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Non-linear singular integral equations analysis for unsteady cascade aeroelasticity applied in turbomachines

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A two-dimensional unsteady cascade aeroelasticity is introduced for the investigation of flow fields of turbomachines (gas or steam). Especially, the velocity field around a cascade of airfoils is determined, while such a problem is reduced to the solution of a non-linear multidimensional singular integral equation when considering harmonic time dependence between the motions of adjacent blades of the turbine. Consequently, a general non-linear model is investigated by proposing an "innovative" and "groundbreaking" method. An application is finally presented by considering a special description of the velocity field and therefore such a field is determined for arbitrary geometry and arbitrary interblade phase angle.

Key words: non-linear multidimensional singular integral equations, two-dimensional unsteady cascade aeroelasticity, unsteady flow, turbomachines.

Notations

Λ	strength of vortices,
ζ_n	location of the n-th airfoil,
t	time,
a	cascade spacing,
c	length of wake,
Z	point on the zeroth blade,
τ	time lag,
$\lambda_a(x)$	vortex strength distribution,
v, u	unsteady velocities,
δ	phase angle.

1. Introduction

Over the last years the big evolution of the jet engine and the high performance axial-flow compressor increased too much the possibilities of turbomachines applied in aircrafts. Also, the turbojet engines were found to experience severe vibration of the rotor blades and part-speed operation. The increasing evolution of aeroelasticity in aircraft turbomachines continues to be under active investigation, driven by the needs of aircraft powerplant and turbine designers.

When considering an airfoil or blade in an axial- flow turbine or compressor which is running at some rotational speed, the blade, because of the aerodynamic and structural performance, has certain geometric properties defined by its length, root and tip fixation, possible mechanical attachment to other blades and by the chord, camber, thickness, stagger and profile shape which are functions of the radial coordinate. Moreover, the blade can be constructed in such a manner that the line of centroids and the line of shear centers are neither radial nor straight, but are defined by schedules of axial and tangential coordinates as functions of radius.

Two- and three-dimensional cascade flow was already formulated in the 1970s by Smith [1], Namba [2], Salan [3], Goldstein [4], Platzer [5], Adamczyk and Goldstein [6], Fleeter [7] and others. In order to apply two-dimensional theory to the aeroelastic problems of real blade systems they used either a representative section analysis or they applied the strip hypothesis; the aerodynamics at one radius is uncoupled from the aerodynamics at any other radius.

In addition to the above references, during the 1980s the methods of two-dimensional unsteady incompressible flow were further extended by Kaza and Kielb [8], Verdon and Caspar [9], Spalart [10], Crawley and Hall [11], Speziale, Sisto and Jonnavithula [12], Platzer and Carta [13], Sisto, Wu, Thangam and Jonnavithula [14] and others. Subject areas which received attention include such topics as finite shock motion, variable shock strength of thick and highly cambered blades, and the effects of curvilinear wakes and vorticity transport.

On the other hand, Ladopoulos [15] proposed a finite-part singular integro-differential equations method arising in two-dimensional aerodynamics and extended his non-linear method [16, 17] for the solution of some important problems of solid mechanics theory. Furthermore, Ladopoulos [18, 19] introduced a two-dimensional fluid mechanics representation analysis for the investigation of inviscid flowfields of unsteady airfoils. More precisely, the velocity and pressure coefficient field around a NACA airfoil was determined, while such a problem was reduced to the solution of a non-linear multidimensional singular integral equation, when the form of the source and vortex strength distribution are dependent on the history of the vorticity and source distribution on the NACA airfoil surface. This method was applied to determine the velocity field around the blades of a vertical axis wind turbine.

Beyond the above sources, Ladopoulos and Zisis [20, 21] applied the non-linear singular integral equations to determine the form of the profiles of a turbomachine in two-dimensional steady flow of an incompressible fluid. On the other hand, in the present investigation the speed field is determined around a cascade of airfoils of a turbomachine (gas or steam) in two-dimensional unsteady and incompressible flow. Such a problem is reduced to the solution of a non-linear multidimensional singular integral equation, when considering harmonic time dependence between the motions of adjacent blades of the turbine. As an application of the proposed method the velocity field of a turbomachine is considered, and therefore this field is determined for arbitrary geometry and arbitrary interblade phase angle.

2. Non-linear unsteady cascade aeroelasticity

The aeroelastic problems of the axial flow turbine (gas or steam) are of increasing interest over the last years. In the turbomachine, the angle of attack of each rotor airfoil at each radius r is compounded of the tangential velocities of the airfoil section due to rotor rotation and through the flow velocity as modified in direction by the upstream stator row (Fig. 1).

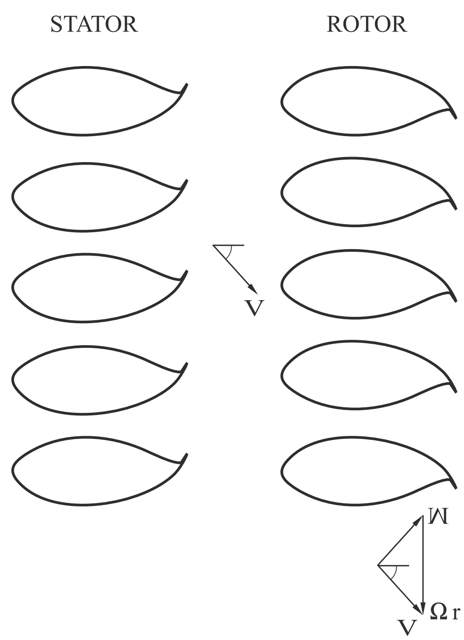

Fig. 1. Velocity triangle in an axial turbine.

Materials of which the blades are made are conventionally aluminum alloys, steel or stainless steel (high nickel and/or chromium content). On the other hand, in some recent applications titanium and beryllium were used too. Moreover, blades of turbomachines can be made of laminated materials such as glass cloth, graphite or metal oxide fibers present in polymeric or metal matrix materials and modeled under pressure to final airfoil contours.

When considering a cascade of airfoils, the fact that the flexible blades may be vibrating means that the relative pitch and stagger may be functions of time and also position in the cascade (Fig. 2).

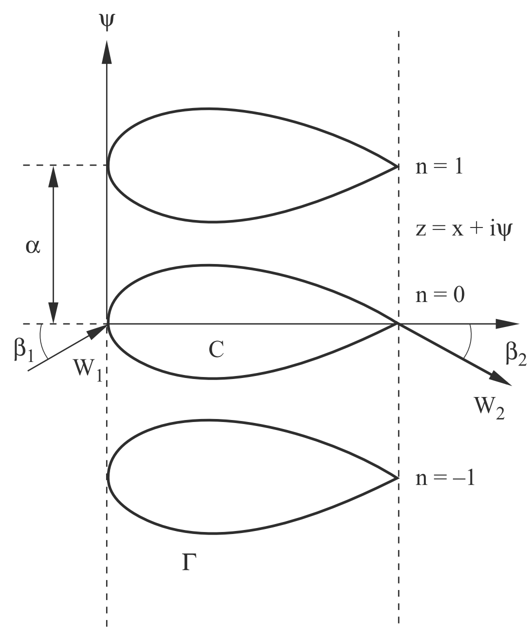

Fig. 2. A lattice of airfoils in an axial turbine.

The velocities induced by an infinite column of vortices of equal strength Λ are given by the relation [20, 21]

$$\delta[u(z) - iv(z)] = \frac{i\Lambda}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{Z - \zeta_n}, \tag{2.1}$$

where ζ_n denotes the location of the n-th airfoil

$$\zeta_n = \xi + inae^{-i\beta} + iY_n(\xi_n, t) + X_n(t) \tag{2.2}$$

in which t is the time and $Y_n \ll a$, $X_n \ll c$, with a the pitch (see Fig. 2) and c the length of the wake (Fig. 2). Also, the point Z is on the zeroth blade:

$$Z = x + iY(x, t) + X(t) \tag{2.3}$$

and the points ξ_n in Eq. (2.2) are equal to

$$\xi_n = \xi + na \sin\beta. \tag{2.4}$$

Beyond the above, the harmonic time dependence between the motions of adjacent blades is given by the relation

$$\Upsilon_n(\xi_n,t) = e^{i\pi\omega\tau}Y(\xi,t) \tag{2.5}$$

with τ the time lag.

Therefore, the singular kernel in Eq. (2.1) takes the form

$$\frac{1}{Z-\zeta_n} = \frac{1}{[x-\xi-inae^{-i\beta}+i[Y(x,t)-Y_n(\xi_n,t)]+X(t)-X_n(t)]} \tag{2.6}$$

and for infinite number of blades it is

$$\sum_{n=-\infty}^{\infty} \frac{1}{Z-\zeta_n} = \frac{1}{x-\xi+i[Y(x,t)-Y(\xi,t)]} + \sum_{n=-\infty}^{\infty}{}^{*} \frac{1}{Z-\zeta_n}. \tag{2.7}$$

For the thin-airfoil theory the first term in Eq. (2.7) is conventionally ignored, and therefore the remaining term can be written as follows:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}{}^{*} \frac{1}{Z-\zeta_n} \approx \sum_{n=-\infty}^{\infty}{}^{*} \frac{1}{x-\xi-inae^{-i\beta}} + i\sum_{n=-\infty}^{\infty}{}^{*} \frac{Y_n(\xi_n,t)-Y(x,t)}{(x-\xi-inae^{-i\beta})^2} \\ + \sum_{n=-\infty}^{\infty}{}^{*} \frac{X_n(t)-X(t)}{(x-\xi-inae^{-i\beta})^2} + \ldots .\end{aligned} \tag{2.8}$$

By combining Eqs. (2.1) and (2.8) and denoting by $\lambda_a(x)$ (a is a cascade spacing) the vortex strength distribution, we obtain for the unsteady induced velocities

$$\begin{aligned}\delta[u^*(x')-iv^*(x')] \approx -\frac{\lambda_\alpha(\xi')\delta\xi'}{2\pi c}R^2\Bigg[\sum_{n=-\infty}^{\infty}{}^{*} \frac{e^{in\omega\tau}\Upsilon(\xi',t)-\Upsilon(x',t)}{(S-in\pi)^2} \\ + \frac{1}{i}\sum_{n=-\infty}^{\infty}{}^{*} \frac{e^{in\omega\tau}X(t)-X(t)}{(S-in\pi)^2}\Bigg],\end{aligned} \tag{2.9}$$

where the variables R and S are equal to

$$\begin{aligned} R &= \pi e^{i\beta}(c/a), \\ S &= R(x'-\xi') \end{aligned} \tag{2.10}$$

and u^*, v^* are time dependent parts of u and v.

Furthermore, let us replace by Q

$$Q = 1 - \frac{\omega\tau}{\pi} \tag{2.11}$$

and let the special case when the blades move perpendicular to their chord lines with the same amplitude along the chord:

$$\Upsilon = -c_1^* e^{i\omega t} = -c_2. \tag{2.12}$$

Furthermore, by integrating over the chord in Eq. (2.9) we obtain the following non-linear multidimensional singular integral equation:

$$u^*(x') - iv^*(x') = \frac{R^2}{2\pi c} \int_0^1 \lambda_a(\xi') \sum_{n=-\infty}^{\infty}{}^* \frac{e^{in\omega\tau} c_2(t) - c_2(t)}{(S - in\pi)^2} \tag{2.13}$$

which further takes the form

$$\begin{aligned} u^*(x') &= -\frac{c_2}{2\pi c} \int_0^1 \lambda_a(\xi')[F - iI]d\xi', \\ v^*(x') &= \frac{c_2}{2\pi c} \int_0^1 \lambda_a(\xi')[G + iH]d\xi', \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} F + iG &= R^2 \frac{Q \sinh S \sinh QS - \cosh S \cosh QS + 1}{\sinh^2 S}, \\ H + iI &= R^2 \frac{Q \sinh S \cosh QS - \cosh S \sinh QS}{\sinh^2 S}. \end{aligned} \tag{2.15}$$

3. Non-linear unsteady aerodynamics application

As an application of the previous outlined theory, consider the case where the velocity field is described as follows, on $y = 0$ for $0 < x < c$, with c the length of the wake:

$$\begin{aligned} v_1(x) + v_2(x) + v_3(x) = &\frac{1}{2\pi} \int_0^\infty [\lambda_1(\xi) + \lambda_2(\xi) + \lambda_3(\xi)] K(\xi - x) d\xi \\ &+ \frac{1}{2\pi} \int_c^\infty [\lambda_1(\xi) + \lambda_2(\xi) + \lambda_3(\xi)] K(\xi - x) d\xi, \end{aligned} \tag{3.1}$$

where $\lambda_a(\xi)$ $(a = 1, 2, 3)$ denotes the vortex strength distribution.

Beyond the above, in Eq. (3.1) v_1 denotes the unsteady upstream velocity convected as a gust with the mean flow and v_2 the unsteady upstream velocity attributable to vibratory displacement of all the blades in the cascade. Also, v_3 is the unsteady upstream velocity relative to the zeroth airfoil occasioned by its harmonic vibration. Furthermore, the vortex distributions λ_1, λ_2, λ_3, which represent the lift distributions on the cascade chord lines, are unsteady; hence, they must give a rise to distributions of free vortices in the wake of each airfoil of the cascade.

The kernel $K(\xi - x)$ in Eq. (3.1) may be written as follows:

$$\frac{1}{2\pi}K(\xi - x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{in\omega\tau}}{\xi - x + inae^{-i\beta}} \tag{3.2}$$

which further takes the form

$$\frac{1}{2\pi}K(\xi - x) = \frac{e^{i}}{2a}\,\frac{\cosh[(1-\delta/\pi)\pi e^{i\beta}(\xi - x)/a] + i\sinh[(1-\delta/\pi)e^{i\beta}(\xi - x)/a]}{\sinh[\pi e^{i\beta}(\xi - x)/a]}, \tag{3.3}$$

where δ denotes the interblade phase angle.

From the previous analysis, it is obvious that the parameters β, a/c and δ are strong determinants of the unsteady aerodynamic reactions. Consequently, in the following Table 1 the values of $u(x)$ and $v(x)$are given for arbitrary geometry (β and a/c) and arbitrary interblade phase angle δ.

Table 1. Velocity values for the airfoils cascade of a turbomachine.

Case No.	a/c	β	δ	$u(x)$	$v(x)$
1	1	0	0.4π	-0.6004	-0.0580
2	1	0.25π	0	-0.4261	-0.2267
3	1	0.25π	0.4π	-0.7318	-0.1558
4	1	0.42π	0.4π	-0.7606	-0.5298
5	1	0.25π	0.8π	-0.8734	-0.0232
6	2	0.25π	0.4π	-0.6903	-0.0991
7	∞	—	—	-0.6963	-0.4343

From Table 1, it follows that the values of speeds are increasing when the angle β is increasing as well and furthermore speeds $u(x)$ are again increasing, when the interblade phase angle δ is increasing. On the other hand, there is a decreasing in the values of the velocity field when increasing the pitch to chord ratio a/c, while $\beta$and δ are remaining the same (see Cases 3 and 6).

4. Conclusions

A general non-linear model was proposed, for the determination of the velocity field around a cascade of airfoils of a turbomachine. Such a problem was reduced to the solution of a non-linear multidimensional singular integral equation, when considering harmonic time dependence between the motions of adjacent blades of the turbine.

Such non-linear singular integral equation methods will be of increasing interest in the future, as these methods are very important for the solution of generalized solid mechanics and fluid mechanics problems. Modern problems of fluid and solid mechanics are much more simplified when solved by general non-linear singular integral equation methods.

Beyond the above, the field of aeroelasticity in turbomachines, continues to be under active investigation, driven by the needs of aircraft powerplant, gas turbine and steam turbine designers. Especially, the design of the new generation turbomachines for the next generation aircrafts will be made possible by highly sophisticated non-linear computational methods.

Consequently, the non-linear singular integral equations method which was successfully used over the last years for the solution of problems of aerodynamics, fluid mechanics, hydraulics, structural analysis and fracture mechanics, etc. will be further used for the design of the next generation turbomachines.

As a future research we propose the extension of the current non-linear method of incompressible flows to compressible flows too. Thus, in such case the Mach number should be used together with other parameters of the compressible flows. As complicated turbomachines for next generation aircraft and spacecraft [22–25] work with very high speeds in compressible flows, our "groundbreaking" non-linear method should be extended to compressible flows too.

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