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Explicit and exact Levinson-type solutions for multilayered plates

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Explicit and exact three-dimensional solutions are derived for the equilibrium problem of plate-like bodies with rectangular or circular cross-sections, made of several layers of transversely isotropic materials, simply supported on the lateral boundary, and loaded on the end faces. The component-wise representation of equilibrium displacement fields has the form of a series, whose typical term is the product of a function of the in-plane coordinates and a function of the transverse coordinate. Examples are presented to demonstrate, by comparison with finite-element solutions, the accuracy achieved when solution series are truncated after few terms.

Key words: three-dimensional exact solutions, multilayered plate-like bodies.

1. Introduction

A central issue in structure mechanics is to assess how well a lower-dimensional model approximates the behavior of the three-dimensional bodies this model is associated with. Explicit solutions of representative three-dimensional problems are rarely available for benchmarking purposes. In the case of linearly elastic plate theory, such a benchmark solution was offered by Levinson [1] in 1985.

Levison considered a three-dimensional plate-like body in the form of a right parallelepiped of rectangular cross-section, comprised of a linearly elastic *isotropic* material, loaded exclusively on the top and bottom ends and with homogeneous complementing conditions on the lateral surface, where the author assumed both the tangential components of displacement and the normal component of the applied traction to be null. Levinson's approach has been extended to treat piezoelectric plate-like bodies [2], the dynamics of electro-elastic plate-like bodies [3], transversely isotropic plate-like bodies with general cross-section [4] and general edge conditions [5], and simply supported plate-like bodies with circular cross-section, made of isotropic [6] and transversely isotropic materials [7].

In the present paper, we derive explicit and exact Levinson-type equilibrium solutions for plate-like bodies composed of several layers of *possibly different* linearly elastic *transversely isotropic* materials, all layers having identical rectangular or circular cross-section.

Levinson laid down the following *a priori* representation for a class of candidate solution displacements:

$$\mathbf{u}(x_1,x_2,x_3) = -g(x_3)\, w,_{\alpha}(x_1,x_2)\mathbf{e}_\alpha + f(x_3)\, w(x_1,x_2)\mathbf{e}_3 \qquad (\alpha = 1,2), \tag{1.1}$$

parameterized by a function w of the in-plane coordinates and two functions f, g of the transverse coordinate. With (1.1), Lamé's equilibrium equations are satisfied for w an eigenfunction of the Laplace operator with homogeneous Dirichlet conditions, and for f, g satisfying a system of two second-order ordinary differential equations, whose solutions depend on four integration constants that are determined by the values of the surface loads acting on the end faces of the body. More precisely, the boundary conditions on the end faces require that the normal and tangential components of the surface loads be expressible in series of the eigenfunctions w and their gradients, respectively; then, for each eigenfunction w, a pair of functions f and g is found, with the integration constants determined by the coefficients of the relative series expansions of the loads.

Our derivation mimics Levinson's, in that we move by taking his representation for the displacement in each layer (see Eq. (2.2)), and we require that the relative triplet of parameter fields satisfies Lamé's equations at interior points of the layer, as well as Levinson-type boundary conditions.

The crucial point of our solution method comes when *interlayer transmission conditions* for the displacement and the traction vectors are posed. We impose continuity for both: (i) displacement continuity forbids sliding or detachment of adjacent layers; (ii) traction continuity is implied by the request that part-wise equilibrium equations be localizable even for plate-like body parts including one or more interfaces, that are singular surfaces for the stress field (see, e.g., [8], Sect. 193). Displacement continuity implies that the eigenfunctions w of different layers correspond to the same eigenvalues and are therefore proportional; traction continuity permits to express the integration constants in functions f, g for a layer in terms of those for one of the adjacent layers, so that, by a process of elimination, a system of four equations is obtained that determines the integration constants of one of the layers in terms of the loads acting on the end faces of the plate-like body; then, an inverse process gives in sequence the integration constants for functions f, g of all other layers.

We consider equilibrium of multilayered plate-like bodies with rectangular or circular cross-section[1]. Our paper is organized as follows. In Section 2 we deduce the exact equilibrium solution for multilayered plate-like bodies with rectangular cross-section, where each layer is subject to tractions on its end faces, while homogeneous complementing conditions of Levinson type hold at its lateral boundary. In Section 3 we solve the corresponding equilibrium problem of multilayered plates with circular cross-section, subject to axisymmetric loads on the end faces; on the lateral boundary, where the transverse and tangential displacements are taken null as in the case of a rectangular cross-section, a system of radial tractions is applied, to guarantee equilibrium (see [6, 7]). For both rectangular and circular cross-sections, the solutions are obtained in series form: in Section 4, we present some examples which, by comparison with finite-element solutions, demonstrate the remarkable accuracy achieved even if our series solutions are truncated after few terms. Section 5 contains our conclusions.

2. Plate-like bodies with rectangular cross-section

Consider a plate-like body $\mathscr{C}$ which is identified with the cylindrical region of height $2h$ it occupies in the three-dimensional space; let (x_1, x_2, x_3) be a Cartesian coordinate system with origin on the mid-section $\mathscr{S}$ of $\mathscr{C}$ and x_3-axis parallel to its generatrix, and denote by $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the coordinate base vectors.

The body $\mathscr{C}$ is assumed to be composed of n layers of transversely isotropic materials. The k-th layer $(1 \leq k \leq n)$ occupies the sub-region $\mathscr{C}^{(k)}$ of $\mathscr{C}$ included between the planes $x_3 = x_3^{(k-1)}$ and $x_3 = x_3^{(k)}$, with $x_3^{(0)} = -h$ and $x_3^{(n)} = h$. The height of the k-th layer is $2h^{(k)} = x_3^{(k)} - x_3^{(k-1)}$, and its mid-section is the set of points for which $x_3 = o^{(k)} = (x_3^{(k)} + x_3^{(k-1)})/2$. The symbols $\mathscr{S}^+$ and $\mathscr{S}^-$ denote the end faces of $\mathscr{C}$, i.e., the cross-sections corresponding to $x_3 = \pm h$; $\partial\mathscr{C}$ denotes the boundary of the body $\mathscr{C}$, $\partial\mathscr{S}$ that of its mid-section $\mathscr{S}$.

[1]The case of rectangular cross-section was solved by Pan [9]. Applying a method already used to study mechanical problems concerning layered bodies, Pan assumes that the displacement field in each layer of the body is composed of (i) a vector in the traversal direction that has the form of a unit vector $\boldsymbol{e}$ times the product of a function of the transverse coordinate and a Fourier sine series S of the in-plane coordinates; and (ii) two vectors parallel to the mid-plane, which are given by the products of functions of the transverse coordinate and, respectively, the gradient of S and the curl of $S\boldsymbol{e}$. Such a displacement field is required to satisfy equilibrium equations, Levinson's type conditions on the boundary of the plate-like body and, at layer interfaces, continuity conditions for the field itself and for the tractions. These requirements reduce the assumed displacement field to that of the Levinson-type equilibrium solution. Our work is more general than Pan's, because we also consider plates with circular cross-section and because we do not pre-assign the form of the dependence of the displacement held on the in-plane coordinates, but instead deduce it by requiring that the equilibrium equations be satisfied.

In this section, we consider the equilibrium of $\mathscr{C}$ in the case in which $\mathscr{S}$ is a rectangle with sides of length l_1 and l_2, parallel to the axes x_1 and x_2. The boundary conditions on the lateral surface $\partial\mathscr{S} \times (-h, h)$ of $\mathscr{C}$ are

$$u_3 = 0, \qquad u_\tau = 0, \qquad t_n = 0, \tag{2.1}$$

where u_3 and u_τ are the transverse and tangential components of the displacement $\boldsymbol{u}$, and t_n is the normal component of the traction; surface loads $\boldsymbol{t}^{\pm}$ are assigned on the end faces $\mathscr{S}^{\pm}$ of $\mathscr{C}$.

In each layer $\mathscr{C}^{(k)}$, $k = 1, \ldots, n$, the displacement field takes the same form as the one assumed in the Levinson's equilibrium solution for a single-layered plate; namely,

$$\begin{aligned} u_\alpha^{(k)}(x_1, x_2, x_3) &= -g^{(k)}(x_3)\, w_{,\alpha}^{(k)}(x_1, x_2), \quad \alpha = 1, 2, \\ u_3^{(k)}(x_1, x_2, x_3) &= f^{(k)}(x_3)\, w^{(k)}(x_1, x_2), \end{aligned} \tag{2.2}$$

where $u_i^{(k)}$ is the restriction of the component $u_i = \boldsymbol{u} \cdot \boldsymbol{e}_i$ to the k-th layer, and $(\,\cdot\,)_{,\alpha}$ denotes partial differentiation with respect to x_α, for $\alpha = 1, 2$. To the displacement (2.2) there corresponds the strain

$$E_{\alpha\beta}^{(k)} = -g^{(k)} w_{,\alpha\beta}^{(k)}, \quad E_{\alpha 3}^{(k)} = \frac{1}{2}\left(f^{(k)} - \frac{dg^{(k)}}{dx_3}\right) w_{,\alpha}^{(k)}, \quad E_{33}^{(k)} = \frac{df^{(k)}}{dx_3} w^{(k)}\,. \tag{2.3}$$

We express the constitutive equation

$$\boldsymbol{S}^{(k)} = \mathbb{C}^{(k)} \boldsymbol{E}^{(k)}, \tag{2.4}$$

of the transversely isotropic elastic material comprising the k-th layer in terms of the moduli $\lambda^{(k)}$, $\mu^{(k)}$, $\widehat{\lambda}^{(k)}$, $\widehat{\mu}^{(k)}$ and $\widetilde{\lambda}^{(k)}$, which are related to the components of the elasticity tensors $\mathbb{C}^{(k)}$ by

$$\lambda^{(k)} = \mathbb{C}_{1122}^{(k)}, \quad \mu^{(k)} = \mathbb{C}_{1212}^{(k)}, \quad \widetilde{\lambda}^{(k)} = \mathbb{C}_{1133}^{(k)}, \quad \widehat{\lambda}^{(k)} = \mathbb{C}_{3333}^{(k)}, \quad \widehat{\mu} = \mathbb{C}_{1313}, \tag{2.5}$$

with

$$\mathbb{C}_{1111}^{(k)} = \mathbb{C}_{1122}^{(k)} + 2\mathbb{C}_{1212}^{(k)} = \lambda^{(k)} + 2\mu^{(k)}. \tag{2.6}$$

Accordingly, the stresses are

$$\begin{aligned} S_{\alpha\beta}^{(k)} &= \left(\widetilde{\lambda}^{(k)} \frac{df^{(k)}}{dx_3} w^{(k)} - \lambda^{(k)} g^{(k)} \Delta w^{(k)}\right) \delta_{\alpha\beta} - 2\mu^{(k)} g^{(k)} w_{,\alpha\beta}^{(k)}, \\ S_{\alpha 3}^{(k)} &= \widehat{\mu}^{(k)} \left(f^{(k)} - \frac{dg^{(k)}}{dx_3}\right) w_{,\alpha}^{(k)}, \\ S_{33}^{(k)} &= -\widetilde{\lambda}^{(k)} g^{(k)} \Delta w^{(k)} + \widehat{\lambda}^{(k)} \frac{df^{(k)}}{dx_3} w^{(k)}, \end{aligned} \tag{2.7}$$

where $\alpha, \beta = 1, 2$, and Δ is the Laplace operator. Equations (2.1), (2.2), and (2.7) imply that, in the k-th layer, the boundary conditions on $\partial\mathscr{S} \times (x^{(k-1)}, x^{(k)})$ are

$$\begin{gathered} w^{(k)}(0, x_2) = w^{(k)}(l_1, x_2) = w^{(k)}(x_1, 0) = w^{(k)}(x_1, l_2) = 0, \\ u_2^{(k)}(0, x_2, x_3) = u_2^{(k)}(l_1, x_2, x_3) = u_1^{(k)}(x_1, 0, x_3) = u_1^{(k)}(x_1, l_2, x_3) = 0, \\ S_{13}^{(k)}(0, x_2, x_3) = S_{13}^{(k)}(l_1, x_2, x_3) = S_{23}^{(k)}(x_1, 0, x_3) = S_{23}^{(k)}(x_1, l_2, x_3) = 0. \end{gathered} \tag{2.8}$$

On the end sections $x_3 = o^{(k)} \pm h^{(k)}$ of $\mathscr{C}^{(k)}$ tractions $\boldsymbol{t}^{(k)\pm} = \boldsymbol{t}^{(k)\pm}(x_1, x_2)$ are applied which, at the interfaces, are the actions exerted by the layer adjacent to that under consideration and, at the end faces of $\mathscr{C}$, are the external surface loads $\boldsymbol{t}^{\pm}$; thus, we have

$$\pm S_{i3}^{(k)}(x_1, x_2, o^{(k)} \pm h^{(k)})\boldsymbol{e}_3 = t_i^{(k)\pm}(x_1, x_2), \qquad i = 1, 2, 3, \tag{2.9}$$

with $\boldsymbol{t}^{(1)-} = \boldsymbol{t}^-$ and $\boldsymbol{t}^{(n)+} = \boldsymbol{t}^+$. Introduction of stresses (2.7) into three-dimensional equilibrium equations with null volume forces, followed by separation of variables, yields

$$\begin{gathered} \Delta w^{(k)} + (\kappa^{(k)})^2 w^{(k)} = 0, \\ \widehat{\lambda}^{(k)} + \frac{df^{(k)}}{dx_3^2} + (\kappa^{(k)})^2(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})\frac{dg^{(k)}}{dx_3} - (\kappa^{(k)})^2\widehat{\mu}^{(k)} f = 0, \\ \widehat{\mu}^{(k)}\frac{dg^{(k)}}{dx_3^2} - (\kappa^{(k)})^2(\lambda^{(k)} + 2\mu^{(k)})g^{(k)} - (\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})\frac{df^{(k)}}{dx_3} = 0, \end{gathered} \tag{2.10}$$

where $\kappa^{(k)}$ is a constant. Continuity of the displacement at the interface between the layers k and $(k+1)$, where $x_3 = x_3^{(k)}$, requires that

$$\begin{gathered} g^{(k)}(x_3^{(k)})w_{,\alpha}^{(k)}(x_1, x_2) = g^{(k+1)}(x_3^{(k)})w_{,\alpha}^{(k+1)}(x_1, x_2), \qquad \alpha = 1, 2, \\ f^{(k)}(x_3^{(k)})w^{(k)}(x_1, x_2) = f^{(k+1)}(x_3^{(k)})w^{(k+1)}(x_1, x_2); \end{gathered} \tag{2.11}$$

Eq. $(2.11)_3$ implies

$$\frac{f^{(k)}(x_3^{(k)})}{f^{(k+1)}(x_3^{(k)})} = \frac{w^{(k+1)}(x_1, x_2)}{w^{(k)}(x_1, x_2)} = \varkappa_{(k+1,k)}, \tag{2.12}$$

where $\varkappa_{(k+1,k)}$ is a constant. Substitution in Eqs. (2.11) gives

$$\begin{gathered} w^{(k+1)}(x_1, x_2) = \varkappa_{(k+1,k)}\, w^{(k)}(x_1, x_2), \\ f^{(k+1)}(x_3^{(k)}) = \frac{1}{\varkappa_{(k+1,k)}}\, f^{(k)}(x_3^{(k)}), \\ g^{(k+1)}(x_3^{(k)}) = \frac{1}{\varkappa_{(k+1,k)}}\, g^{(k)}(x_3^{(k)}). \end{gathered} \tag{2.13}$$

Equation $(2.13)_1$ shows that both the functions $w^{(k+1)}$ and $w^{(k)}$ must satisfy Eq. $(2.10)_1$ with the boundary condition $(2.8)_1$, and thus that

$$\kappa^{(k+1)} = \kappa^{(k)} = \kappa. \tag{2.14}$$

The functions

$$w^{(k)} = w^{(k)}(x_1, x_2; p, q) = W^{(k)}(p, q) \sin \frac{p\pi x_1}{l_1} \sin \frac{q\pi x_2}{l_2}, \tag{2.15}$$

where $W^{(k)}(p, q)$ is a constant with p and q integers, satisfy the boundary condition $(2.8)_1$; they satisfy also Eq. $(2.10)_1$ provided κ^2 has the value

$$\kappa^2 = \kappa^2(p, q) = \frac{p^2\pi^2}{l_1^2} + \frac{q^2\pi^2}{l_2^2}. \tag{2.16}$$

Equation $(2.13)_1$ then yields

$$W^{(k+1)}(p, q) = \varkappa_{(k+1,k)} W^{(k)}(p, q), \tag{2.17}$$

and for each pair (p, q) there is only one independent constant $W^{(k)}$, which can be chosen to be that of the layer l containing the mid-section $(x_3 = 0)$ of $\mathscr{C}$. The value of $W^{(l)}(p, q)$ can be determined by requiring that $w^{(l)}$ be the transverse component of displacement of the points of the mid-plane of $\mathscr{C}$, i.e., in view of $(2.2)_2$, by the condition

$$f^{(l)}(0) = 1. \tag{2.18}$$

For each fixed pair (p, q), and for each fixed layer index k, the functions $f^{(k)} = f^{(k)}(x_3; p, q)$ and $g^{(k)} = g^{(k)}(x_3; p, q)$ must satisfy Eqs. $(2.10)_{2,3}$; equivalently, $f^{(k)}$ must satisfy the following equation:

$$\begin{aligned} \widehat{\lambda}^{(k)}\widehat{\mu}^{(k)} \frac{d^4 f^{(k)}}{dx_3^4} - \kappa^2\big(\widehat{\lambda}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}) - \widetilde{\lambda}^{(k)}(\widetilde{\lambda}^{(k)} + 2\widehat{\mu}^{(k)})\big)\frac{d^2 f^{(k)}}{dx_3^2} \\ + \kappa^4 \widehat{\mu}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}) f^{(k)} = 0, \end{aligned} \tag{2.19}$$

and, once $f^{(k)}$ is known, $g^{(k)}$ is given by

$$g^{(k)} = -\frac{\widehat{\lambda}^{(k)}\widehat{\mu}^{(k)} \dfrac{d^3 f^{(k)}}{dx_3^3} + \kappa^2 \widetilde{\lambda}^{(k)}(\widetilde{\lambda}^{(k)} + 2\widehat{\mu}^{(k)})\dfrac{df^{(k)}}{dx_3}}{\kappa^4(\lambda^{(k)} + 2\mu^{(k)})(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})}. \tag{2.20}$$

The functions $f^{(k)}$ and $g^{(k)}$ can be written in the form

$$\begin{aligned} f^{(k)}(x_3; p, q) &= \boldsymbol{a}_f^{(k)}(x_3; p, q) \cdot \boldsymbol{c}^{(k)}(p, q), \\ g^{(k)}(x_3; p, q) &= \boldsymbol{a}_g^{(k)}(x_3; p, q) \cdot \boldsymbol{c}^{(k)}(p, q), \end{aligned} \tag{2.21}$$

where, by omitting indication of the dependence on p and q,

$$
\boldsymbol{c}^{(k)} = \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \\ c_3^{(k)} \\ c_4^{(k)} \end{bmatrix}, \qquad \boldsymbol{a}_f^{(k)}(x_3) = \begin{bmatrix} \cosh(\beta_1^{(k)} x_3) \\ \sinh(\beta_1^{(k)} x_3) \\ \cosh(\beta_2^{(k)} x_3) \\ \sinh(\beta_2^{(k)} x_3) \end{bmatrix},
$$

(2.22)

$$
\boldsymbol{a}_g^{(k)}(x_3) = \frac{1}{\kappa^2(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})} \begin{bmatrix} (\kappa^2 \widehat{\mu}^{(k)}/\beta_1^{(k)} - \beta_1^{(k)} \widehat{\lambda}^{(k)}) \sinh(\beta_1^{(k)} x_3) \\ (\kappa^2 \widehat{\mu}^{(k)}/\beta_1^{(k)} - \beta_1^{(k)} \widehat{\lambda}^{(k)}) \cosh(\beta_1^{(k)} x_3) \\ (\kappa^2 \widehat{\mu}^{(k)}/\beta_2^{(k)} - \beta_2^{(k)} \widehat{\lambda}^{(k)}) \sinh(\beta_2^{(k)} x_3) \\ (\kappa^2 \widehat{\mu}^{(k)}/\beta_2^{(k)} - \beta_2^{(k)} \widehat{\lambda}^{(k)}) \cosh(\beta_2^{(k)} x_3) \end{bmatrix}.
$$

Here, $\boldsymbol{c}^{(k)} = \boldsymbol{c}^{(k)}(p,q)$ is the vector of the integration constants of Eq. (2.19). The vectors $\boldsymbol{a}_f^{(k)} = \boldsymbol{a}_f^{(k)}(x_3; p, q)$ and $\boldsymbol{a}_g^{(k)} = \boldsymbol{a}_g^{(k)}(x_3; p, q)$ are functions of the x_3-coordinate and the integers p and q; the quantities $\beta_1^{(k)}$ and $\beta_2^{(k)}$ are the absolute values of the roots $\pm\beta_1^{(k)}$, $\pm\beta_2^{(k)}$ of the algebraic equation associated with the differential equation (2.19),

(2.23)

$$
\beta_1^{(k)} = \beta_1^{(k)}(p,q) = \kappa(p,q) \sqrt{\frac{-b^{(k)} + \sqrt{{b^{(k)}}^2 - 4a^{(k)}c^{(k)}}}{2a^{(k)}}},
$$

$$
\beta_2^{(k)} = \beta_2^{(k)}(p,q) = \kappa(p,q) \sqrt{\frac{-b^{(k)} - \sqrt{{b^{(k)}}^2 - 4a^{(k)}c^{(k)}}}{2a^{(k)}}},
$$

where $a^{(k)}$, $b^{(k)}$, and $c^{(k)}$ depend on the elastic moduli of the material comprising the layer and are defined as

(2.24)

$$
a^{(k)} = \widehat{\lambda}^{(k)} \widehat{\mu}^{(k)}, \qquad b^{(k)} = -\widehat{\lambda}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}) + \widetilde{\lambda}^{(k)}(\widetilde{\lambda}^{(k)} + 2\widehat{\mu}^{(k)}),
$$

$$
c^{(k)} = \widehat{\mu}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}).
$$

Clearly, $\beta_1^{(k)}$ and $\beta_2^{(k)}$ also depend on the material of which the layer is formed, and Eqs. (2.23)–(2.24) show that they assume the same values in layers whose elastic moduli are proportional. In terms of the vectors $\boldsymbol{c}^{(k)}$, $\boldsymbol{a}_f^{(k)}$, and $\boldsymbol{a}_g^{(k)}$, kinematical conditions (2.11), for $k = 1, \ldots, n-1$, are

(2.25)

$$
\boldsymbol{a}_f^{(k)}(x_3^{(k)}) \cdot \boldsymbol{c}^{(k)} = \varkappa_{(k+1,k)}\, \boldsymbol{a}_f^{(k+1)}(x_3^{(k)}) \cdot \boldsymbol{c}^{(k+1)},
$$

$$
\boldsymbol{a}_g^{(k)}(x_3^{(k)}) \cdot \boldsymbol{c}^{(k)} = \varkappa_{(k+1,k)}\, \boldsymbol{a}_g^{(k+1)}(x_3^{(k)}) \cdot \boldsymbol{c}^{(k+1)},
$$

where the dependence of all the vectors on the pair (p,q) is assumed known.

The form of the functions $w^{(k)}$ in Eq. (2.15) and that of the stresses $S_{i3}^{(k)}$ in constitutive equations (2.7) show that, in order to make possible that the boundary conditions (2.9) be satisfied, the surface loads applied on the end faces of $\mathscr{C}$ must be expressible in appropriate series of trigonometric functions. In this case, the applied loads, for each pair (p,q), are

$$\boldsymbol{t}^{\pm}(x_1,x_2)=\tau^{\pm}\,\nabla\left(\sin\frac{p\pi x_1}{l_1}\sin\frac{q\pi x_2}{l_2}\right)+\sigma^{\pm}\sin\frac{p\pi x_1}{l_1}\sin\frac{q\pi x_2}{l_2}\boldsymbol{e}_3, \tag{2.26}$$

where $\tau^{\pm}=\tau^{\pm}(p,q)$, $\sigma^{\pm}=\sigma^{\pm}(p,q)$, ∇ is the gradient operator, and the dependence of $\boldsymbol{t}^{\pm}$ on (p,q) is understood.

In view of (2.17) and (2.7), the instance when the traction vector $\boldsymbol{t}=\boldsymbol{S}\boldsymbol{e}_3$ be continuous at the interfaces between adjacent layers is expressed by the conditions

$$\begin{aligned}&\widehat{\mu}^{(k)}\left(f^{(k)}-\frac{dg^{(k)}}{dx_3}\right)w_{,\alpha}^{(k)}\boldsymbol{e}_\alpha\\&\qquad=\widehat{\mu}^{(k+1)}\left(f^{(k+1)}(x_3^{(k)})-\frac{dg^{(k+1)}}{dx_3}\right)\varkappa_{(k+1,k)}w_{,\alpha}^{(k)}\boldsymbol{e}_\alpha,\\&\left(\kappa^2\widetilde{\lambda}^{(k)}g^{(k)}+\widehat{\lambda}^{(k)}\frac{df^{(k)}}{dx_3}\right)w^{(k)}\\&\qquad=\left(\kappa^2\widetilde{\lambda}^{(k+1)}g^{(k+1)}+\widehat{\lambda}^{(k+1)}\frac{df^{(k+1)}}{dx_3}\right)\varkappa_{(k+1,k)}w^{(k)},\end{aligned} \tag{2.27}$$

for $k=1,\ldots,n-1$.

In order to write Eqs. (2.9) and (2.27) in a more concise form, we introduce the vectors $\boldsymbol{a}_\sigma^{(k)}=\boldsymbol{a}_\sigma^{(k)}(x_3;p,q)$ and $\boldsymbol{a}_\tau^{(k)}=\boldsymbol{a}_\tau^{(k)}(x_3;p,q)$. By leaving out dependence on p and q, the definitions of these vectors are

$$\boldsymbol{a}_\sigma^{(k)}(x_3)=\frac{\widehat{\mu}^{(k)}}{\widetilde{\lambda}^{(k)}+\widehat{\mu}^{(k)}}\begin{bmatrix}(\kappa^2\widetilde{\lambda}^{(k)}/\beta_1^{(k)}+\beta_1^{(k)}\widehat{\lambda}^{(k)})\sinh(\beta_1^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}/\beta_1^{(k)}+\beta_1^{(k)}\widehat{\lambda}^{(k)})\cosh(\beta_1^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}/\beta_2^{(k)}+\beta_2^{(k)}\widehat{\lambda}^{(k)})\sinh(\beta_2^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}/\beta_2^{(k)}+\beta_2^{(k)}\widehat{\lambda}^{(k)})\cosh(\beta_2^{(k)}x_3)\end{bmatrix}, \tag{2.28}$$

$$\boldsymbol{a}_\tau^{(k)}(x_3)=\frac{\widehat{\mu}^{(k)}}{\kappa^2(\widetilde{\lambda}^{(k)}+\widehat{\mu}^{(k)})}\begin{bmatrix}(\kappa^2\widetilde{\lambda}^{(k)}+(\beta_1^{(k)})^2\widehat{\lambda}^{(k)})\cosh(\beta_1^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}+(\beta_1^{(k)})^2\widehat{\lambda}^{(k)})\sinh(\beta_1^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}+(\beta_2^{(k)})^2\widehat{\lambda}^{(k)})\cosh(\beta_2^{(k)}x_3)\\(\kappa^2\widetilde{\lambda}^{(k)}+(\beta_2^{(k)})^2\widehat{\lambda}^{(k)})\sinh(\beta_2^{(k)}x_3)\end{bmatrix}. \tag{2.29}$$

Taking (2.26) into account, for each pair (p,q) conditions (2.9) and (2.27) can be written

$$\text{(2.30)}\qquad \begin{aligned} \boldsymbol{a}_\sigma^{(n)}(+h)\cdot\boldsymbol{c}^{(n)} &= \frac{\sigma^+}{W^{(n)}}, &\qquad \boldsymbol{a}_\tau^{(n)}(+h)\cdot\boldsymbol{c}^{(n)} &= \frac{\tau^+}{W^{(n)}},\\ \boldsymbol{a}_\sigma^{(1)}(-h)\cdot\boldsymbol{c}^{(1)} &= -\frac{\sigma^-}{W^{(1)}}, &\qquad \boldsymbol{a}_\tau^{(1)}(-h)\cdot\boldsymbol{c}^{(1)} &= -\frac{\tau^-}{W^{(1)}}, \end{aligned}$$

$$\text{(2.31)}\qquad \begin{aligned} \boldsymbol{a}_\sigma^{(k)}(x_3^{(k)})\cdot\boldsymbol{c}^{(k)} &= \varkappa_{(k+1.k)}\boldsymbol{a}_\sigma^{(k+1)}(x_3^{(k)})\cdot\boldsymbol{c}^{(k+1)},\\ \boldsymbol{a}_\tau^{(k)}(x_3^{(k)})\cdot\boldsymbol{c}^{(k)} &= \varkappa_{(k+1.k)}\boldsymbol{a}_\sigma^{(k+1)}(x_3^{(k)})\cdot\boldsymbol{c}^{(k+1)}. \end{aligned}$$

Boundary conditions (2.30) and continuity conditions (2.25) and (2.31), for $k = 1,\ldots,n-1$, form a system of $4n$ equations for the $4n$ unknown components of the n vectors $\boldsymbol{c}^{(k)}$. Let $\boldsymbol{A}^{(k)} = \boldsymbol{A}^{(k)}(x_3;p,q)$ be the matrices whose rows are the components of the vectors $(2.22)_{2,3}$, (2.28), and (2.29),

$$\boldsymbol{A}^{(k)} = \begin{bmatrix} \boldsymbol{a}_f^{(k)}\\ \boldsymbol{a}_g^{(k)}\\ \boldsymbol{a}_\sigma^{(k)}\\ \boldsymbol{a}_\tau^{(k)} \end{bmatrix}.$$

In terms of these matrices, the continuity conditions (2.25) and (2.31) are written as

$$\text{(2.32)}\qquad \boldsymbol{A}^{(k)}(x_3^{(k)})\,\boldsymbol{c}^{(k)} = \varkappa_{(k+1,k)}\boldsymbol{A}^{(k+1)}(x_3^{(k)})\,\boldsymbol{c}^{(k+1)}.$$

For each (p,q), Eqs. (2.32) can be solved to obtain $\boldsymbol{c}^{(k+1)}$ in terms of $\boldsymbol{c}^{(k)}$. Then, by substitution, Eqs. (2.30) are reduced to a system of four equations for the components of $\boldsymbol{c}^{(1)}$ and, when these are known, the $\boldsymbol{c}^{(k)}$, for $k = 2,\ldots,n$, are obtained in sequence from (2.32).

Finally, we observe that, in the displacement components (2.2), the constants $W^{(k)}$ in the expression (2.15) of the function $w^{(k)}$ cancel with the constants $1/W^{(k)}$ which, in view of Eqs. (2.17) and (2.32), appear in the expressions of the functions $f^{(k)}$ e $g^{(k)}$. Thus, the displacement components are independent of $W^{(k)}$ and $\varkappa_{(k+1,k)}$.

3. Plate-like bodies with circular cross-section

We now deduce a Levinson-type solution for multilayered plate-like bodies with circular cross-sections, subject to axisymmetric deformations.

An explicit Levinson-type solution for the equilibrium of a simply-supported circular cylinder made of an isotropic material and subject to axisymmetric deformations was given in [6]. The solution was extended in [7] to the case of a circular cylinder made of a transversely isotropic material.

It was shown in [4] that, for a simply-supported plate-like body $\mathscr{C}$ of general cross-section, a Levinson-type equilibrium solution requires the presence, on the lateral surface, of normal tractions which are proportional to the curvature of the cross-sectional contour $\partial\mathscr{S}$. Such tractions are null when $\mathscr{C}$ has a rectangular cross-section and have magnitude that varies along the thickness but is independent of the position on $\partial\mathscr{S}$ in axisymmetric deformations of a plate with a circular cross-section.

Consider a multilayered plate-like body $\mathscr{C}$, and let (r, θ, z) and $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ be the coordinates and the base vectors of a cylindrical system with origin at the center of the mid-section $\mathscr{S}$ of $\mathscr{C}$ and z-axis orthogonal to $\mathscr{S}$. In axisymmetric deformations and for null volume forces, the three-dimensional equilibrium equations reduce to

$$S_{rr,r} + S_{rz,z} + \frac{1}{r}(S_{rr} - S_{\theta\theta}) = 0, \qquad S_{zr,r} + S_{zz,z} + \frac{1}{r}S_{zr} = 0. \tag{3.1}$$

It is assumed that the body $\mathscr{C}$ is acted upon by tractions $\boldsymbol{t}^{\pm}(r)$ applied at the end faces,

$$\boldsymbol{S}\boldsymbol{e}_z = \pm\boldsymbol{t}^{\pm}, \qquad \text{on } \mathscr{S}^{\pm}; \tag{3.2}$$

and that the boundary conditions on the lateral surface are

$$u_z = 0, \qquad S_{rr} = \hat{t}_r(z), \qquad \text{on } \partial\mathscr{S} \times (-h, h), \tag{3.3}$$

where u_z is the transverse component of the displacement and $\hat{t}_r$ is the applied radial traction, needed for the equilibrium in a Levinson-type solution.

In view of the symmetry, in each layer $\mathscr{C}^{(k)}$, that is included between the cross-sections at $z = z^{(k-1)}$ and $z = z^{(k)}$, the displacement field is assumed to be in the form

$$u_r^{(k)}(r, z) = -g^{(k)}(z)w_{,r}^{(k)}(r), \qquad u_z^{(k)}(r, z) = f^{(k)}(z)w^{(k)}(r), \tag{3.4}$$

to which there correspond the nonzero strains

$$\begin{aligned} E_{rr}^{(k)} &= -g^{(k)}w_{,rr}^{(k)}, \qquad & E_{\theta\theta}^{(k)} &= -\frac{1}{r}g^{(k)}w_{,r}^{(k)}, \\ E_{zz}^{(k)} &= \frac{df^{(k)}}{dz}w^{(k)}, \qquad & E_{rz}^{(k)} &= \frac{1}{2}\left(f^{(k)} - \frac{dg^{(k)}}{dz}\right)w_{,r}^{(k)}, \end{aligned} \tag{3.5}$$

and, for layers composed of transverse isotropic materials, the stresses

$$
\begin{aligned}
S_{rr}^{(k)} &= -(\lambda^{(k)} + 2\mu^{(k)})g^{(k)}w_{,rr}^{(k)} - \lambda^{(k)}\frac{1}{r}g^{(k)}w_{,r}^{(k)} + \widetilde{\lambda}^{(k)}\frac{df^{(k)}}{dz}w^{(k)},\\
S_{\theta\theta}^{(k)} &= -\lambda^{(k)}g^{(k)}w_{,rr}^{(k)} - (\lambda^{(k)} + 2\mu^{(k)})\frac{1}{r}g^{(k)}w_{,r}^{(k)} + \widetilde{\lambda}^{(k)}\frac{f^{(k)}}{dz}w^{(k)},\\
S_{zz}^{(k)} &= -\widetilde{\lambda}^{(k)}g^{(k)}\left(w_{,rr}^{(k)} + \frac{1}{r}w_{,r}^{(k)}\right) + \widehat{\lambda}^{(k)}\frac{df^{(k)}}{dz}w^{(k)},\\
S_{rz}^{(k)} &= \widehat{\mu}^{(k)}\left(f^{(k)} - \frac{dg^{(k)}}{dz}\right)w_{,r}^{(k)}.
\end{aligned}
\tag{3.6}
$$

A development similar to the one previously performed for the plate-like body with rectangular cross-section shows that the following results hold. The function $w^{(k)}$ must satisfy the equation

$$
w_{,rr}^{(k)} + \frac{1}{r}w_{,r}^{(k)} + (\kappa^{(k)})^2 w^{(k)} = 0,
\tag{3.7}
$$

with the boundary condition

$$
w^{(k)}(R) = 0,
\tag{3.8}
$$

where R is the radius of $\mathscr{S}$. The continuity of the displacement at the interfaces between layers implies that, for $k = 1, \dots, n-1$,

$$
\begin{aligned}
w^{(k+1)}(r) &= \varkappa_{(k+1.k)}w^{(k)}(r),\\
f^{(k+1)}(z^{(k)}) &= \frac{1}{\varkappa_{(k+1.k)}}\,f^{(k)}(z^{(k)}),\\
g^{(k+1)}(z^{(k)}) &= \frac{1}{\varkappa_{(k+1.k)}}\,g^{(k)}(z^{(k)}),
\end{aligned}
\tag{3.9}
$$

where $\varkappa_{(k+1.k)}$ is a constant; it follows from (3.7) and $(3.9)_1$ that

$$
\kappa^{(k+1)} = \kappa^{(k)} = \kappa.
\tag{3.10}
$$

The solution of (3.7)–(3.8) is

$$
w^{(k)}(r;m) = W^{(k)}(m)J_0(\kappa(m)r/R), \qquad m = 1, 2, \dots,
\tag{3.11}
$$

where $W^{(k)}(m)$ is a constant, J_0 is the Bessel function of the first kind and order 0, and for each m, the number $\kappa(m)$ is the m-th positive zero of J_0,

$$
J_0(\kappa(m)) = 0, \qquad m = 1, 2, \dots.
\tag{3.12}
$$

In view of $(3.9)_1$, it is

$$W^{(k+1)}(m) = \varkappa_{(k+1.k)} W^{(k)}(m); \tag{3.13}$$

hence, for each m, there is only one independent constant $W^{(k)}$, whose value can be fixed by the condition

$$f^{(l)}(0) = 1, \tag{3.14}$$

corresponding to the request that $w^{(l)}(m)$ be the transverse displacement of the points of the mid-section of $\mathscr{C}$, assumed to be in the layer $\mathscr{C}^{(l)}$.

For each layer index k and for each m, the functions $f^{(k)} = f^{(k)}(z;m)$ and $g^{(k)} = g^{(k)}(z;m)$ are such that

$$\begin{aligned} &\widehat{\lambda}^{(k)}\widehat{\mu}^{(k)}\frac{d^4 f^{(k)}}{dz^4} - \kappa^2(\widehat{\lambda}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}) - \widetilde{\lambda}^{(k)}(\widetilde{\lambda}^{(k)} + 2\widehat{\mu}^{(k)}))\frac{d^2 f^{(k)}}{dz^2} \\ &\qquad\qquad + \kappa^4 \widehat{\mu}^{(k)}(\lambda^{(k)} + 2\mu^{(k)}) f^{(k)} = 0, \\ &g^{(k)} = -\frac{\widehat{\lambda}^{(k)}\widehat{\mu}^{(k)}\dfrac{d^3 f^{(k)}}{dz^3} + \kappa^2 \widetilde{\lambda}^{(k)}(\widetilde{\lambda}^{(k)} + 2\widehat{\mu}^{(k)})\dfrac{df^{(k)}}{dz}}{\kappa^4(\lambda^{(k)} + 2\mu^{(k)})(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})}. \end{aligned} \tag{3.15}$$

The functions $f^{(k)}$ and $g^{(k)}$ can be written as

$$\begin{aligned} f^{(k)}(z;m) &= \boldsymbol{a}_f^{(k)}(z;m) \cdot \boldsymbol{c}^{(k)}(m), \\ g^{(k)}(z;m) &= \boldsymbol{a}_g^{(k)}(z;m) \cdot \boldsymbol{c}^{(k)}(m), \end{aligned} \tag{3.16}$$

where, with the dependence on m agreed upon known,

$$\begin{aligned} &\boldsymbol{c}^{(k)} = \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \\ c_3^{(k)} \\ c_4^{(k)} \end{bmatrix}, \qquad \boldsymbol{a}_f^{(k)}(z) = \begin{bmatrix} \cosh(\beta_1^{(k)} z) \\ \sinh(\beta_1^{(k)} z) \\ \cosh(\beta_2^{(k)} z) \\ \sinh(\beta_2^{(k)} z) \end{bmatrix}, \\ &\boldsymbol{a}_g^{(k)}(z) = \frac{1}{\kappa^2(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})} \begin{bmatrix} (\kappa^2\widehat{\mu}^{(k)}/\beta_1^{(k)} - \beta_1^{(k)}\widehat{\lambda}^{(k)}) \sinh(\beta_1^{(k)} z) \\ (\kappa^2\widehat{\mu}^{(k)}/\beta_1^{(k)} - \beta_1^{(k)}\widehat{\lambda}^{(k)}) \cosh(\beta_1^{(k)} z) \\ (\kappa^2\widehat{\mu}^{(k)}/\beta_2^{(k)} - \beta_2^{(k)}\widehat{\lambda}^{(k)}) \sinh(\beta_2^{(k)} z) \\ (\kappa^2\widehat{\mu}^{(k)}/\beta_2^{(k)} - \beta_2^{(k)}\widehat{\lambda}^{(k)}) \cosh(\beta_2^{(k)} z) \end{bmatrix}. \end{aligned} \tag{3.17}$$

In these equations the quantities $\beta_1^{(k)}$ and $\beta_2^{(k)}$ are the absolute values of the roots $\pm\beta_1^{(k)}$, $\pm\beta_2^{(k)}$ of the algebraic equation associated with $(3.15)_1$,

$$
\begin{aligned}
\beta_1^{(k)} &= \beta_1^{(k)}(m) = \kappa(m)\sqrt{\frac{-b^{(k)} + \sqrt{{b^{(k)}}^2 - 4a^{(k)}c^{(k)}}}{2a^{(k)}}},\\
\beta_2^{(k)} &= \beta_2^{(k)}(m) = \kappa(m)\sqrt{\frac{-b^{(k)} - \sqrt{{b^{(k)}}^2 - 4a^{(k)}c^{(k)}}}{2a^{(k)}}},
\end{aligned}
\tag{3.18}
$$

with $a^{(k)}$, $b^{(k)}$, $b^{(k)}$ defined as in Eq. (2.24). In terms of the vectors (3.17) the displacement continuity conditions $(3.9)_{2,3}$ become

$$
\begin{aligned}
\boldsymbol{a}_f^{(k)}(z^{(k)}) \cdot \boldsymbol{c}^{(k)} &= \varkappa_{(k+1,k)}\, \boldsymbol{a}_f^{(k+1)}(z^{(k)}) \cdot \boldsymbol{c}^{(k+1)},\\
\boldsymbol{a}_g^{(k)}(z^{(k)}) \cdot \boldsymbol{c}^{(k)} &= \varkappa_{(k+1,k)}\, \boldsymbol{a}_g^{(k+1)}(z^{(k)}) \cdot \boldsymbol{c}^{(k+1)}.
\end{aligned}
\tag{3.19}
$$

The expressions of $w^{(k)}$ in Eq. (3.11) and of the stresses in equations $(3.6)_{3,4}$ show that, in order the boundary conditions (3.2) on the end faces $\mathscr{S}^{\pm}$ of $\mathscr{C}$ can be satisfied, the surface loads $\boldsymbol{t}^{\pm} = \boldsymbol{t}^{\pm}(r; m)$ must be expressible in Bessel series and, for each m, they must have the form

$$
\boldsymbol{t}^{\pm}(r) = -\tau^{\pm} J_{0,\,r}(\kappa r/R)\boldsymbol{e}_r + \sigma^{\pm} J_0(\kappa r/R)\boldsymbol{e}_z,
\tag{3.20}
$$

where the dependence on m of $\boldsymbol{t}^{\pm}$, $\tau^{\pm}$, $\sigma^{\pm}$, and κ is known. By $(3.9)_1$ and $(3.6)_{3,4}$, the continuity conditions of the traction vector $\boldsymbol{t} = \boldsymbol{S}\boldsymbol{e}_z$ at the interfaces between layers can be expressed as

$$
\begin{aligned}
&\widehat{\mu}^{(k)}\Big(f^{(k)}(z^{(k)}) - \frac{dg^{(k)}}{dz}(z^{(k)})\Big)w_{,r}^{(k)} =\\
&\qquad = \varkappa_{(k+1.k)}\widehat{\mu}^{(k+1)}\Big(f^{(k+1)}(z^{(k)}) - \frac{dg^{(k+1)}}{dz}(z^{(k)})\Big)w_{,r}^{(k)},\\
&\widetilde{\lambda}^{(k)}g^{(k)}(z^{(k)})\frac{1}{r}\frac{d}{dr}(rw_{,r}^{(k)}) + \widehat{\lambda}^{(k)}\frac{df^{(k)}}{dz}(z^{(k)})w^{(k)} =\\
&\qquad = \varkappa_{(k+1.k)}\Big(\widetilde{\lambda}^{(k+1)}g^{(k+1)}(z^{(k)})\frac{1}{r}\frac{d}{dr}(rw_{,r}^{(k)}) + \widehat{\lambda}^{(k)}\frac{df^{(k)}}{dz}(z^{(k)})w^{(k))}\Big).
\end{aligned}
\tag{3.21}
$$

In view of writing boundary and continuity conditions in a more concise form, we introduce the vectors $\boldsymbol{a}_\sigma^{(k)} = \boldsymbol{a}_\sigma^{(k)}(z;m)$ and $\boldsymbol{a}_\tau^{(k)} = \boldsymbol{a}_\tau^{(k)}(z;m)$, which, for each m, are defined by

$$
\boldsymbol{a}_\sigma^{(k)}(z) = \frac{\widehat{\mu}^{(k)}}{\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)}}
\begin{bmatrix}
(\kappa^2\widetilde{\lambda}^{(k)}/\beta_1^{(k)} + \beta_1^{(k)}\widehat{\lambda}^{(k)})\sinh(\beta_1^{(k)}z)\\
(\kappa^2\widetilde{\lambda}^{(k)}/\beta_1^{(k)} + \beta_1^{(k)}\widehat{\lambda}^{(k)})\cosh(\beta_1^{(k)}z)\\
(\kappa^2\widetilde{\lambda}^{(k)}/\beta_2^{(k)} + \beta_2^{(k)}\widehat{\lambda}^{(k)})\sinh(\beta_2^{(k)}z)\\
(\kappa^2\widetilde{\lambda}^{(k)}/\beta_2^{(k)} + \beta_2^{(k)}\widehat{\lambda}^{(k)})\cosh(\beta_2^{(k)}z)
\end{bmatrix},
\tag{3.22}
$$

$$(3.23)\qquad \boldsymbol{a}_{\tau}^{(k)}(z) = \frac{\widehat{\mu}^{(k)}}{\kappa^2(\widetilde{\lambda}^{(k)} + \widehat{\mu}^{(k)})} \begin{bmatrix} (\kappa^2\widetilde{\lambda}^{(k)} + (\beta_1^{(k)})^2\widehat{\lambda}^{(k)})\cosh(\beta_1^{(k)}z) \\ (\kappa^2\widetilde{\lambda}^{(k)} + (\beta_1^{(k)})^2\widehat{\lambda}^{(k)})\sinh(\beta_1^{(k)}z) \\ (\kappa^2\widetilde{\lambda}^{(k)} + (\beta_2^{(k)})^2\widehat{\lambda}^{(k)})\cosh(\beta_2^{(k)}z) \\ (\kappa^2\widetilde{\lambda}^{(k)} + (\beta_2^{(k)})^2\widehat{\lambda}^{(k)})\sinh(\beta_2^{(k)}z) \end{bmatrix}.$$

Making use of these definitions and taking (3.20) into account the boundary conditions (3.2) and the continuity conditions (3.21) are

$$(3.24)\qquad \begin{aligned} \boldsymbol{a}_{\sigma}^{(n)}(+h)\cdot\boldsymbol{c}^{(n)} &= \frac{\sigma^+}{W^{(n)}}, &\qquad \boldsymbol{a}_{\tau}^{(n)}(+h)\cdot\boldsymbol{c}^{(n)} &= \frac{\tau^+}{W^{(n)}}, \\ \boldsymbol{a}_{\sigma}^{(1)}(-h)\cdot\boldsymbol{c}^{(1)} &= -\frac{\sigma^-}{W^{(1)}}, &\qquad \boldsymbol{a}_{\tau}^{(1)}(-h)\cdot\boldsymbol{c}^{(1)} &= -\frac{\tau^-}{W^{(1)}}, \end{aligned}$$

$$(3.25)\qquad \begin{aligned} \boldsymbol{a}_{\sigma}^{(k)}(z^{(k)})\cdot\boldsymbol{c}^{(k)} &= \varkappa_{(k+1.k)}\boldsymbol{a}_{\sigma}^{(k+1)}(z^{(k)})\cdot\boldsymbol{c}^{(k+1)}, \\ \boldsymbol{a}_{\tau}^{(k)}(z^{(k)})\cdot\boldsymbol{c}^{(k)} &= \varkappa_{(k+1.k)}\boldsymbol{a}_{\sigma}^{(k+1)}(z^{(k)})\cdot\boldsymbol{c}^{(k+1)}. \end{aligned}$$

For each m, boundary conditions (3.24) and continuity conditions (3.19) and (3.25) written for $k = 1, \ldots, n-1$, form a system of $4n$ equations for the $4n$ unknown components of the n vectors $\mathbf{c}^{(k)}$. The solution of this system is obtained in the same way as in the case of plate-like bodies with rectangular cross-sections examined in the previous section.

The expression $(3.6)_1$ of the stress S_{rr} shows that the radial tractions, that must be applied at the mantle to assure the equilibrium of $\mathscr{C}$, are

$$(3.26)\qquad t^{(k)}(z) = -2\mu^{(k)}g^{(k)}(z)w_{,rr}(R) = -\frac{2}{R}\mu^{(k)}\kappa W^{(k)}J_1(\kappa)g^{(k)}(z), \qquad z^{(k)} \le z \le z^{(k+1)}, \quad k = 1,\ldots,n-1.$$

When the layers forming $\mathscr{C}$ are symmetric with respect to the mid-plane, the tractions (3.26) have a zero resultant on the segments of the lateral surface parallel to the z-axis.

4. Comparison with finite-element solutions

The exact solutions derived in previous sections for equilibrium problems of multilayered plate-like bodies, are expressed as series of products of a function of the in-plane coordinates and a function of the transverse coordinate. The functions expressing dependence on the in-plane coordinates are the terms of a trigonometric and a Bessel series, respectively for plate-like bodies with rectangular and circular cross-sections. In this section we make a comparison between the results given by the solution series truncated after a few terms, and the results obtained from a finite-element three-dimensional analysis of the equilibrium problems.

We consider plate-like bodies symmetric with respect to the mid-section, that are formed of three layers having the same thickness $2h/3$, and assume that the total thickness of the bodies is $2h = 200$ mm.

We use technical elastic moduli to characterize the mechanical properties of the transversely isotropic materials forming the layers, and we denote by E the Young modulus for the directions x_1 and x_2, by $\overline{E}$ the Young modulus for the direction x_3, by ν and $\overline{\nu}$ the Poisson ratios relative to the pairs of directions (x_2, x_1) and (x_3, x_1), respectively; finally we denote by $\overline{G}$ the tangential modulus for the directions x_1 and x_3. The relationships between the technical moduli and the components of the elasticity tensor $\mathbb{C}$ are

$$
\begin{aligned}
&\mathbb{C}_{1111} = \frac{E(E - \overline{E}\overline{\nu}^2)}{(1+\nu)(E(1-\nu) - 2\overline{E}\overline{\nu}^2)}, && \mathbb{C}_{1122} = \frac{E(E\nu + \overline{E}\overline{\nu}^2)}{(1+\nu)(E(1-\nu) - 2\overline{E}\overline{\nu}^2)},\\
&\mathbb{C}_{1133} = \frac{\overline{E}E\overline{\nu}}{E(1-\nu) - 2\overline{E}\overline{\nu}^2}, && \mathbb{C}_{1212} = \frac{E}{2(1+\nu)},\\
&\mathbb{C}_{1313} = \overline{G}, && \mathbb{C}_{3333} = \frac{\overline{E}E(1-\nu)}{E(1-\nu) - 2\overline{E}\overline{\nu}^2}.
\end{aligned}
\tag{4.1}
$$

In all the examples, the outer layers, labelled with the number 2, are made of the same material whose moduli are

$$
\begin{aligned}
&E^{(2)} = 1.7 \times 10^5\,\mathrm{N/mm}^2, && G^{(2)} = 0.25 \times E^{(2)},\\
&\overline{E}^{(2)} = 0.75 \times E^{(2)}, && \nu^{(2)} = \overline{\nu}^{(2)} = 0.25,
\end{aligned}
$$

and the inner layer, labelled with the number 1, is made of a material whose moduli are

$$
\begin{aligned}
&E^{(1)} = \frac{1}{25}\,E^{(2)}, && G^{(1)} = 0.25 \times E^{(1)},\\
&\overline{E}^{(1)} = 0.75 \times E^{(1)}, && \nu^{(1)} = \overline{\nu}^{(1)} = 0.25.
\end{aligned}
$$

We use accurate finite element solutions as reference to evaluate the approximation given by the Levinson-type solutions truncated after the first term, after the terms whose indices p and q (see (2.15)) or m (see (3.11)) are not greater than 3, and after the terms whose indices p and q (see (2.15)) or m (see (3.11)) are not greater than 5. In the figures we use the following conventions:

— ≡ 3D FE solution (by COMSOL Multiphysics);

△ ≡ Levinson-type solution truncated after the first term;

○ ≡ Levinson-type solution truncated after the terms whose indices p, q or m are not greater than 3;

□ ≡ Levinson-type solution truncated after the terms whose indices p, q or m are not greater than 5.

For a sake of simplicity, we will denote by w the transverse displacement at the mid-plane, i.e., the values of functions $u_3^{(1)}(x_1, x_2, 0)$ or $u_z^{(1)}(r, 0)$.

In the first example, we consider a plate with rectangular cross-section whose sides have length $l_1 = 1500$ and $l_2 = 2000$ mm, and are parallel to the x_1 and x_2 axes, respectively. We consider three loading conditions in which on the upper end face $\mathscr{S}^+$ of the body acts: (i) a uniform load $\sigma = 10$ MPa; (ii) a force $Q = 25$ kN applied at the centre P_1 of the section; (iii) a force $Q = 25$ kN applied at the point P_2, which is on a diagonal of the section at the distance of one quarter of the diagonal length from the corner. In Figures 1, 2 and 3, for the three loading conditions, we plot:

— the transverse displacement over the thickness of the plate versus the adimensional abscissa, x_α/l_α, along lines, through the center of the mid-section, parallel to its sides;

— the stress S_{11} over $E^{(1)}$ versus the adimensional abscissa $x_3/(2h)$, along a transverse fibre that, in the undeformed configuration, intersects the upper end face at the point P_2 for the first two loading conditions, and at the point P_1 for the third loading condition.

In the second example, we consider a plate with circular cross-section, whose radius is $R = 1000$ mm, under the action of a uniform load $\sigma = 10$ MPa. The Bessel series expression of σ is

$$\sigma(r) = \sum_{m=1}^{\infty} \sigma_m^+ \, J_0(\kappa_m \, r/R), \tag{4.2}$$

where κ_m is the m-th positive root of the Bessel function J_0, and the coefficients σ_m^+ are

$$\sigma_m^+ = \frac{2\sigma}{\big(R\, J_1(\kappa_m)\big)^2} \int_0^R J_0(\kappa_m \, r/R) \, r \, \mathrm{d}r. \tag{4.3}$$

The load σ is accompanied by radial tractions on the lateral surface of the plate-like body that, for each term of the Bessel series expansion (4.2), have the expression (3.26). In the finite-element analysis, the tractions corresponding to the whole load σ are assumed to be those given by the Levinson-type series solution truncated after the tenth term.

For the circular plate, we plot in Fig. 4, the transverse displacement over the thickness of the plate versus the adimensional abscissa r/R, along a diameter of the mid-section, and the stress S_{rr} over $E^{(1)}$ versus the adimensional abscissa $z/(2h)$ along a transverse fibre which intersects the mid-section at a point having distance $3R/4$ from the centre.

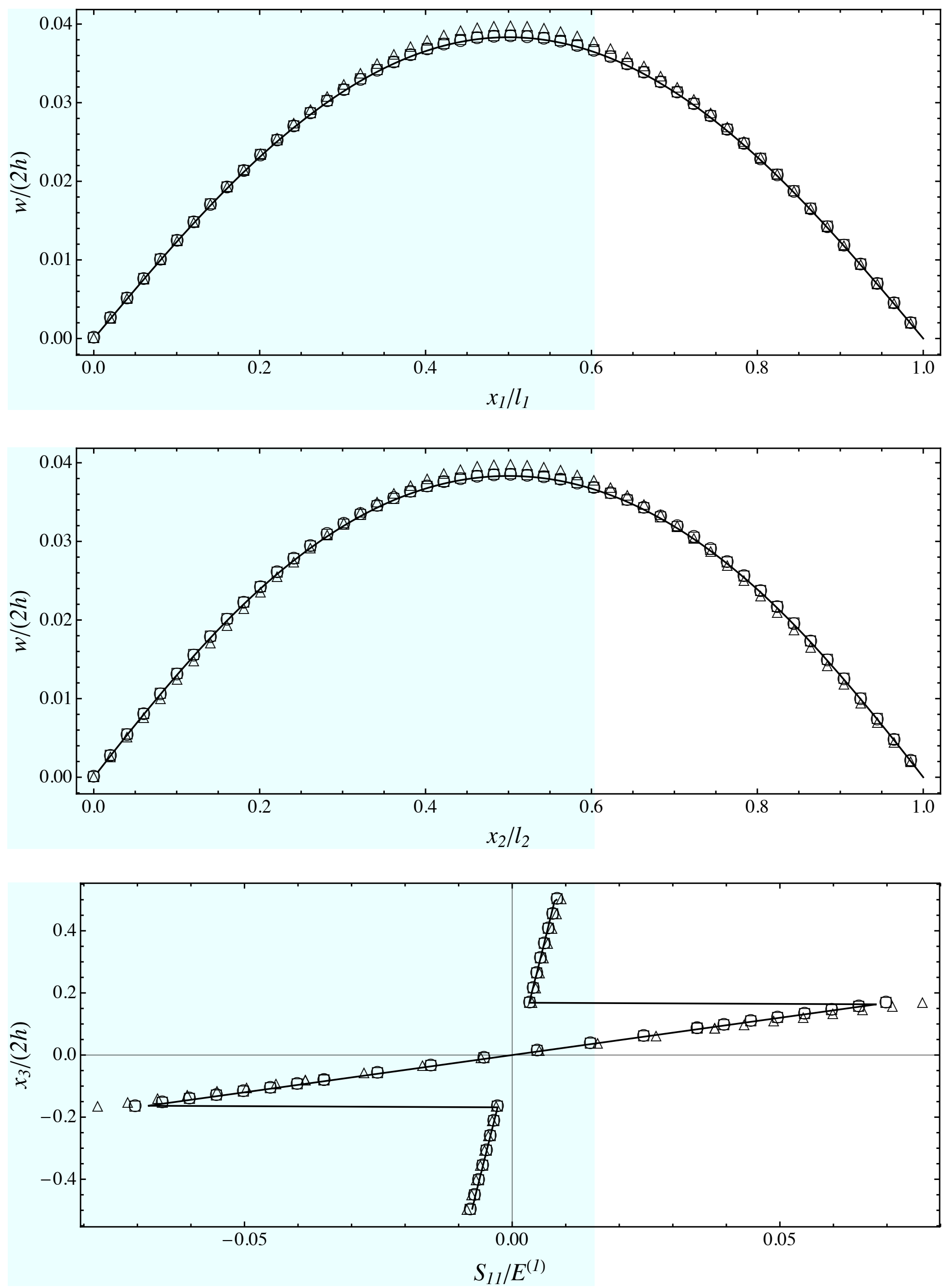

Fig. 1. Uniform load.

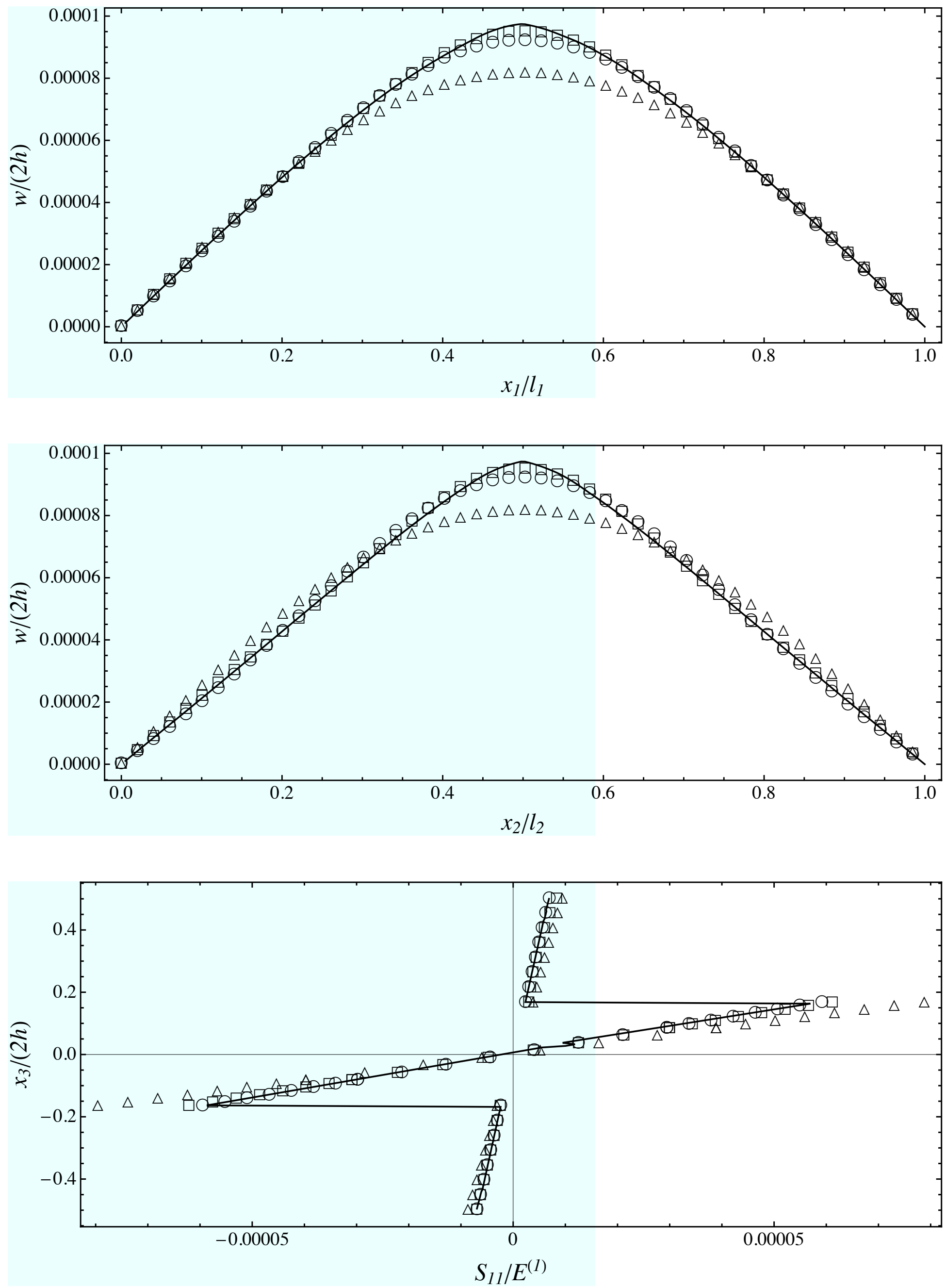

Fig. 2. Force applied at the centre of the upper end face.

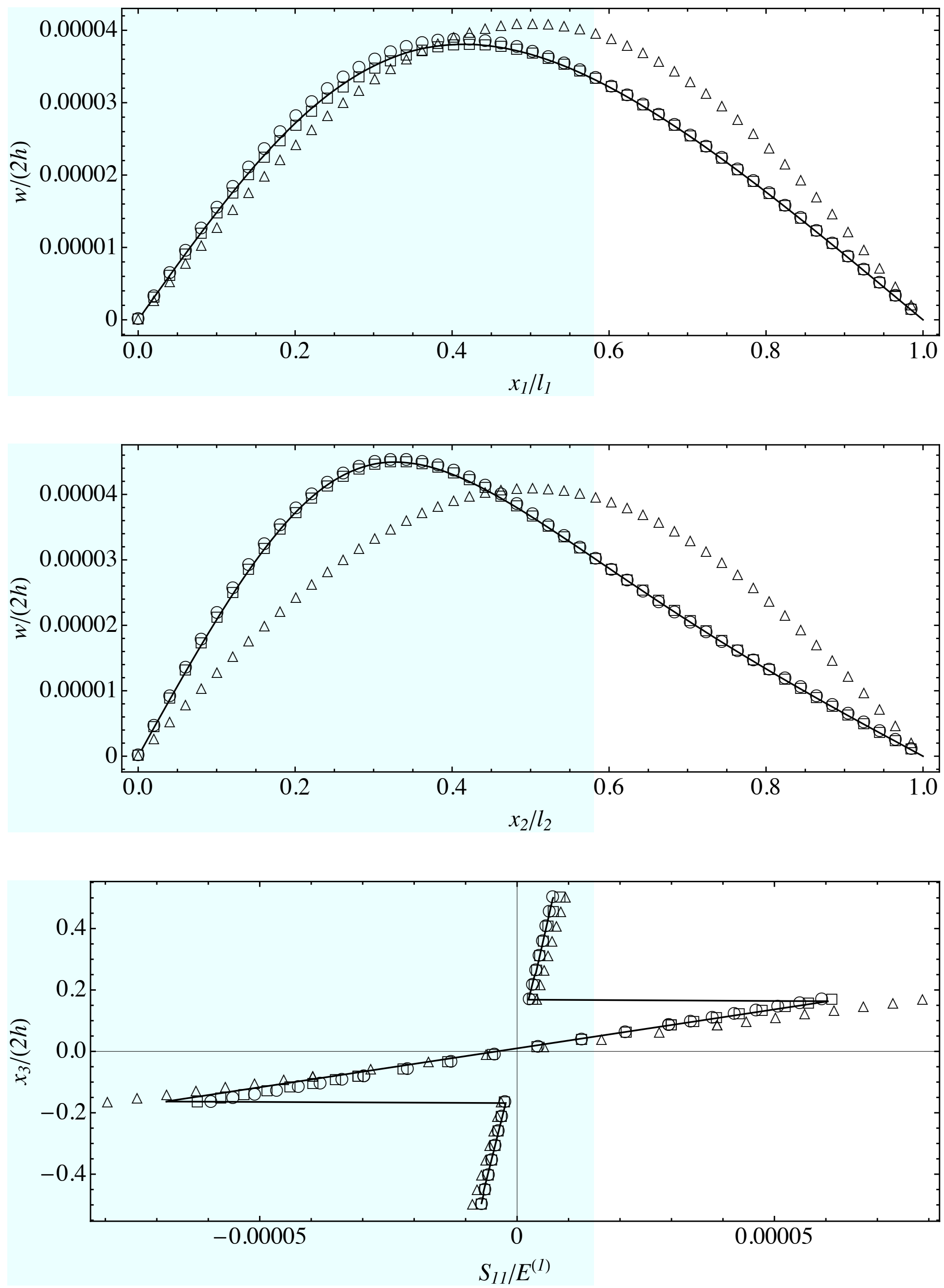

Fig. 3. Force applied at a quarter of the length of a diagonal on the upper end face.

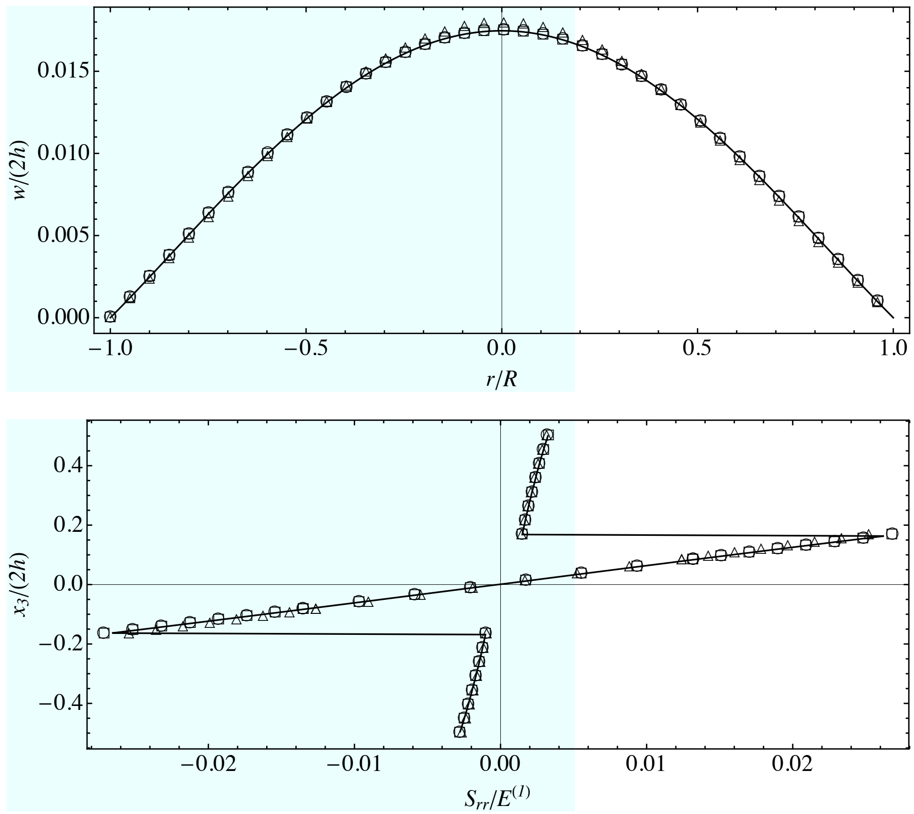

Fig. 4. Uniform load.

The figures show that, although the accuracy of the truncated solutions is better when the body is subject to a uniform load, the solutions truncated after the terms whose indices are not greater than 3 give a good approximation of the reference results in all the considered cases.

5. Conclusions

We derived explicit and exact three-dimensional equilibrium solutions for plate-like bodies consisting of several layers of transversely isotropic materials, with both rectangular and circular cross-sections.

Under the assumption that each layer deforms as a Levinson plate, and that displacement and tractions are continuous across the interfaces between layers, we obtained equilibrium displacement fields in the form of series of products of a function of the in-plane coordinates and a function of the transverse coordinate.

To test the convergence of such series solutions, we compared the approximate values for displacement and stress they deliver under truncation after a few terms with the values given by accurate three-dimensional finite-element analyses. Layered plates with both rectangular and circular cross-sections were considered, subject to either a uniform load or a force concentrated at the center of the upper end-face. In all cases, we found that retaining the first three terms of the series is sufficient to guarantee a satisfactory approximation.

A reviewer of a former version of our present work raised two interesting questions: the one, as to what changes would ensue from letting the layer thickness vary; the other, as to what would happen if the number of layers were made larger and larger. We plan to take up these issues in a future paper. For the moment, we are only able to offer some numerical results concerning a preliminary sampling of the effects of having five instead of three layers of different thickness. With reference to Fig. 5, we plot the maximum deflection of the mid-section

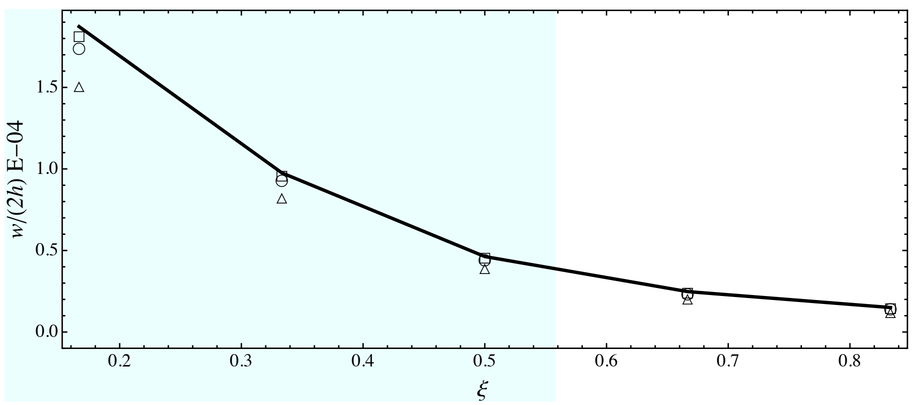

Fig. 5. Force applied at the centre of the upper end face. Maximum deflection at the mid-plane cross-section. Three-layer plate-like body.

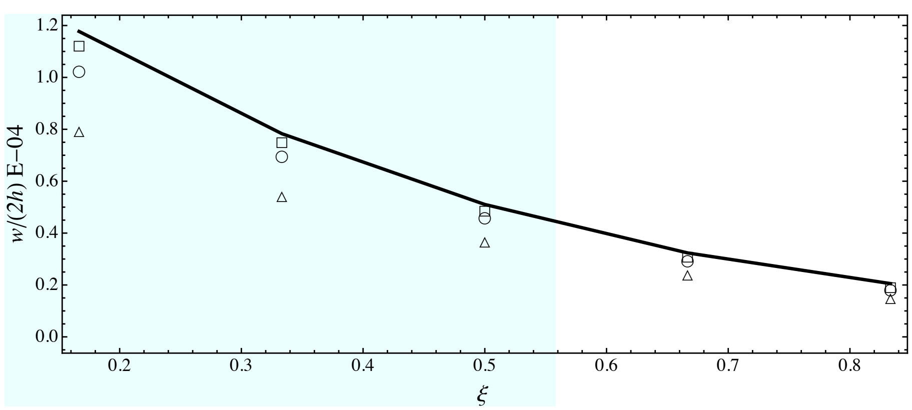

Fig. 6. Force applied at the centre of the upper end face. Maximum deflection at the mid-plane cross-section. Five-layer plate-like body.

of the same ($2h$-thick) rectangular plate-like body analyzed before, subject to a force applied at the center of the upper end face, by varying the thickness $2\xi h$ of the central (stiffest) layer.

We then consider a five-layer plate-like body that can be thought of as obtained by cutting the body of the previous example at the mid-plane and inserting an additional layer made of the same material as that of external (less stiff) layers. We denote again by $2h$ the body thickness and by ξh the thickness of each of the two (stiffest) intermediate layers. In Fig. 6 we plot the same quantities of the previous example.

Both tests show that solutions truncated after order-5 terms approximate well the reference solution, to which they get closer and closer as the thickness of the stiffest layers becomes larger.

References

1. M. Levinson, *The simply supported rectangular plate: An exact, three dimensional, linear elasticity solution*, J. Elasticity, **15**, 283–291, 1985.
2. P. Bisegna, F. Maceri, *An exact three-dimensional solution for piezoelectric plates*, J. Appl. Mech., **63**, 628–638, 1996.
3. V. Nicotra, A. Tiero, *Exact solutions for linearly electroelastic plate-like bodies*, J. Elasticity, **74**, 249–263, 2004.
4. V. Nicotra, P. Podio-Guidugli, A. Tiero, *Exact equilibrium solutions for linearly elastic plate-like bodies*, J. Elasticity, **56**, 231–245, 1999.
5. P. Nardinocchi, P. Podio-Guidugli, *Levinson-type benchmarks for slide-clamped and elastically supported plates*, J. Elasticity, **73**, 211–220, 2003.
6. V. Nicotra, *Benchmark solutions for simply supported plates of arbitrary shape. The case of circular plates* [in Italian], Quaderno IAC, Vol. 4, 2000.
7. M. Lembo, P. Podio-Guidugli, *How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body*, Int. J. Solids and Structures, **44**, 231–245, 2007.
8. C. Truesdell, R. Toupin, *The Classical Field Theories*, [in:] *Handbuch der Physik*, S. Flügge [Ed.], Springer, Berlin, 1960.
9. E. Pan, *An exact solution for transversely isotropic, simply supported and layered rectangular plates*, J. Elasticity, **25**, 101–116, 1991.

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