sum_{j=i+1}^{i+m-1} [(j-i-1)^{2-\alpha} - 2(j-i)^{2-\alpha} + (j-i+1)^{2-\alpha}] U'(X_j) \}$$

$$\cong \frac{(\Delta X)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=i}^{i+m+1} U_j w_1(i, j),$$

where the coefficients $w_1(i, j)$ have the form

$$(5.7) \quad w_1(i, j) = \begin{cases} -(m-1)^{2-\alpha} + (m+\alpha-2)m^{1-\alpha} & \text{for } j = i+m+1, \\ -(m-2)^{2-\alpha} + 3(m-1)^{2-\alpha} \\ \quad -(m+\alpha-2)m^{1-\alpha} - m^{2-\alpha} & \text{for } j = i+m, \\ -(j-i-2)^{2-\alpha} + 3(j-i-1)^{2-\alpha} \\ \quad -3(j-i)^{2-\alpha} + (j-i+1)^{2-\alpha} & \text{for } j = i+2, \dots, i+m-1 \wedge j \neq i+1, \\ 2^{2-\alpha} - 3 & \text{for } j = i+1, \\ 1 & \text{for } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

In a similar way, we determine the discrete form of other derivatives occurring in the expression (5.2), and we have

$$(5.8) \quad {}^C_{X_{i-m-1}} D_X^\alpha U|_{X=X_{i-1}} \cong \frac{(\Delta X)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=i-m-1}^i U_j v_2(i, j),$$

$$(5.9) \quad {}^C_X D_{X_{i+m-1}}^\alpha U|_{X=X_{i-1}} \cong \frac{(\Delta X)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=i-1}^{i+m} U_j w_2(i, j),$$

where the coefficients $v_2(i, j)$ and $w_2(i, j)$ are as follows:

$$(5.10) \quad v_2(i, j) = \begin{cases} -(m-1)^{2-\alpha} + (m+\alpha-2)m^{1-\alpha} & \text{for } j = i-m-1, \\ -(m-2)^{2-\alpha} + 3(m-1)^{2-\alpha} \\ -(m+\alpha-2)m^{1-\alpha} - m^{2-\alpha} & \text{for } j = i-m, \\ -(i-j-2)^{2-\alpha} + 3(i-j-1)^{2-\alpha} \\ -3(i-j)^{2-\alpha} + (i-j+1)^{2-\alpha} & \text{for } j = i-m+1, \dots, i-2 \wedge j \neq i-1, \\ 2^{2-\alpha} - 3 & \text{for } j = i-1 \\ 1 & \text{for } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(5.11) \quad w_2(i, j) = \begin{cases} -(m-1)^{2-\alpha} + (m+\alpha-2)m^{1-\alpha} & \text{for } j = i+m, \\ -(m-2)^{2-\alpha} + 3(m-1)^{2-\alpha} \\ -(m+\alpha-2)m^{1-\alpha} - m^{2-\alpha} & \text{for } j = i+m-1, \\ -(j-i-1)^{2-\alpha} + 3(j-i)^{2-\alpha} \\ -3(j-i+1)^{2-\alpha} + (j-i+2)^{2-\alpha} & \text{for } j = i+1, \dots, i+m-2 \wedge j \neq i, \\ 2^{2-\alpha} - 3 & \text{for } j = i, \\ 1 & \text{for } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the formulas (5.2), (5.4), (5.6), (5.8) and (5.9), we can describe a discrete form of the fractional operator $\frac{\partial}{\partial X}(\text{Grad } \tilde{U}_X)$ as

$$\begin{aligned}
 (5.12) \quad \frac{\partial}{\partial X}(\text{Grad } \tilde{U}_X)|_{X=X_i} & \cong \frac{\ell^{\alpha-1}}{(4-2\alpha)(\Delta X)^{1+\alpha}} \left(\sum_{j=i-m}^{i+1} U_j v_1(i, j) - \sum_{j=i}^{i+m+1} U_j w_1(i, j) \right. \\
 & \quad \left. - \sum_{j=i-m-1}^i U_j v_2(i, j) + \sum_{j=i-1}^{i+m} U_j w_2(i, j) \right).
 \end{aligned}$$

Finally, we present the discrete form of the considered problem (5.1). For calculation of displacements U_0, U_1, \dots, U_N we need to solve the system of $N + 1$ linear equations. For every grid node X_i , where $i = 0, \dots, N$, we can write the following equations:

$$(5.13) \quad \begin{cases} U_0 = 0, \\ \sum_{j=i-m}^{i+1} U_j v_1(i, j) - \sum_{j=i}^{i+m+1} U_j w_1(i, j) \\ \quad - \sum_{j=i-m-1}^i U_j v_2(i, j) + \sum_{j=i-1}^{i+m} U_j w_2(i, j) = -\frac{b}{F \cdot E}, \\ U_N = 0.01l. \end{cases}$$

where

$$(5.14) \quad F = \frac{\ell^{\alpha-1}}{(4-2\alpha)(\Delta X)^{1+\alpha}}.$$

The approximation for fractional strains can be written as

- for X_0 we use forward difference for derivatives in Eqs. (5.3) and (5.6)

$$(5.15) \quad \overset{\diamond}{\varepsilon} = \text{Grad } \tilde{U}_X|_{X=X_0} \cong F \Delta X \left[\sum_{j=-m}^1 U_j v_1(0, j) - \sum_{j=0}^{m+1} U_j w_1(0, j) \right],$$

- for $X_1 \div X_{N-1}$ we use central difference for derivatives in Eqs. (5.3) and (5.6)

$$\begin{aligned}
 (5.16) \quad \overset{\diamond}{\varepsilon} & = \text{Grad } \tilde{U}_X|_{X=X_i} \\
 & \cong \frac{F \Delta X}{2} \left[\sum_{j=i-m-1}^{i+1} U_j [v_1(i, j) + v_2(i, j)] - \sum_{j=i-1}^{i+m+1} U_j [w_1(i, j) + w_2(i, j)] \right].
 \end{aligned}$$

- for X_N we use backward difference for derivatives in Eqs. (5.3) and (5.6)

$$(5.17) \quad \overset{\diamond}{\varepsilon} = \text{Grad } \tilde{U}_X|_{X=X_N} \cong F \Delta X \left[\sum_{j=N-m-1}^N U_j v_2(N, j) - \sum_{j=N-1}^{N+m} U_j w_2(N, j) \right].$$

As an illustrative example let us consider the case for $m = 2$. We obtain from Eq. (5.2)

$$(5.18) \quad \frac{\partial}{\partial X}(\text{Grad } \tilde{U})|_{X=X_i} \cong F[BU_{i-3} + (C - 2B)U_{i-2} + (B - 2C + 2)U_{i-1} \\ + (2C - 4)U_i + (B - 2C + 2)U_{i+1} \\ + (C - 2B)U_{i+2} + BU_{i+3}],$$

where

$$B = 1 - \alpha 2^{1-\alpha}, \\ C = 2^{2-\alpha} - 2.$$

Similarly, for fractional strains we have:

- for X_0

$$(5.19) \quad \overset{\diamond}{\varepsilon} = \text{Grad } \tilde{U}|_{X=X_0} \\ \cong F\Delta X[-BU_{-2} + (B - C)U_{-1} + (C - 2)U_0 + (2 - C)U_1 \\ + (C - B)U_2 + BU_3],$$

- for $X_1 \div X_{N-1}$

$$(5.20) \quad \overset{\diamond}{\varepsilon} = \text{Grad } \tilde{U}|_{X=X_i} \\ \cong \frac{F\Delta X}{2}[-BU_{i-3} - CU_{i-2} + (B - 2)U_{i-1} + (2 - B)U_{i+1} \\ + CU_{i+2} + BU_{i+3}],$$

- for X_N

$$(5.21) \quad \overset{\diamond}{\varepsilon} = \text{Grad } \tilde{U}|_{X=X_N} \\ \cong F\Delta X[-BU_{N-3} + (B - C)U_{N-2} + (C - 2)U_{N-1} + (2 - C)U_N \\ + (C - B)U_{N+1} + BU_{N+2}].$$

5.3. Numerical results

In Figs. 6, 7 and 8 (right side) the computations for BVP defined by Eqs (5.1) for $b/E = 0.1$ are presented. Comparing with classical solution $\alpha = 1$ we see that fractional continua enable us to obtain a family of solutions dependently on chosen α (the order of fractional kinematics) and length scale ℓ .

It is observed that by decreasing ℓ one captures classical solution almost independently of α . This situation is somehow equivalent to local theory. On

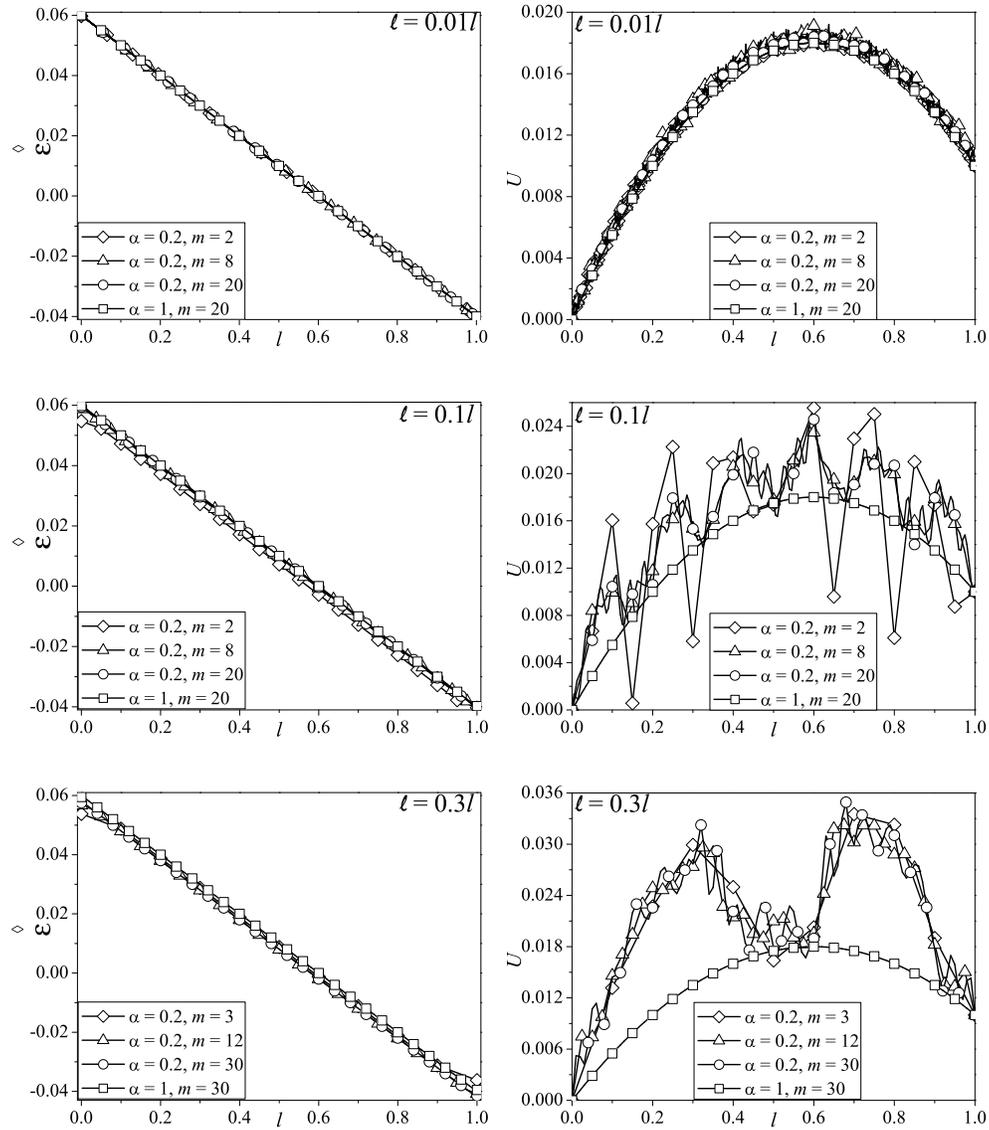


FIG. 6. The comparison of strains distribution and displacements through the length of the body for $\alpha = 0.2$, and selected ℓ and m for $b/E = 0.1$.

the other hand, by making ℓ bigger one observes stronger discrepancy, thus the surrounding of the point of interest influences the results more considerably.

One should notice that for $\alpha = 0.2$ ‘oscillations’ of displacements can occur cf. Fig. 6. In this sense, it seems that for specific physical process some α are not allowed.

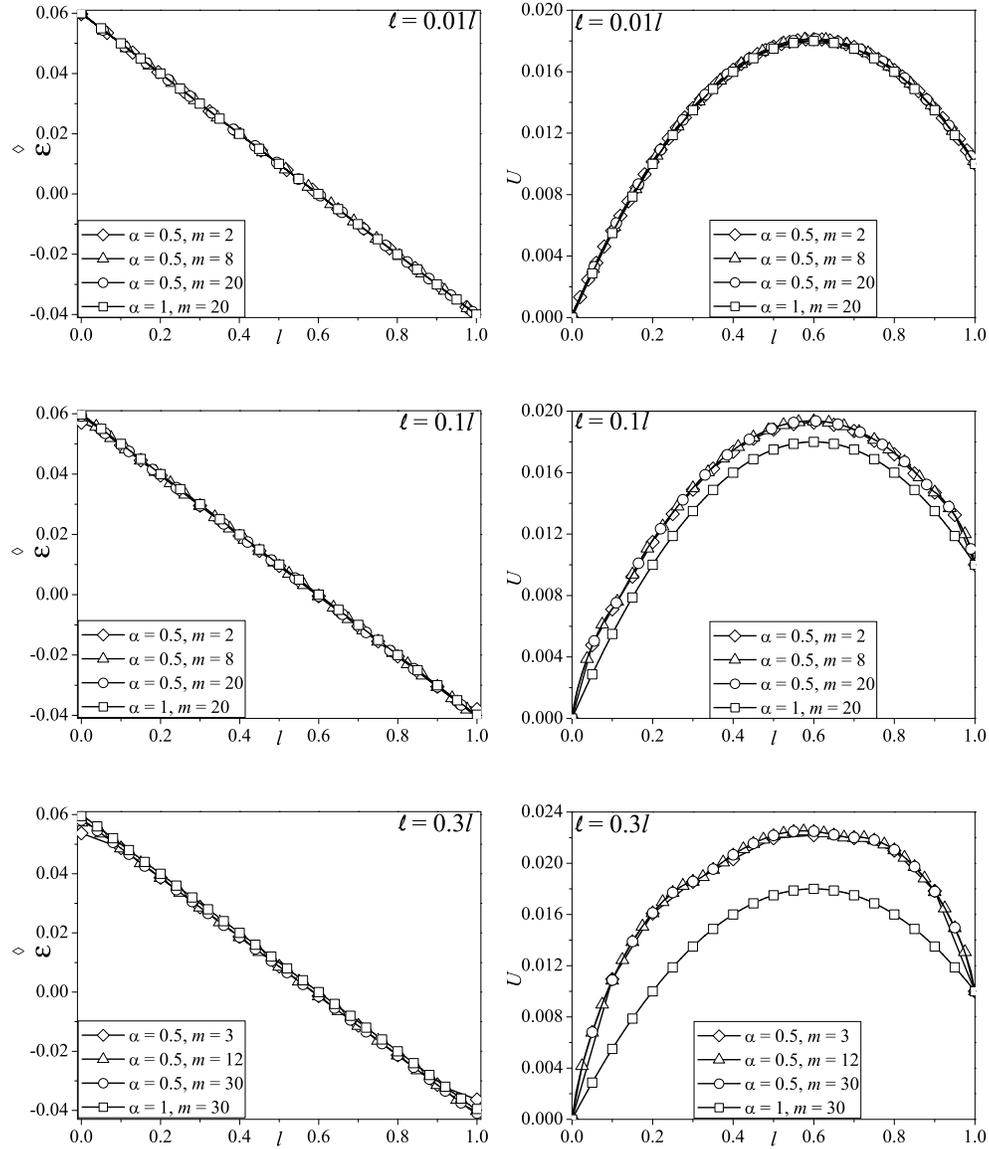


FIG. 7. The comparison of strains distribution and displacements through the length of the body for $\alpha = 0.5$, and selected l and m for $b/E = 0.1$.

Figures 6, 7 and 8 (left side) present the strains distribution through the length l of the analysed problem. We observe that for $\alpha \in (0, 1)$, in general, different values of strain appear comparing with classical solution for $\alpha = 1$. Thus, α and l induce the change of the stiffness of the body.

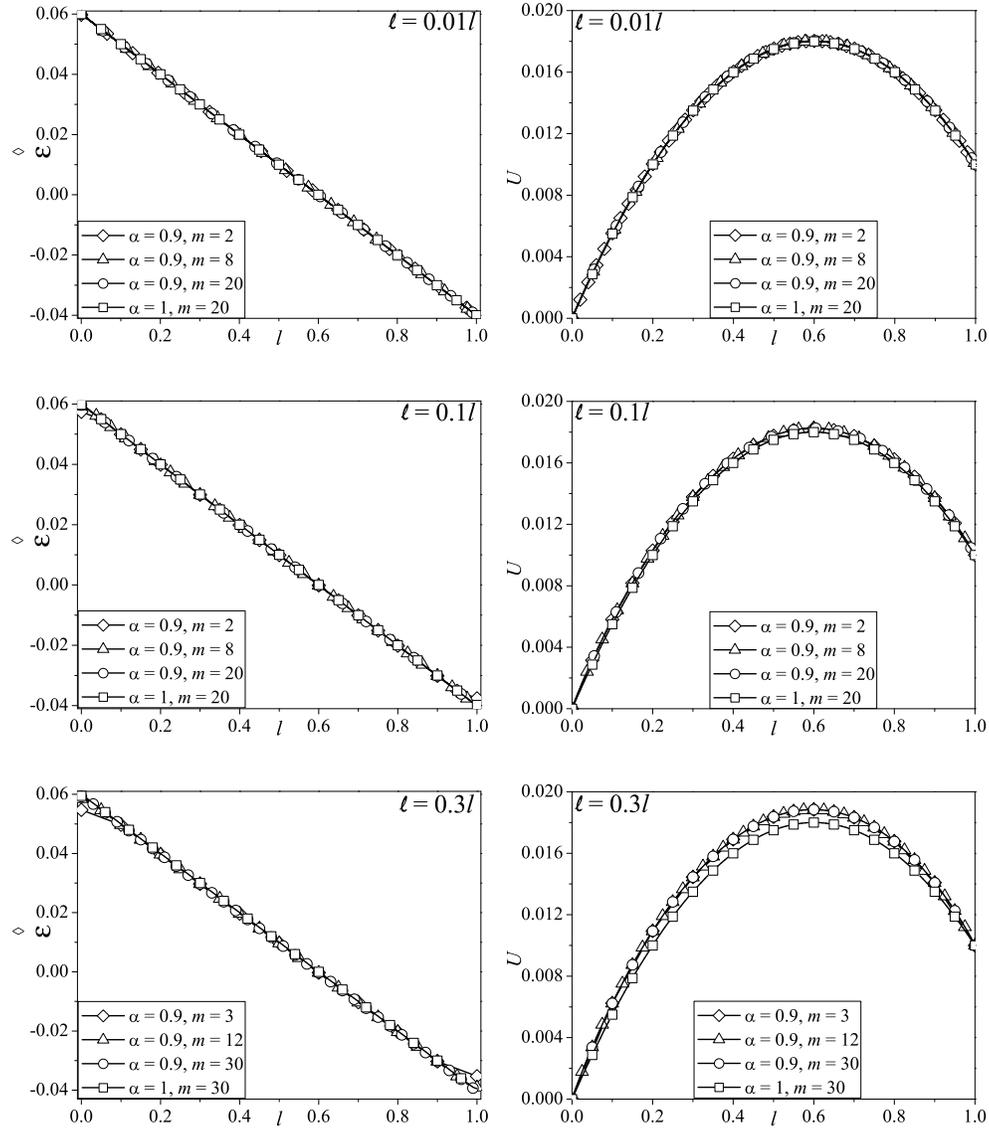


FIG. 8. The comparison of strains distribution and displacements through the length of the body for $\alpha = 0.9$, and selected ℓ and m for $b/E = 0.1$.

6. Conclusions

In this paper the concept of fractional continua is presented. This concept shows a new way, contrary to the existing models, of utilising fractional calculus in continuum mechanics to formulate *non-local* theory. Fractional counterparts

of commonly known measures such as left and right Cauchy–Green’s tensors, left and right stretch tensors etc. are introduced. It is also presented that objectivity requirements hold in exactly the same way as those in the classical description.

Based on the results of the boundary value problem showing one-dimensional deformation of fractional linear elastic body, the role of the non-local kinematics based on the fractional differential operators is presented. The numerical solution is obtained using generalised finite difference method. It is concluded that fractional continua dependently of their order and length scale can provide the opportunity for deeper insight into the analysed problem than classical approach. In this sense, the order of fractional continua and length scale become additional material parameters, whose value should be identified for specific type of material under consideration.

It is clear that the classical formulation is recovered as a special case of the introduced generalisation.

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