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Solvability of a theory of anti-plane shear with partially coated boundaries

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We consider the anti-plane shear of an elastic solid whose boundary is partially reinforced by a thin solid film represented by the union of a finite number of open curves. The solvability of the resulting boundary value problems is complicated by the presence of end-point conditions which must be satisfied at the ends of each section of the reinforcing film. In order to avoid complicated solvability conditions which carry no clear physical meaning, we modify the boundary integral equation method using an equivalent (lower-order) reinforcement condition which leads to the desired solvability results for the corresponding boundary value problems.

Key words: elastic reinforcement, linear elasticity, thin film, partial coating, surface energy.
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1. Introduction

The modelling of the mechanical behaviour of materials in which the separate mechanics of the bounding surface is known to significantly affect the overall deformation of the bulk continues to attract considerable interest in the literature. This class of problems is of particular interest to researchers working in the emerging area of nanomechanics in which the effects of surface energy have been included in continuum models in an attempt to understand the size-dependency of material properties at the nano-scale (see, for example, [1–3]).

In many cases, the contribution of surface mechanics can be captured by studying the mathematically equivalent problem of a solid whose boundary is coated by or reinforced with a thin solid film. Essentially, this idea of boundary reinforcement allows for the incorporation of the effects of surface stresses on the boundary of the solid through a higher order boundary condition which then forms part of a series of non-standard boundary value problems for the corresponding displacement field.

Boundary value problems for Laplace's equation involving a reinforcement boundary condition arise, for example, in the corresponding linear theory of anti-plane shear deformations of an elastic solid. In [4], the authors considered such boundary value problems but only in the case when the reinforced part of the boundary consisted of a finite number of sufficiently smooth closed curves. The more general case in which the reinforced section of the boundary can be represented by a finite number of open curves is of practical relevance [5] since it allows for the modelling of a wider class of problems. Unfortunately, this more general case is associated with reinforcement boundary conditions posed over a series of open arcs and the associated end-point conditions to be satisfied at the ends of each arc mean that the standard boundary integral equation approach cannot be applied in this particular case without imposing solvability conditions which carry no clear physical meaning. Instead, we proceed by utilizing an alternative lower-order form of the reinforcement boundary condition which is designed to automatically incorporate the corresponding end-point conditions. This particular form of the reinforcement boundary condition allows for the application of the boundary integral equation method albeit in generalized form. We find that the corresponding interior and exterior boundary value problems are reduced directly to Fredholm integral equations of the second kind from which solvability results are deduced in the appropriate classical function spaces.

2. Preliminaries

We consider the equilibrium of a deformable solid occupying a cylindrical region whose generators are parallel to the x_3-axis of a rectangular cartesian coordinate system. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. A state of anti-plane shear is characterized by a displacement field $u = (u_1, u_2, u_3)$ of the form

$$u_1(x_1, x_2, x_3) = u_2(x_1, x_2, x_3) = 0, \qquad u_3(x_1, x_2, x_3) = w(x),$$

where the out-of-plane displacement u_3 is a function w of $x \equiv (x_1, x_2)$ on the cross-section S of the cylinder. We assume that S is occupied by a homogeneous and isotropic elastic material with shear modulus $\mu_1 > 0$. The boundary ∂S of S is described by the union of a finite number of sufficiently smooth closed curves. We regard a subset Γ (consisting of a finite number m of sufficiently smooth open curves L_i with endpoints a_i and b_i, such that L_j and L_k have no point in common for $j \neq k$, $i, j, k = 1, \ldots, m$) of ∂S as being coated with a thin, homogeneous and isotropic elastic film with separate shear modulus $\mu_2 > 0$ (see Fig. 1).

It is well-known that, in the absence of body forces, the governing equation for the anti-plane displacement field w is given by

$$\Delta w(x) = 0, \ x \in S, \tag{2.1}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator in $\mathbb{R}^2$.

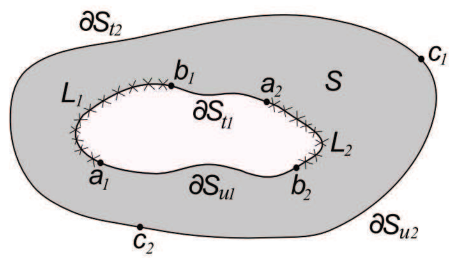

FIG. 1. Interior problem.

The boundary condition on the (reinforced) subset Γ of the boundary ∂S couples the response of the solid to that of the coating on Γ and is given by [4]:

$$\frac{1}{\mu_1}\sigma_{3n} = \frac{\partial w}{\partial n} = -h\frac{\mu_2}{\mu_1}\frac{d^2 w}{ds^2} + g. \tag{2.2}$$

Here, Γ is parametrized by arclength s and we denote by $n(s)$ the unit outward normal to Γ at $s \in \Gamma$, h is the thickness of the coating, $\partial(\ )/\partial n = n\cdot\nabla$ represents the normal derivative, σ_{3n} is the usual stress component in the Cartesian basis and g represents a prescribed traction along Γ.

In addition to (2.2), we must impose conditions at the end-points of Γ. Natural end-point conditions describe 'free-ends' at which the appropriate shearing force given by $h\mu_2\dfrac{dw}{ds}$ must vanish. Consequently, we impose conditions of the form

$$\frac{dw}{ds}(a_i) = \frac{dw}{ds}(b_i) = 0, \qquad i = 1, \ldots, m. \tag{2.3}$$

Alternatives to the end-point conditions (2.3) include the cases when one or both of the end-points of the coating are fixed, for example,

$$\begin{aligned} \frac{dw}{ds}(a_i) &= 0, \qquad w(b_i) = 0, \qquad \text{or} \\ w(a_i) &= 0, \qquad w(b_i) = 0, \qquad i = 1, \ldots, m. \end{aligned} \tag{2.4}$$

Finally, we recall that the fundamental solution for the Laplace operator is given by

$$D(x, y) = -\frac{1}{2\pi}\ln|x - y|,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote generic points in $\mathbb{R}^2$.

3. Alternative form of the reinforcement conditions

The reinforcement condition (2.2) is required over open arcs and when coupled with the end-point conditions (2.3) or (2.4) leads to a nonstandard boundary value problem whose analysis is not accommodated by the methods used in [4]. Instead, we proceed by integrating (2.2) along the reinforcement using the accompanying end-point conditions (from (2.3) or (2.4)) to evaluate the constants of integration. In this way, we incorporate the reinforcement condition (2.2) and the corresponding end-point conditions into an equivalent single lower-order boundary condition on the reinforcement.

We begin by writing (2.2) and (2.3) as

$$\begin{aligned} &\frac{d^2w}{ds^2}(x) = -\frac{1}{h\mu_2}(\sigma_{3n}(x) - t(x)) = -\frac{1}{h\mu_2}\mathcal{S}(x), \qquad x \in \varGamma, \\ &\frac{dw}{ds}(a_i) = \frac{dw}{ds}(b_i) = 0, \qquad i = 1, \ldots, m, \end{aligned} \tag{3.1}$$

where $t(x) = h(\mu_2/\mu_1)g(x)$. Integrate $(3.1)_1$ over the interval $[a_i, x]$, $x \in \varGamma$:

$$\frac{dw}{ds}(x) - \frac{dw}{ds}(a_i) = -\frac{1}{h\mu_2}\int_{a_i}^{x} \mathcal{S}(t)ds_t, \qquad x \in L_i.$$

Using the end-point conditions $(3.1)_2$, we obtain

$$\frac{dw}{ds}(x) = -\frac{1}{h\mu_2}\int_{a_i}^{x} \mathcal{S}(t)ds_t, \qquad x \in L_i. \tag{3.2}$$

To satisfy the remaining condition $\frac{dw}{ds}(b_i) = 0$, $i = 1, \ldots, m$, from (3.2) it is necessary and sufficient that

$$\int_{a_i}^{b_i} \mathcal{S}(t)ds_t = 0, \qquad i = 1, \ldots, m, \tag{3.3}$$

which expresses the requirement that $\mathcal{S}$ be a self-equilibrating system of tractions along the arcs L_i, $i = 1, \ldots, m$. It is clear that (3.2) and (3.3) are equivalent to (3.1).

Integrating (3.2) again, we obtain

$$w(x) - w(a_i) = -\frac{1}{h\mu_2}\int_{a_i}^{x}\int_{a_i}^{t_2} \mathcal{S}(t_1)ds_{t_1}ds_{t_2}, \qquad x, t_2 \in L_i,\ i = 1, \ldots, m. \tag{3.4}$$

Here $w(a_i)$ are constrained by conditions (3.3) and must therefore be chosen accordingly. In fact, (3.3) requires that

$$w(a_i) = \frac{1}{|L_i|}\int\limits_{a_i}^{b_i}(w(x) + \frac{1}{h\mu_2}\int\limits_{a_i}^{x}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2} + \lambda\mathcal{S}(x))ds_x, \qquad x, t_2 \in L_i,\ i = 1, \dots, m, \tag{3.5}$$

where λ is a suitably chosen parameter introduced to ensure that the term $\lambda\mathcal{S}$ is dimensionally correct.

It is seen that the reinforcement conditions (3.1) are equivalent to the Dirichlet condition (3.4) in which $w(a_i)$ are given by (3.5). For convenience, we write the specific values $w(a_i)$ from (3.5) as $\mathcal{C}^i$, $i = 1, \dots, m$. Then, from (3.4), we have the following Dirichlet boundary condition equivalent to (3.1):

$$w(x) = -\frac{1}{h\mu_2}\int\limits_{a_i}^{x}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2} + \mathcal{C}^i, \qquad x, t_2 \in L_i,\ i = 1, \dots, m. \tag{3.6}$$

3.1. The case of 'free-fixed' and 'fixed-fixed' end-point conditions

We can also consider end-point conditions given by (2.4). In the case of $(2.4)_1$, we again arrive at Eq. (3.4) with the $w(a_i)$ constrained by the requirement that $w(b_i) = 0$. Consequently, we choose

$$w(a_i) = \frac{1}{h\mu_2}\int\limits_{a_i}^{b_i}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2}, \qquad i = 1, \dots, m. \tag{3.7}$$

Using (3.7) in (3.4) we again obtain a Dirichlet condition of the form (3.6) with the constants $\mathcal{C}^i$, $i = 1, \dots, m$ given by (3.7).

In the case of $(2.4)_2$, an integration over the interval $[a_i, x]$ $x \in L_i$, $i = 1, \dots, m$, again brings us to

$$\frac{dw}{ds}(x) - \frac{dw}{ds}(a_i) = -\frac{1}{h\mu_2}\int\limits_{a_i}^{x}\mathcal{S}(t)ds_t, \qquad x \in L_i.$$

But now $\frac{dw}{ds}(a_i) \neq 0$, so we integrate again over the interval $[a_i, x]$:

$$w(x) - w(a_i) - \frac{dw}{ds}(a_i)(x - a_i) = -\frac{1}{h\mu_2}\int\limits_{a_i}^{x}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2}, \qquad x \in L_i.$$

Applying the condition $w(a_i) = 0$,

$$w(x) = \frac{dw}{ds}(a_i)(x - a_i) - \frac{1}{h\mu_2} \int\limits_{a_i}^{x} \int\limits_{a_i}^{t_2} \mathcal{S}(t_1) ds_{t_1} ds_{t_2}, \qquad x \in L_i. \tag{3.8}$$

We choose the value of $\frac{dw}{ds}(a_i)$ to satisfy the condition $w(b_i) = 0$, i.e.:

$$w(b_i) = \frac{dw}{ds}(a_i)(b_i - a_i) - \frac{1}{h\mu_2} \int\limits_{a_i}^{b_i} \int\limits_{a_i}^{t_2} \mathcal{S}(t_1) ds_{t_1} ds_{t_2} = 0,$$

so that

$$\frac{dw}{ds}(a_i) = \frac{1}{h\mu_2(b_i - a_i)} \int\limits_{a_i}^{b_i} \int\limits_{a_i}^{t_2} \mathcal{S}(t_1) ds_{t_1} ds_{t_2}. \tag{3.9}$$

Eq. (3.8) with Eq. (3.9) now leads to the following Dirichlet condition on Γ:

$$w(x) = \frac{(x - a_i)}{h\mu_2(b_i - a_i)} \int\limits_{a_i}^{b_i} \int\limits_{a_i}^{t_2} \mathcal{S}(t_1) ds_{t_1} ds_{t_2} - \frac{1}{h\mu_2} \int\limits_{a_i}^{t_2} \int\limits_{a_i}^{x} \mathcal{S}(t_1) ds_{t_1} ds_{t_2}, \qquad x \in L_i,$$

or in the form of a Dirichlet condition similar to (3.6), we have

$$w(x) = C^i(x - a_i) - \frac{1}{h\mu_2} \int\limits_{a_i}^{t_2} \int\limits_{a_i}^{x} \mathcal{S}(t_1) ds_{t_1} ds_{t_2}, \qquad x \in L_i, \tag{3.10}$$

where the constants $\mathcal{C}^i$, $i = 1, \ldots, m$ are given by the values of $\frac{dw}{ds}(a_i)$ from (3.9).

4. Mixed boundary-value problems

In the formulation of the corresponding boundary value problems, we first consider the case when S is a bounded domain enclosed by sufficiently smooth boundary ∂S. Write $\partial S = \partial S_1 \cup \Gamma$ where the curve ∂S_1 represents the non-reinforced section of ∂S. We divide ∂S_1 into two sets of open curves $\partial S_u = \partial S_{u1} \cup \partial S_{u2}$, $\partial S_t = \partial S_{t1} \cup \partial S_{t2}$ (generally, they can have common points c_1, c_2) and let Γ represent two open curves L_1 and L_2 with endpoints a_1, b_1 and a_2, b_2, respectively, such that they have no point in common. The case where ∂S_1 is divided into more than four parts and where Γ consists of a finite number (> 2)

of open curves is treated similarly without any significant modifications to our method. One of the domains satisfying this boundary-value problem is shown on Fig. 1.

The interior mixed boundary value problem requires that we find $w \in C^2(S) \cap C^1(\bar{S} \setminus \gamma)$ such that (2.1) is satisfied in S and

$$
\begin{aligned}
&w(x) = w^{(0)}(x), && x \in \partial S_u,\\
&\frac{\partial w(x)}{\partial n(x)} = t^{(0)}(x), && x \in \partial S_t,\\
&\frac{\partial w(x)}{\partial n(x)} = -h\frac{\mu_2}{\mu_1} d_x^2 w(x) + g(x), && x \in \Gamma.
\end{aligned}
\tag{4.1}
$$

Here, $d_x^2 = d_x d_x$ where $d_x \equiv d/ds_x$ denotes the directional derivative with respect to $s(x)$ along Γ, $\gamma = \{a_i, b_i, c_i\}$ $i = 1, 2$ and $w^{(0)}$, $t^{(0)}$ are, respectively, functions of prescribed displacement and stress on ∂S_u and ∂S_t. In addition, we require the end-point conditions (2.3) (or (2.4)).

The exterior problem is posed similarly except that S is now an unbounded domain in which we require, additionally, the standard asymptotic condition that as $r = |x| \to \infty$,

$$w = w^* + d, \tag{4.2}$$

where $w^* = O(r^{-1})$ and d is an arbitrary constant.

4.1. Uniqueness result

THEOREM 1. *Both the interior and exterior problems have at most one solution.*

Proof. The result follows immediately using classical techniques [6] by applying Green's first identity for the Laplace operator over a positively oriented boundary ∂S to the difference of any two solutions of either the interior or exterior mixed boundary value problem. □

4.2. Reduction to singular integro-differential equations

Without loss of generality, we consider the following mixed reinforcement problem.

Find $w \in C^2(S) \cap C^1(\bar{S} \setminus \gamma)$ such that (2.1) is satisfied in S and

$$
\begin{aligned}
&w(x) = 0, && x \in \partial S_u,\\
&\frac{\partial w(x)}{\partial n(x)} = 0, && x \in \partial S_t,\\
&\frac{\partial w(x)}{\partial n(x)} = -h\frac{\mu_2}{\mu_1} d_x^2 w(x) + g(x), && x \in \Gamma,\\
&\frac{dw}{ds}(a_i) = \frac{dw}{ds}(b_i) = 0, && i = 1, 2.
\end{aligned}
\tag{4.3}
$$

When S is bounded, (4.3) will describe the interior problem. When S is infinite, (4.3), with the added requirement (4.2), will describe the exterior problem.

4.2.1. Interior problem with natural ('free') end-point conditions. If we apply the standard boundary integral equation method [6] and seek the solution of the interior problem in the form of a single-layer potential, we obtain the following singular integro-differential equation:

$$
\frac{1}{2}\varphi(x) + \frac{h\mu_2}{2\pi\mu_1} \int_\Gamma e^{i\theta(x)} \frac{\varphi(y)}{(x-y)^2} dy = \int_\Gamma \Lambda(x,y)\varphi(y) ds_y + g(x), \quad x \in \Gamma, \tag{4.4}
$$

where, $\Lambda(x, y)$ is a weakly singular kernel. Following standard procedures, we can show that the imposition of two supplementary conditions on any solution φ of (4.4), namely

$$
\varphi(a_i) = \varphi(b_i) = 0, \tag{4.5}
$$

(see, for example, [7]), reveals that the singular operator associated with (4.4) has zero index. This allows for the application of Fredholm's alternative and the establishment of solvability results for the corresponding boundary value problems.

However, the conditions (4.5) have no apparent physical meaning in the context of this theory of reinforcement and are thus inconvenient in any existence theory.

To address this issue, we will show that if the reinforcement condition (3.1) is replaced with the alternative lower-order form (3.6), we can establish the required solvability results without resorting to conditions similar to (4.5).

To this end, we introduce the (in general, multiply-connected) domain Ω_I with sufficiently smooth boundary $\partial\Omega_I$ constructed so that

$$
\text{(i) } S \in \Omega_I, \qquad \text{(ii) } \Gamma \in \Omega_I, \qquad \partial S_u \cup \partial S_t \subseteq \partial\Omega_I.
$$

We write

$$\Omega_I = S + M_1 \cup M_2, \qquad \partial\Omega_I = \partial S_u \cup \partial S_t \cup S_1 \cup S_2,$$

where the subregions M_1 and M_2 are enclosed by S_1, L_1 and S_2, L_2, respectively. The domains S_i $(i = 1, 2)$ are divided, in turn, into two parts: S_{i1} and S_{i2} such that $S_{i1} \cap L_i$ consists of the endpoints L_i and $S_{i2} \cap L_i$ is the empty set.

Introduce the function P_1 satisfying

$$\begin{aligned}
\Delta P_1(x,y) &= 0, \qquad x \in \Omega_I,\\
P_1(x,y) &= -\frac{1}{2\pi}\frac{\partial \ln|x-y|}{\partial n(y)}, \qquad x \in \partial\Omega_I \setminus (\partial S_t + S_1 \cup S_2),\\
\frac{\partial P_1(x,y)}{\partial n(x)} &= -\frac{1}{2\pi}\frac{\partial^2 \ln|x-y|}{\partial n(x)\partial n(y)}, \qquad x \in \partial S_t + S_1 \cup S_2.
\end{aligned}$$

It is well-known [8] that P_1 exists uniquely for each $y \in \Gamma$ in the class $C^2(\Omega_I) \cap C^1(\bar{\Omega}_I \setminus \gamma)$.

Now seek the solution of the interior mixed reinforcement problem in the form of a modified double layer potential

$$w(x) = (W\varphi)(x) = \int\limits_{\Gamma} [\frac{\partial D(x,y)}{\partial n(y)} - P_1(x,y)]\varphi(y)ds_y, \qquad x \in S. \tag{4.6}$$

It is not difficult to show that all conditions of the interior problem are satisfied except for the reinforcement boundary condition (3.6) which leads to the following integral equation:

$$\begin{aligned}
&\frac{1}{2}\varphi(x) - \int\limits_{a_i}^{b_i}\left[\frac{\partial D(x,y)}{\partial n(y)} - P_1(x,y)\right]\varphi(y)ds_y \\
&= \frac{1}{h\mu_2}\int\limits_{a_i}^{x}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2} \\
&\quad - \frac{1}{|L_i|}\int\limits_{a_i}^{b_i}(w(x) + \frac{1}{h\mu_2}\int\limits_{a_i}^{x}\int\limits_{a_i}^{t_2}\mathcal{S}(t_1)ds_{t_1}ds_{t_2} + \lambda\mathcal{S}(x))ds_x, \quad x, t_2 \in L_i.
\end{aligned} \tag{4.7}$$

Using standard results from [6], we can write (4.7) in the equivalent form of a Fredholm equation of the second kind:

$$\frac{1}{2}\varphi(x) + \int\limits_{a_i}^{b_i} K(x,y)\varphi(y)ds_y = T^*(x), \qquad x \in L_i. \tag{4.8}$$

Here, $K(x,y)$ is a Fredholm kernel and $T^*(x)$ is determined by the data $g(x)$ prescribed on Γ.

Consequently, Fredholm's theorems hold for (4.8) and its corresponding adjoint equation. According to the properties of a standard double-layer potential for the Laplace equation as well as those of the function P_1, it follows that if we can show that (4.8) has a unique solution in $C^{1,\alpha}(\Gamma)$ whenever $T^* \in C^{1,\alpha}(\Gamma)$, then the modified potential (4.6) must be the unique solution of the interior reinforcement problem. To this end, we have the following theorem.

THEOREM 2. *The homogeneous equation* $(4.8)^0$ *(i.e.* (4.8) *with* $T^*(x) \equiv 0$) *has only the trivial solution.*

P r o o f. Let $\varphi_0 (\in C^{1,\alpha}(\Gamma))$ be a solution of $(4.8)^0$. Then,

$$w_0 = (W\varphi_0)(x) = \int\limits_{\Gamma} \left[\frac{\partial D(x,y)}{\partial n(y)} - P_1(x,y) \right] \varphi_0(y) ds_y \quad x \in S,$$

solves the homogeneous interior problem $(17)^0$ (i.e. (4.3) with $g \equiv 0$). The Uniqueness Theorem now yields $w_0 = (W\varphi_0)(x) = 0$, $x \in S$. Consequently, $\dfrac{\partial w_0(x)}{\partial n(x)} = 0$, $x (\in S) \to \Gamma$. By the Lyapunov–Tauber theorem for the double layer potential 16), we have that $\dfrac{\partial w_0(x)}{\partial n(x)} = 0$, $x \ (\in \Omega_I \setminus S) \to \Gamma$. Using the definition of $P_1(x,y)$ this means that

$$\begin{aligned} \triangle w_0(x) &= 0, \quad x \in \Omega_1 \setminus S, \\ w_0(x) &= 0, \quad x \in S_{12} \cup S_{22}, \\ \frac{\partial w_0(x)}{\partial n(x)} &= 0, \quad x \in S_{11} \cup S_{21} \cup \Gamma. \end{aligned}$$

By the uniqueness result for the classical interior mixed problem for Laplace's equation (see [8]), $w_0(x) = 0$ in the bounded domain $\Omega_I \setminus S$. Hence $(W\varphi_0)$ vanishes on both sides of the boundary Γ. The jump relations arising from the double layer potential [6] now yield that necessarily

$$W^+(\varphi_0) - W^-(\varphi_0) = \varphi_0 = 0 \quad \text{on} \quad \Gamma,$$

which completes the proof. □

This theorem now allows us to prove the main existence result for the boundary value problem (4.3).

THEOREM 3. *The interior problem* (4.3) *with reinforced boundary* Γ *has a unique solution whenever* $g \in C^{1,\alpha}(\Gamma)$, $0 < \alpha < 1$. *This solution is given by* (4.6) *with* $\varphi \in C^{1,\alpha}(\Gamma)$, $0 < \alpha < 1$, *the unique solution of* (4.8) *whenever* $T^* \in C^{1,\alpha}(\Gamma)$.

P r o o f. From what has been said above, Fredholm's theorems hold for (4.8) and its associated (adjoint) system. From Theorem 2, the homogeneous system $(4.8)^0$ has only the trivial solution. Hence, by Fredholm's theorems and results on smoothness of solution for equations of the type (4.8) [6], we have that (4.8) always has a unique solution $\varphi \in C^{1,\alpha}(\Gamma)$ whenever $T^* \in C^{1,\alpha}(\Gamma)$. Finally, (4.6) is the unique solution of (4.3) with $\varphi \in C^{1,\alpha}(\Gamma)$ delivered from (4.8). □

4.2.2. Exterior problem with natural end-point conditions The exterior problem from (4.3) is treated similarly except that now since S is an unbounded domain, any solution must also satisfy the asymptotic condition given by (4.2). We proceed as for the interior problem and introduce the (in general, multiply-connected) infinite domain Ω_E with sufficiently smooth boundary $\partial\Omega_E$ such that

(i) $S \subset \Omega_E$; (ii) $\Gamma \subset \Omega_E$; (iii) $(\partial S_u \cup \partial S_t) \subseteq \partial\Omega_E$; (iv) $\{0\} \notin \bar{\Omega}_E$.

We write

$$\Omega_E = S + M_1 \cup M_2, \qquad \partial\Omega_E = \partial S_u \cup \partial S_t \cup S_1 \cup S_2,$$

where subregions M_1 and M_2 are enclosed by S_1, L_1 and S_2, L_2, respectively. S_i $(i = 1, 2)$ are divided, in turn, into two parts: S_{i1} and S_{i2} such that $S_{i1} \cap L_i$ consists of the endpoints L_i and $S_{i2} \cap L_i$ is empty set.We then seek a solution in the form

$$w(x) = (W\varphi)(x) = \int\limits_{\Gamma} \left[\frac{\partial D(x,y)}{\partial n(y)} - P_2(x,y)\right]\varphi(y)ds_y, \qquad x \in S, \tag{4.9}$$

where the function $P_2(x, y)$ (for each $y \in \Gamma$) is the unique solution of the following mixed boundary value problem in $C^2(\Omega_E) \cap C^1(\bar{\Omega}_E \setminus \gamma)$ satisfying (see [8]):

$$\begin{aligned}
\Delta P_2(x,y) &= 0, \qquad x \in \Omega_E,\\
P_2(x,y) &= -\frac{1}{2\pi}\frac{\partial \ln|x-y|}{\partial n(y)}, \qquad x \in \partial\Omega_E \setminus (\partial S_t + S_1 \cup S_2),\\
\frac{\partial P_2(x,y)}{\partial n(x)} &= -\frac{1}{2\pi}\frac{\partial^2 \ln|x-y|}{\partial n(x)\partial n(y)}, \qquad x \in \partial S_t + S_1 \cup S_2.
\end{aligned}$$

Here φ is again an unknown density-function of the Hölder class $C^{1,\alpha}(\Gamma)$, $\alpha \in (0, 1)$, defined on Γ. The fact that $(W\varphi)$ from (4.9) satisfies the asymptotic condition (4.2) follows from the asymptotic behaviour of $\dfrac{\partial D(x,y)}{\partial n(y)}$ as $|x| \to \infty$ and the definition of the function P_2 which is chosen specifically to satisfy the boundary value problem above.

It is again clear that (4.9) satisfies all the conditions of the exterior problem from (4.3) except the reinforcement boundary condition (3.6). Proceeding as above for the interior problem we again obtain Fredholm-type equations for φ on Γ, almost identical to (4.8), for which the analogue of Theorem 2 can be established. Moreover, Theorem 3 also holds for the exterior problem, except that the unique solution is now given by (4.9).

Remark 1. The 'free-fixed' and 'fixed-fixed' end-point conditions lead only to insignificant changes in detail in the ensuing integral equations of the form (4.8). Consequently, it is a relative simple matter to write down similar existence results for these particular cases.

Finally, we remark that the implementation of the above theory is the subject of a further paper [9] in which the authors study asymptotic solutions near the crack tip of an interface crack whose crack faces are coated with a thin reinforcing film of separate elastic material. We find that the effect of the reinforcement is to reduce the order of the stress singularity at the crack tip from the classical $O(r^{-\frac{1}{2}})$ to $O(\ln r)$. In addition, we demonstrate that the reinforcement induces a displacement field which is smooth locally and bounded at the crack tip.

5. Conclusions

We consider an elastic solid whose boundary is partially reinforced by a thin elastic coating represented by the union of a finite number of open curves. By integrating the non-standard higher-order boundary condition on the reinforced section of the boundary, we obtain an alternative lower-order form of boundary condition which automatically incorporates the end-point conditions. This allows for the establishment of appropriate existence results for the corresponding mixed boundary value problems via the theory of Fredholm integral equations.

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