Arch. Mech., **65**, 5, pp. 367–390, Warszawa 2013

Fundamental solutions in the full coupled theory of elasticity for solids with double porosity

M. SVANADZE[1)], S. DE CICCO[2)]

[1)] *Institute for Fundamental and Interdisciplinary Mathematics Research*
Ilia State University
K. Cholokashvili Ave., 3/5
0162 Tbilisi, Georgia
e-mail: svanadze@gmail.com

[2)] *Dipartimento di Costruzioni e Metodi Matematici in Architettura*
Università di Napoli, "Federico II"
Via Monteoliveto 3
80134 Napoli, Italy
e-mail: simona.decicco@unina.it

This paper discusses the full coupled linear theory of elasticity for solids with double porosity. The system of the governing equations is based on the equations of motion, conservation of fluid mass, the constitutive equations and Darcy's law for material with double porosity. Four spatial cases of the dynamical equations are considered: equations of steady vibrations, equations in Laplace transform space, equations of quasi-static and equations of equilibrium. The fundamental solutions of the systems of these partial differential equations (PDEs) are constructed by means of elementary functions and finally, the basic properties of these solutions are established.

Key words: double porosity, fundamental solutions, steady vibrations, Darcy's law.
Mathematics Subject Classification: 74F10, 35E05.

1. Introduction

The theory of consolidation for single-porosity materials was formulated in [1]. The Biot system is formally equivalent to the classical coupled quasi-static system of thermoelasticity which describes the deformation and heat flow through an elastic solid. The model for consolidation requires the quasi-static assumption that the equations of motion are replaced by the corresponding equilibrium equations. One important generalization of this theory that has been studied extensively began with the work [2], where a fissured porous medium is characterized as two completely overlapping flow regions: one representing the porous matrix, and the other the fissure network.

The theory of consolidation for elastic materials with double porosity was presented in [3–5]. The theory of Aifantis unifies the earlier proposed models of Barenblatt for porous media with double porosity [2] and Biot's model for porous media with single porosity [1].

However, Aifantis' quasi-static theory ignored the cross-coupling effects between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solid with double porosity by several authors [6–12]. The significance of the cross-coupling effects on the pore and fracture fluid pressure response of double porosity media was highlighted in [9], and it is shown that by neglecting the microscopic coupling between the volumetric deformations of the two-pore system many of the characteristic features of flow and deformation in double porous media cannot be simulated.

In [10, 11], the phenomenological equations of the quasi-static theory for double porosity media are established and the method to determine the relevant coefficients is presented. The governing equations in the quasi-static case for fluid-saturated double porosity media are derived in [12]. In these papers [10–12] the cross-coupled terms were included in Darcy's law for solids with double porosity.

The double porosity concept was extended for multiple porosity media in [13, 14]. The basic equations of the thermo-hydro-mechanical coupling theory for elastic materials with double porosity were presented in [15–17]. The theory of multiporous media, as originally developed for the mechanics of naturally fractured reservoirs, has found applications in blood perfusion. The double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity. An extensive review of the results in the theory of bone poroelasticity can be found in the survey papers [18–20]. For a history of developments and a review of main results in the theory of porous media see [21].

In the governing equations of the above mentioned theories of poroelasticity the inertial term was neglected and the quasi-static problems were investigated. The fully dynamic system to describe deformation in single-porosity media was developed in [22–24]. Obviously, the inertial effect play a pivotal role in investigation of various problems of vibrations and wave propagation through double porosity media. Therefore, it is important to study a full dynamic model for materials with double porosity. In the present paper, we shall consider flow and deformation processes of the double-porosity media in the case when the inertia effect is included, and in the four spatial cases (steady vibrations, Laplace transform space, quasi-static and equilibrium) of the dynamical theory, the fundamental solutions of the governing system of PDEs will be constructed.

The fundamental solutions have occupied a special place in the theory of PDEs. They are encountered in many mathematical, mechanical, physical and engineering applications. Indeed, the application of fundamental solutions to a recently developed area of boundary element methods has provided a distinct advantage in the fact that an integral representation of solution of a boundary value problem by fundamental solution is often more easily solved by numerical methods than a differential equation with specified boundary and initial conditions. Recent advances in the area of boundary element methods, where the theory of fundamental solutions plays a pivotal role, has provided a prominent place in research of problems in the theories of PDEs, applied mathematics, continuum mechanics and quantum physics. The fundamental solutions in the linear theories of elasticity and thermoelasticity for materials with microstructures are constructed by means of elementary functions by several authors [25–32]. The fundamental solution in dynamic poroelasticity for materials with single porosity is constructed in [33–35]. For historical and bibliographical material on the fundamental solutions see [36].

This paper is concerned with the full coupled linear theory of elasticity for solids with double porosity. The system of the governing equations is based on the equations of motion [22–24], conservation of fluid mass [6–9], the constitutive equations [3–8] and Darcy's law for material with double porosity [10–12]. Four spatial cases of the dynamical equations are considered: equations of steady vibrations, equations in Laplace transform space, equations of quasi-static and equations of equilibrium. The fundamental solutions of the systems of these PDEs are constructed by means of elementary (harmonic, biharmonic, metaharmonic) functions. Finally, the basic properties of the fundamental solutions are established.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space R^3, let t denote the time variable, $t \geq 0$, $\mathbf{u}'(\mathbf{x}, t)$ is the displacement vector in a solid, $\mathbf{u}' = (u'_1, u'_2, u'_3)$; $p'_1(\mathbf{x}, t)$ and $p'_2(\mathbf{x}, t)$ are the pore and fissure fluid pressures, respectively.

We assume that the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range $(1, 2, 3)$, and the dot denotes differentiation with respect to t.

The governing system of field equations in the full coupled linear theory of elasticity for solids with double porosity consists of the following equations.

1) *The equations of motion* [22–24]

$$t_{lj,j} = \rho(\ddot{u}'_l - F'_l), \qquad l = 1, 2, 3, \tag{2.1}$$

where t_{lj} is the component of total stress tensor, ρ is the reference mass density, $\rho > 0$, $\mathbf{F}' = (F'_1, F'_2, F'_3)$ is the body force per unit mass.

2) *The equations of fluid mass conservation* [6–9]

$$\operatorname{div}\mathbf{v}^{(1)} + \dot{\zeta}_1 + \beta_1 \dot{e}_{rr} + \gamma(p'_1 - p'_2) = 0, \tag{2.2}$$

and

$$\operatorname{div}\mathbf{v}^{(2)} + \dot{\zeta}_2 + \beta_2 \dot{e}_{rr} - \gamma(p'_1 - p'_2) = 0, \tag{2.3}$$

where $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are the fluid flux vectors for the pores and fissures, respectively; e_{lj} is the component of strain tensor,

$$e_{lj} = \frac{1}{2}\left(u'_{l,j} + u'_{j,l}\right), \qquad l, j = 1, 2, 3, \tag{2.4}$$

β_1 and β_2 are the effective stress parameters, γ is the internal transport coefficient (leakage parameter) and corresponds to a fluid transfer rate with respect to the intensity of flow between the pores and fissures, $\gamma > 0$; ζ_1 and ζ_2 are the increments of fluid (volumetric strain) in the pores and fissures, respectively, and defined by

$$\zeta_1 = \alpha_1 p'_1 + \alpha_{12} p'_2, \qquad \zeta_2 = \alpha_{21} p'_1 + \alpha_2 p'_2, \tag{2.5}$$

α_1 and α_2 measure the compressibilities of the pore and fissure systems, respectively; α_{12} and α_{21} are the cross-coupling compressibility for fluid flow at the interface between the two-pore systems at a microscopic level [6–9]. However, the coupling effect (α_{12} and α_{21}) is often neglected (see, e.g., [3–5]).

3) The constitutive equations (extending Terzaghi's effective stress concept to double porosity) [3–8]

$$t_{lj} = t'_{lj} - (\beta_1 p'_1 + \beta_2 p'_2)\delta_{lj}, \qquad l, j = 1, 2, 3, \tag{2.6}$$

where $t'_{lj} = 2\mu e_{lj} + \lambda e_{rr}\delta_{lj}$ is the component of effective stress tensor, λ and μ are the Lamé constants, δ_{lj} is the Kronecker delta.

4) *Darcy's law for material with double porosity* [10–12]

$$\begin{aligned} \mathbf{v}^{(1)} &= -\frac{1}{\mu'}\left(\kappa_1 \operatorname{grad} p'_1 + \kappa_{12} \operatorname{grad} p'_2\right) - \rho_1 \mathbf{s}^{(1)}, \\ \mathbf{v}^{(2)} &= -\frac{1}{\mu'}\left(\kappa_{21} \operatorname{grad} p'_1 + \kappa_2 \operatorname{grad} p'_2\right) - \rho_2 \mathbf{s}^{(2)}, \end{aligned} \tag{2.7}$$

where μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively; κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; ρ_1, $\mathbf{s}^{(1)}$ and ρ_2, $\mathbf{s}^{(2)}$ are the densities of fluid, the external forces (such as gravity) for the pore and fissure phases, respectively. The cross-coupling terms of (2.7) with coefficients κ_{12} and κ_{21} are considered by several authors [10–12]. However, the latter coupling effect (κ_{12} and κ_{21}) is often neglected (see, e.g., [3–5]).

Substituting equations (2.4)–(2.7) into (2.1)–(2.3), we obtain the following system of equations of motion in the full coupled linear theory of elasticity for solids with double porosity expressed in terms of the displacement vector $\mathbf{u}'$ and the pressures p_1' and p_2':

$$\begin{aligned} &\mu\Delta\mathbf{u}' + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}' - \beta_1\,\mathrm{grad}\,p_1' - \beta_2\,\mathrm{grad}\,p_2' = \rho(\ddot{\mathbf{u}}' - \mathbf{F}'),\\ &k_1\Delta p_1' + k_{12}\Delta p_2' - \alpha_1\dot{p}_1' - \alpha_{12}\dot{p}_1' - \gamma(p_1' - p_2') - \beta_1\,\mathrm{div}\,\dot{\mathbf{u}}' = -\rho_1\,\mathrm{div}\,\mathbf{s}^{(1)},\\ &k_{21}\Delta p_1' + k_2\Delta p_2' - \alpha_{21}\dot{p}_1' - \alpha_2\dot{p}_2' + \gamma(p_1' - p_2') - \beta_2\,\mathrm{div}\,\dot{\mathbf{u}}' = -\rho_2\,\mathrm{div}\,\mathbf{s}^{(2)}, \end{aligned} \tag{2.8}$$

where Δ is the Laplacian operator, $k_j = \kappa_j/\mu'$ $(j = 1,2)$, $k_{12} = \kappa_{12}/\mu'$, $k_{21} = \kappa_{21}/\mu'$.

For the body force $\mathbf{F}'$, the external forces $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are assumed to be absent, and for the displacement vector $\mathbf{u}'$, the pressures p'_1 and p'_2 are postulated to have a harmonic time variation, that is,

$$\{\mathbf{u}', p_1', p_2'\}(\mathbf{x}, t) = \mathrm{Re}\,[\{\mathbf{u}, p_1, p_2\}(\mathbf{x})e^{-i\omega t}],$$

then from system of equations of motion (2.8) we obtain the following system of homogeneous equations of *steady vibrations* in the full coupled linear theory of elasticity for solids with double porosity:

$$\begin{aligned} \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta_1\,\mathrm{grad}\,p_1 - \beta_2\,\mathrm{grad}\,p_2 + \rho\omega^2\mathbf{u} &= \mathbf{0},\\ (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 + i\omega\beta_1\,\mathrm{div}\,\mathbf{u} &= 0,\\ (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 + i\omega\beta_2\,\mathrm{div}\,\mathbf{u} &= 0, \end{aligned} \tag{2.9}$$

where $a_j = i\omega\,\alpha_j - \gamma$, $a_{lj} = i\omega\,\alpha_{lj} + \gamma$ $(l, j = 1,2)$; ω is the oscillation frequency, $\omega > 0$.

If $\mathbf{F}' = \mathbf{s}^{(1)} = \mathbf{s}^{(2)} = \mathbf{0}$, then the system (2.8) in the Laplace transform space can be rewritten as

$$\begin{aligned} \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta_1\,\mathrm{grad}\,p_1 - \beta_2\,\mathrm{grad}\,p_2 - \rho\tau^2\mathbf{u} &= \mathbf{0},\\ (k_1\Delta + b_1)p_1 + (k_{12}\Delta + b_{12})p_2 - \tau\beta_1\,\mathrm{div}\,\mathbf{u} &= 0,\\ (k_{21}\Delta + b_{21})p_1 + (k_2\Delta + b_3)p_2 - \tau\beta_2\,\mathrm{div}\,\mathbf{u} &= 0, \end{aligned} \tag{2.10}$$

where $b_j = -\tau\alpha_j - \gamma$, $b_{lj} = -\tau\alpha_{lj} + \gamma$ $(l, j = 1,2)$; τ is a complex number and $\mathrm{Re}\,\tau > 0$. It is easy to verify that the system (2.10) may be obtained from the system (2.9) by replacing ω by $-i\tau$. The system (2.10) plays an important auxiliary role in the study of dynamic problems of the full coupled linear theory of elasticity for solids with double porosity. As in the classical theories of elasticity and thermoelasticity (see [37]), Eqs. (2.10) will be called the equations of *pseudo-oscillations* in the full coupled linear theory of elasticity for solids with double porosity.

Neglecting inertial effect ($\rho = 0$) in (2.8), we obtain the system of homogeneous equations of steady vibrations in the full coupled linear *quasi-static* theory of elasticity for solids with double porosity

$$
\begin{gathered}
\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta_1\,\mathrm{grad}\,p_1 - \beta_2\,\mathrm{grad}\,p_2 = \mathbf{0},\\
(k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 + i\omega\beta_1\,\mathrm{div}\,\mathbf{u} = 0,\\
(k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 + i\omega\beta_2\,\mathrm{div}\,\mathbf{u} = 0.
\end{gathered}
\tag{2.11}
$$

Obviously, in the static case ($\omega = 0$), from (2.9) we get the following system of homogeneous equations in the full coupled linear *equilibrium* theory of elasticity for solids with double porosity

$$
\begin{gathered}
\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta_1\,\mathrm{grad}\,p_1 - \beta_2\,\mathrm{grad}\,p_2 = \mathbf{0},\\
(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0,\\
(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0.
\end{gathered}
\tag{2.12}
$$

We introduce the second-order matrix differential operators with constant coefficients:

1) $\mathbf{A}^{(s)}(\mathbf{D_x}) = (A^{(s)}_{lj}(\mathbf{D_x}))_{5\times5}$,

$$A^{(s)}_{lj}(\mathbf{D_x}) = (\mu\Delta + \rho\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l\partial x_j},$$

$$A^{(s)}_{l;m+3}(\mathbf{D_x}) = -\beta_m\frac{\partial}{\partial x_l},\quad A^{(s)}_{m+3;l}(\mathbf{D_x}) = i\omega\beta_m\frac{\partial}{\partial x_l},$$

$$A^{(s)}_{44}(\mathbf{D_x}) = k_1\Delta + a_1,\qquad A^{(s)}_{45}(\mathbf{D_x}) = k_{12}\Delta + a_{12},$$

$$A^{(s)}_{54}(\mathbf{D_x}) = k_{21}\Delta + a_{21},\qquad A^{(s)}_{55}(\mathbf{D_x}) = k_2\Delta + a_2,\qquad m=1,2,\ l,j=1,2,3.$$

2) $\mathbf{A}^{(p)}(\mathbf{D_x}) = (A^{(p)}_{lj}(\mathbf{D_x}))_{5\times5}$,

$$A^{(p)}_{lj}(\mathbf{D_x}) = (\mu\Delta - \rho\tau^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l\partial x_j},$$

$$A^{(p)}_{l;m+3}(\mathbf{D_x}) = -\beta_m\frac{\partial}{\partial x_l},\qquad A^{(p)}_{m+3;l}(\mathbf{D_x}) = -\tau\beta_m\frac{\partial}{\partial x_l},$$

$$A^{(p)}_{44}(\mathbf{D_x}) = k_1\Delta + b_1,\qquad A^{(p)}_{45}(\mathbf{D_x}) = k_{12}\Delta + b_{12},$$

$$A^{(p)}_{54}(\mathbf{D_x}) = k_{21}\Delta + b_{21},\quad A^{(p)}_{55}(\mathbf{D_x}) = k_2\Delta + b_2,\quad m=1,2,\ l,j=1,2,3.$$

3) $\mathbf{A}^{(q)}(\mathbf{D_x}) = (A^{(q)}_{lj}(\mathbf{D_x}))_{5\times 5}, \qquad A^{(q)}_{lj}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\dfrac{\partial^2}{\partial x_l \partial x_j},$

$$A^{(q)}_{l;m+3}(\mathbf{D_x}) = A^{(s)}_{l;m+3}(\mathbf{D_x}), \qquad A^{(q)}_{m+3;n}(\mathbf{D_x}) = A^{(s)}_{m+3;n}(\mathbf{D_x}),$$
$$m=1,2, \quad l,j=1,2,3, \quad n=1,2,\dots,5.$$

4) $\mathbf{A}^{(e)}(\mathbf{D_x}) = (A^{(e)}_{lj}(\mathbf{D_x}))_{5\times 5}, \qquad A^{(e)}_{lj}(\mathbf{D_x}) = A^{(q)}_{lj}(\mathbf{D_x}),$

$$A^{(e)}_{l;m+3}(\mathbf{D_x}) = A^{(q)}_{l;m+3}(\mathbf{D_x}), \qquad A^{(e)}_{m+3;l}(\mathbf{D_x}) = 0,$$
$$A^{(e)}_{44}(\mathbf{D_x}) = k_1\Delta - \gamma, \qquad A^{(e)}_{45}(\mathbf{D_x}) = k_{12}\Delta + \gamma,$$
$$A^{(e)}_{54}(\mathbf{D_x}) = k_{21}\Delta + \gamma, \qquad A^{(e)}_{55}(\mathbf{D_x}) = k_2\Delta - \gamma, \qquad m=1,2,\ l,j=1,2,3.$$

5) $\mathbf{A}^{(o)}(\mathbf{D_x}) = (A^{(o)}_{lj}(\mathbf{D_x}))_{5\times 5}, \qquad A^{(o)}_{lj}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\dfrac{\partial^2}{\partial x_l \partial x_j},$

$$A^{(o)}_{44}(\mathbf{D_x}) = k_1\Delta, \quad A^{(o)}_{45}(\mathbf{D_x}) = k_{12}\Delta, \quad A^{(o)}_{54}(\mathbf{D_x}) = k_{21}\Delta,$$
$$A^{(o)}_{55}(\mathbf{D_x}) = k_2\Delta, \quad A^{(o)}_{l;m+3}(\mathbf{D_x}) = A^{(o)}_{m+3;l}(\mathbf{D_x}) = 0, \quad m=1,2,\ l,j=1,2,3.$$

It is easily seen that the systems (2.9)–(2.12) can be written as

$$\mathbf{A}^{(s)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{A}^{(p)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0},$$
$$\mathbf{A}^{(q)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{A}^{(e)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

respectively, where $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ is the five-component vector function and $\mathbf{x} \in R^3$.

The matrix differential operator $\mathbf{A}^{(o)}(\mathbf{D_x})$ is called the *principal part* of the operator $\mathbf{A}^{(r)}(\mathbf{D_x})$, where $r = s,p,q,e$.

DEFINITION 1. The operator $\mathbf{A}^{(r)}(\mathbf{D_x})$ is said to be elliptic if [36]

$$\det \mathbf{A}^{(o)}(\boldsymbol{\xi}) \neq 0,$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $|\boldsymbol{\xi}| \neq 0$, $r = s,p,q,e$.

Obviously, we have

$$\det \mathbf{A}^{(o)}(\boldsymbol{\xi})$$
$$= \det \begin{pmatrix} \mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 & 0 & 0 \\ (\lambda+\mu)\xi_1\xi_2 & \mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\xi_2^2 & (\lambda+\mu)\xi_2\xi_3 & 0 & 0 \\ (\lambda+\mu)\xi_1\xi_3 & (\lambda+\mu)\xi_2\xi_3 & \mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\xi_3^2 & 0 & 0 \\ 0 & 0 & 0 & k_1|\boldsymbol{\xi}|^2 & k_{12}|\boldsymbol{\xi}|^2 \\ 0 & 0 & 0 & k_{21}|\boldsymbol{\xi}|^2 & k_2|\boldsymbol{\xi}|^2 \end{pmatrix}_{5\times 5}$$
$$= \mu^2 \mu_0\, k\, |\boldsymbol{\xi}|^{10},$$

where $\mu_0 = \lambda + 2\mu$, $k = k_1 k_2 - k_{12} k_{21}$. Hence, $\mathbf{A}^{(r)}(\mathbf{D_x})$ is an elliptic differential operator if and only if

$$\mu\, \mu_0\, k \neq 0, \tag{2.13}$$

where $r = s, p, q, e$. We will suppose that the assumption (2.13) holds true.

Definition 2. The fundamental matrix of operator $\mathbf{A}^{(r)}(\mathbf{D_x})$ is the matrix $\mathbf{\Gamma}^{(r)}(\mathbf{x}) = (\Gamma^{(r)}_{lj}(\mathbf{x}))_{5\times 5}$ satisfying condition (in the class of generalized functions) [36]

$$\mathbf{A}^{(r)}(\mathbf{D_x})\mathbf{\Gamma}^{(r)}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{2.14}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{5\times 5}$ is the unit matrix, $\mathbf{x} \in R^3$, and $r = s, p, q, e$.

The matrices $\mathbf{\Gamma}^{(s)}, \mathbf{\Gamma}^{(p)}, \mathbf{\Gamma}^{(q)}$ and $\mathbf{\Gamma}^{(e)}$ are called the fundamental solutions of systems (2.9), (2.10), (2.11) and (2.12), respectively.

In this article the matrices $\mathbf{\Gamma}^{(s)}$, $\mathbf{\Gamma}^{(p)}$, $\mathbf{\Gamma}^{(q)}$ and $\mathbf{\Gamma}^{(e)}$ are constructed in terms of elementary functions, and some of their basic properties are established.

3. Fundamental solution of the system of steady vibrations equations

First we construct the matrix $\mathbf{\Gamma}^{(s)}$. We consider the system of nonhomogeneous equations

$$\begin{aligned} \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + i\omega\beta_1\,\mathrm{grad}\,p_1 + i\omega\beta_2\,\mathrm{grad}\,p_2 + \rho\omega^2\mathbf{u} &= \mathbf{f},\\ (k_1\Delta + a_1)p_1 + (k_{21}\Delta + a_{21})p_2 - \beta_1\,\mathrm{div}\,\mathbf{u} &= f_1,\\ (k_{12}\Delta + a_{12})p_1 + (k_2\Delta + a_2)p_2 - \beta_2\,\mathrm{div}\,\mathbf{u} &= f_2, \end{aligned} \tag{3.1}$$

where $\mathbf{f}$ is a three-component vector function, f_1 and f_2 are scalar functions on R^3. As one may easily verify, the system (3.1) may be written in the form

$$\mathbf{A}^{(s)^T}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{3.2}$$

where $\mathbf{A}^{(s)^T}$ is the transpose of matrix $\mathbf{A}^{(s)}$, $\mathbf{F} = (\mathbf{f}, f_1, f_2)$ is a five-component vector function and $\mathbf{x} \in R^3$.

Applying the operator div to $(3.1)_1$ from system (3.1) we obtain

$$\begin{aligned} (\mu_0\Delta + \rho\omega^2)\,\mathrm{div}\,\mathbf{u} + i\omega\beta_1\Delta p_1 + i\omega\beta_2\Delta p_2 &= \mathrm{div}\,\mathbf{f},\\ (k_1\Delta + a_1)p_1 + (k_{21}\Delta + a_{21})p_2 - \beta_1\,\mathrm{div}\,\mathbf{u} &= f_1,\\ (k_{12}\Delta + a_{12})p_1 + (k_2\Delta + a_2)p_2 - \beta_2\,\mathrm{div}\,\mathbf{u} &= f_2. \end{aligned} \tag{3.3}$$

From (3.3) we have

$$\mathbf{B}(\Delta)\mathbf{V}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}), \tag{3.4}$$

where $\mathbf{V} = (\operatorname{div}\mathbf{u}, p_1, p_2)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) = (\operatorname{div}\mathbf{f}, f_1, f_2)$ and

$$\mathbf{B}(\Delta) = (B_{lj}(\Delta))_{3\times 3} = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & i\omega\beta_1\Delta & i\omega\beta_2\,\Delta \\ -\beta_1 & k_1\Delta + a_1 & k_{21}\Delta + a_{21} \\ -\beta_2 & k_{12}\Delta + a_{12} & k_2\Delta + a_2 \end{pmatrix}_{3\times 3}.$$

We introduce the notation

$$\Lambda_1(\Delta) = \frac{1}{k\mu_0}\det\mathbf{B}(\Delta).$$

It is easily seen that $\Lambda_1(-\xi) = 0$ is a cubic algebraic equation and there exists three roots λ_1^2, λ_2^2 and λ_3^2 (with respect to ξ). Then we have

$$\Lambda_1(\Delta) = (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2).$$

The system (3.4) implies

$$\Lambda_1(\Delta)\,\mathbf{V} = \boldsymbol{\Phi}, \tag{3.5}$$

where

$$\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_3), \qquad \Phi_j = \frac{1}{k\mu_0}\sum_{l=1}^{3} B_{lj}^{*}\,\varphi_l, \qquad j = 1, 2, 3 \tag{3.6}$$

and B_{lj}^{*} is the cofactor of element B_{lj} of the matrix $\mathbf{B}$.

Now applying the operators $\Lambda_1(\Delta)$ to $(3.1)_1$ and taking into account (3.5), we obtain

$$\Lambda_2(\Delta)\mathbf{u} = \mathbf{F}_1, \tag{3.7}$$

where $\Lambda_2(\Delta) = \Lambda_1(\Delta)(\Delta + \lambda_4^2)$, $\lambda_4^2 = \rho\omega^2/\mu$ and

$$\mathbf{F}_1 = \frac{1}{\mu}\left[\Lambda_1(\Delta)\mathbf{f} - (\lambda + \mu)\operatorname{grad}\Phi_1 - i\omega\beta_1\operatorname{grad}\Phi_2 - i\omega\beta_2\operatorname{grad}\Phi_3\right]. \tag{3.8}$$

On the basis of (3.5) and (3.7) we get

$$\boldsymbol{\Lambda}^{(s)}(\Delta)\mathbf{U}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}), \tag{3.9}$$

where $\boldsymbol{\psi} = (\mathbf{F}_1, \Phi_2, \Phi_3)$ is five-component vector and

$$\boldsymbol{\Lambda}^{(s)}(\Delta) = (\Lambda_{lj}^{(s)}(\Delta))_{5\times 5}, \qquad \Lambda_{11}^{(s)}(\Delta) = \Lambda_{22}^{(s)}(\Delta) = \Lambda_{33}^{(s)}(\Delta) = \Lambda_2(\Delta),$$

$$\Lambda_{44}^{(s)}(\Delta) = \Lambda_{55}^{(s)}(\Delta) = \Lambda_1(\Delta), \qquad \Lambda_{lj}^{(s)}(\Delta) = 0, \qquad l, j = 1, 2, \dots, 5,\ l \neq j.$$

We introduce the notations

$$
\begin{aligned}
n_{j1}(\Delta) &= -\frac{1}{k\mu\mu_0}[(\lambda+\mu)B^*_{j1}(\Delta)+i\omega\beta_1 B^*_{j2}(\Delta)+i\omega\beta_2 B^*_{j3}(\Delta)],\\
n_{jl}(\Delta) &= \frac{1}{k\mu_0}B^*_{jl}(\Delta), \qquad j=1,2,3,\ l=2,3.
\end{aligned} \tag{3.10}
$$

In view of (3.6) and (3.10), from (3.8) we have

$$
\begin{aligned}
\mathbf{F}_1 &= \left[\frac{1}{\mu}\Lambda_1(\Delta)\mathbf{I}+n_{11}(\Delta)\,\mathrm{grad}\,\mathrm{div}\right]\mathbf{f}+n_{21}(\Delta)\,\mathrm{grad} f_1+n_{31}(\Delta)\,\mathrm{grad} f_2,\\
\Phi_m &= n_{1m}(\Delta)\,\mathrm{div}\mathbf{f}+n_{2m}(\Delta)f_1+n_{3m}(\Delta)f_2, \qquad m=2,3,
\end{aligned} \tag{3.11}
$$

where $\mathbf{I}=(\delta_{lj})_{3\times 3}$ is the unit matrix.

Thus, from (3.11) we have

$$
\boldsymbol{\psi}(\mathbf{x})=\mathbf{L}^{(s)^T}(\mathbf{D}_\mathbf{x})\mathbf{F}(\mathbf{x}), \tag{3.12}
$$

where

$$
\begin{aligned}
&\mathbf{L}^{(s)}(\mathbf{D}_\mathbf{x}) = (L^{(s)}_{lj}(\mathbf{D}_\mathbf{x}))_{5\times 5},\\
&L^{(s)}_{lj}(\mathbf{D}_\mathbf{x}) = \frac{1}{\mu}\Lambda_1(\Delta)\,\delta_{lj}+n_{11}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L^{(s)}_{l;m+2}(\mathbf{D}_\mathbf{x}) = n_{1m}(\Delta)\frac{\partial}{\partial x_l}, \qquad L^{(s)}_{m+2;l}(\mathbf{D}_\mathbf{x}) = n_{m1}(\Delta)\frac{\partial}{\partial x_l},\\
&L^{(s)}_{m+2;4}(\mathbf{D}_\mathbf{x}) = n_{m2}(\Delta),\\
&L^{(s)}_{m+2;5}(\mathbf{D}_\mathbf{x}) = n_{m3}(\Delta), \qquad l,j=1,2,3,\ m=2,3.
\end{aligned} \tag{3.13}
$$

By virtue of (3.2) and (3.12), from (3.9) it follows that $\boldsymbol{\Lambda}\mathbf{U}=\mathbf{L}^{(s)^T}\mathbf{A}^{(s)^T}\mathbf{U}$. It is obvious that $\mathbf{L}^{(s)^T}\mathbf{A}^{(s)^T}=\boldsymbol{\Lambda}$ and, hence,

$$
\mathbf{A}^{(s)}(\mathbf{D}_\mathbf{x})\mathbf{L}^{(s)}(\mathbf{D}_\mathbf{x})=\boldsymbol{\Lambda}^{(s)}(\Delta). \tag{3.14}
$$

We assume that $\lambda_l^2\neq\lambda_j^2$, where $l,j=1,2,3,4$ and $l\neq j$. Let

$$
\begin{aligned}
&\mathbf{Y}^{(s)}(\mathbf{x}) = (Y^{(s)}_{lm}(\mathbf{x}))_{5\times 5},\\
&Y^{(s)}_{11}(\mathbf{x}) = Y^{(s)}_{22}(\mathbf{x}) = Y^{(s)}_{33}(\mathbf{x}) = \textstyle\sum_{j=1}^4 \eta_{2j}\gamma^{(s)}_j(\mathbf{x}),\\
&Y^{(s)}_{44}(\mathbf{x}) = Y^{(s)}_{55}(\mathbf{x}) = \sum_{j=1}^3 \eta_{1j}\gamma^{(s)}_j(\mathbf{x}),\\
&Y^{(s)}_{lm}(\mathbf{x}) = 0, \qquad l\neq m,\ l,m=1,2,\dots,5,
\end{aligned} \tag{3.15}
$$

where

$$\gamma_j^{(s)}(\mathbf{x}) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \tag{3.16}$$

is the fundamental solution of Helmholtz' equation, i.e., $(\Delta + \lambda_j^2)\gamma_j^{(s)}(\mathbf{x}) = \delta\,(\mathbf{x})$ and

$$\eta_{1m} = \prod_{l=1,\,l\neq m}^{3} (\lambda_l^2 - \lambda_m^2)^{-1}, \quad \eta_{2j} = \prod_{l=1,\,l\neq j}^{4} (\lambda_l^2 - \lambda_j^2)^{-1}, \quad m = 1,2,3,\ j = 1,2,3,4.$$

LEMMA 1. *The matrix* $\mathbf{Y}^{(s)}$ *is the fundamental solution of operator* $\mathbf{\Lambda}^{(s)}(\Delta)$, *that is,*

$$\mathbf{\Lambda}^{(s)}(\Delta)\mathbf{Y}^{(s)}\,(\mathbf{x}) = \delta\,(\mathbf{x})\,\mathbf{J}, \tag{3.17}$$

where $\mathbf{x} \in R^3$.

P r o o f. It suffices to show that $Y_{11}^{(s)}$ and $Y_{44}^{(s)}$ are the fundamental solutions of operators $\Lambda_2(\Delta)$ and $\Lambda_1(\Delta)$, respectively, i.e.,

$$\Lambda_2(\Delta)Y_{11}^{(s)}\,(\mathbf{x}) = \delta\,(\mathbf{x}) \tag{3.18}$$

and

$$\Lambda_1(\Delta)Y_{44}^{(s)}\,(\mathbf{x}) = \delta\,(\mathbf{x})\,.$$

Taking into account the equalities

$$\begin{aligned}
&\eta_{11} + \eta_{12} + \eta_{13} = 0, \qquad \eta_{12}(\lambda_1^2 - \lambda_2^2) + \eta_{13}(\lambda_1^2 - \lambda_3^2) = 0,\\
&\eta_{13}(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2) = 1,\\
&(\Delta + \lambda_l^2)\gamma_j^{(s)}(\mathbf{x}) = \delta\,(\mathbf{x}) + (\lambda_l^2 - \lambda_j^2)\gamma_j^{(s)}(\mathbf{x}), \qquad l,j = 1,2,3,\ \mathbf{x} \in R^3,
\end{aligned}$$

we have

$$\begin{aligned}
\Lambda_1(\Delta)Y_{44}^{(s)}(\mathbf{x}) &= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\sum_{j=1}^{3}\eta_{1j}[\delta(\mathbf{x}) + (\lambda_1^2 - \lambda_j^2)\gamma_j^{(s)}(\mathbf{x})]\\
&= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\sum_{j=2}^{3}\eta_{1j}(\lambda_1^2 - \lambda_j^2)\gamma_j^{(s)}(\mathbf{x})\\
&= (\Delta + \lambda_3^2)\sum_{j=2}^{3}\eta_{1j}(\lambda_1^2 - \lambda_j^2)[\delta(\mathbf{x}) + (\lambda_2^2 - \lambda_j^2)\gamma_j^{(s)}(\mathbf{x})]\\
&= (\Delta + \lambda_3^2)\gamma_3^{(s)}(\mathbf{x}) = \delta(\mathbf{x}).
\end{aligned}$$

Equation (3.18) is proved quite similarly. □

We introduce the matrix

$$\mathbf{\Gamma}^{(s)}(\mathbf{x}) = \mathbf{L}^{(s)}(\mathbf{D_x})\mathbf{Y}^{(s)}(\mathbf{x}). \tag{3.19}$$

Using identities (3.14) and (3.17) from (3.19) we get

$$\mathbf{A}^{(s)}(\mathbf{D_x})\mathbf{\Gamma}^{(s)}(\mathbf{x}) = \mathbf{A}^{(s)}(\mathbf{D_x})\mathbf{L}^{(s)}(\mathbf{D_x})\mathbf{Y}^{(s)}(\mathbf{x}) = \mathbf{\Lambda}^{(s)}(\Delta)\mathbf{Y}^{(s)}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Hence, $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ is the solution of (2.14). We have thereby proved the following theorem.

THEOREM 1. *If the condition* (2.13) *is satisfied, then the matrix* $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ *defined by* (3.19) *is the fundamental solution of system* (2.9), *where the matrices* $\mathbf{L}^{(s)}(\mathbf{D_x})$ *and* $\mathbf{Y}^{(s)}(\mathbf{x})$ *are given by* (3.13) *and* (3.15), *respectively.*

Obviously, each element $\Gamma_{lj}^{(s)}(\mathbf{x})$ of the matrix $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ is represented in the following form:

$$\Gamma_{lj}^{(s)}(\mathbf{x}) = L_{lj}^{(s)}(\mathbf{D_x})Y_{11}^{(s)}(\mathbf{x}), \qquad \Gamma_{lm}^{(s)}(\mathbf{x}) = L_{lm}^{(s)}(\mathbf{D_x})Y_{44}^{(s)}(\mathbf{x}), \tag{3.20}$$
$$l = 1, 2, \ldots, 5,\ j = 1, 2, 3,\ m = 4, 5.$$

REMARK 1. In the case $k_{12} = k_{21} = \alpha_{12} = \alpha_{21} = \rho = 0$, the matrix $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ is constructed and its basic properties are established in [30].

REMARK 2. On the basis of operator $\mathbf{L}^{(s)}(\mathbf{D_x})$ and (3.19) we can obtain the Galerkin type representation of solution of system (2.9) (for details see [38, 39]).

REMARK 3. The operator $\mathbf{A}^{(s)}(\mathbf{D_x})$ is not self adjoined. Obviously, it is possible to construct the fundamental solution of adjoined operator in quite similar manner.

It is easy to verify that the fundamental solution $\mathbf{\Gamma}^{(p)}(\mathbf{x})$ of the system of equations of pseudo-oscillations (2.10), the fundamental matrix of the operator $\mathbf{A}^{(p)}(\mathbf{D_x})$, may be obtained from the matrix $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ by replacing ω by $-i\tau$. We have the following result.

THEOREM 2. *If the condition* (2.13) *is satisfied, then the matrix* $\mathbf{\Gamma}^{(p)}(\mathbf{x})$ *defined by*

$$\mathbf{\Gamma}^{(p)}(\mathbf{x}) = \mathbf{L}^{(p)}(\mathbf{D_x})\mathbf{Y}^{(p)}(\mathbf{x})$$

is the fundamental solution of system (2.10), *where* $\mathbf{L}^{(p)}(\mathbf{D_x})$ *and* $\mathbf{Y}^{(p)}(\mathbf{x})$ *are obtained from the matrix* $\mathbf{L}^{(s)}(\mathbf{x})$ *and* $\mathbf{Y}^{(s)}(\mathbf{x})$ *by replacing* ω *by* $-i\tau$, *respectively.*

REMARK 4. The matrices $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ and $\mathbf{\Gamma}^{(p)}(\mathbf{x})$ are constructed by 4 metaharmonic functions (solutions of the Helmholtz equation) $\gamma_m^{(s)}$ $(m = 1, 2, 3, 4)$ (see (3.16)).

4. Fundamental solutions of the systems of quasi-static and equilibrium equations

By the method, developed in the previous section, we can construct the fundamental solutions of the systems of quasi-static equations (2.11) and equilibrium equations (2.12).

In what follows we shall use the notations

1)

$$\mathbf{C}(\Delta) = (C_{lj}(\Delta))_{3\times3} = \begin{pmatrix} \mu_0\Delta & i\omega\beta_1\Delta & i\omega\beta_2\,\Delta \\ -\beta_1 & k_1\Delta + a_1 & k_{21}\Delta + a_{21} \\ -\beta_2 & k_{12}\Delta + a_{12} & k_2\Delta + a_2 \end{pmatrix}_{3\times3},$$

$$\mathbf{C}'(\Delta) = \left(C'_{lj}(\Delta)\right)_{3\times3} = \begin{pmatrix} \mu_0 & i\omega\beta_1 & i\omega\beta_2 \\ -\beta_1 & k_1\Delta + a_1 & k_{21}\Delta + a_{21} \\ -\beta_2 & k_{12}\Delta + a_{12} & k_2\Delta + a_2 \end{pmatrix}_{3\times3}.$$

2)

$$\begin{aligned} &\mathbf{\Lambda}^{(q)}(\Delta) = (\Lambda^{(q)}_{lj}(\Delta))_{5\times5}, \\ &\Lambda^{(q)}_{11}(\Delta) = \Lambda^{(q)}_{22}(\Delta) = \Lambda^{(q)}_{33}(\Delta) = \Delta\Lambda_3(\Delta), \\ &\Lambda^{(q)}_{44}(\Delta) = \Lambda^{(q)}_{55}(\Delta) = \Lambda_3(\Delta), \qquad \Lambda^{(q)}_{lj}(\Delta) = 0, \quad l,j = 1,2,\dots,5,\ l \neq j, \end{aligned}$$

where $\Lambda_3(\Delta) = \Delta(\Delta + \xi_1^2)(\Delta + \xi_2^2)$; ξ_1^2 and ξ_2^2 are the roots of equation (with respect to ξ) $\det \mathbf{C}'(-\xi) = 0$.

3)

$$\begin{aligned} &m_{j1}(\Delta) = -\frac{1}{k\mu\mu_0}[(\lambda+\mu)C^*_{j1}(\Delta) + i\omega\beta_1 C^*_{j2}(\Delta) + i\omega\beta_2 C^*_{j3}(\Delta)], \\ &m_{jl}(\Delta) = \frac{1}{k\mu_0}C^*_{jl}(\Delta), \quad j = 1,2,3, l = 2,3, \end{aligned}$$

where C^*_{lj} is the cofactor of element C_{lj} of the matrix $\mathbf{C}$.

4)

$$\begin{aligned} &\mathbf{L}^{(q)}(\mathbf{D_x}) = (L^{(q)}_{lj}(\mathbf{D_x}))_{5\times5}, \\ &L^{(q)}_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1(\Delta)\delta_{lj} + m_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j}, \\ &L^{(q)}_{l;n+2}(\mathbf{D_x}) = m_{1n}(\Delta)\frac{\partial}{\partial x_l}, \quad L^{(q)}_{n+2;l}(\mathbf{D_x}) = m_{n1}(\Delta)\frac{\partial}{\partial x_l}, \\ &L^{(q)}_{n+2;4}(\mathbf{D_x}) = m_{n2}(\Delta), \qquad L^{(q)}_{55}(\mathbf{D_x}) = m_{33}(\Delta), \quad l,j = 1,2,3,\ n = 2,3. \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&\mathbf{Y}^{(q)}(\mathbf{x}) = (Y^{(q)}_{lj}(\mathbf{x}))_{5\times 5},\\
&Y^{(q)}_{11}(\mathbf{x}) = Y^{(q)}_{22}(\mathbf{x}) = Y^{(q)}_{33}(\mathbf{x}) = \sum_{n=1}^{2}[c_n\gamma_n(\mathbf{x}) + c_{n+2}\gamma^{(q)}_n(\mathbf{x})],\\
&Y^{(q)}_{44}(\mathbf{x}) = Y^{(q)}_{55}(\mathbf{x}) = c_2\gamma_1(\mathbf{x}) + \sum_{n=1}^{2} d_n\gamma^{(q)}_n(\mathbf{x}), \qquad Y^{(q)}_{lj}(\mathbf{x}) = 0,\\
&\qquad\qquad l \neq j,\ l,j = 1,2,\ldots,5,
\end{aligned}
\tag{4.2}
$$

where

$$
\gamma_1(\mathbf{x}) = -\frac{1}{4\pi\,|\mathbf{x}|}, \quad \gamma_2(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \qquad \gamma^{(q)}_n(\mathbf{x}) = -\frac{e^{i\xi_n|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \quad n = 1,2,
\tag{4.3}
$$

and

$$
\begin{aligned}
c_1 &= -\frac{\xi_1^2+\xi_2^2}{\xi_1^4\xi_2^4}, & c_2 &= \frac{1}{\xi_1^2\xi_2^2},\\
c_3 &= \frac{1}{\xi_1^4(\xi_2^2-\xi_1^2)}, & c_4 &= \frac{1}{\xi_2^4(\xi_1^2-\xi_2^2)},\\
d_1 &= \frac{1}{\xi_1^2(\xi_1^2-\xi_2^2)}, & d_2 &= \frac{1}{\xi_2^2(\xi_2^2-\xi_1^2)}.
\end{aligned}
$$

We introduce the matrix

$$
\mathbf{\Gamma}^{(q)}(\mathbf{x}) = \mathbf{L}^{(q)}(\mathbf{D_x})\,\mathbf{Y}^{(q)}(\mathbf{x}).
\tag{4.4}
$$

THEOREM 3. *If the condition* (2.13) *is satisfied, then the matrix* $\mathbf{\Gamma}^{(q)}(\mathbf{x})$ *defined by* (4.4) *is the fundamental solution of system of quasi-static equations* (2.11), *where the matrices* $\mathbf{L}^{(q)}(\mathbf{D_x})$ *and* $\mathbf{Y}^{(q)}(\mathbf{x})$ *are given by* (4.1) *and* (4.2), *respectively.*

REMARK 1. The matrix $\mathbf{\Gamma}^{(q)}(\mathbf{x})$ is constructed by harmonic (γ_1), biharmonic (γ_2) and metaharmonic ($\gamma_1^{(q)}$ and $\gamma_2^{(q)}$) functions (see (4.3)).

Now we introduce the following notations:

1)

$$
\begin{aligned}
&\mathbf{L}^{(e)}(\mathbf{D_x}) = (L^{(e)}_{lj}(\mathbf{D_x}))_{5\times 5}, \quad L^{(e)}_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\left(\Delta\delta_{lj} - \frac{\lambda+\mu}{\mu_0}\frac{\partial^2}{\partial x_l \partial x_j}\right),\\
&L^{(e)}_{l4}(\mathbf{D_x}) = \frac{1}{k\mu_0}[(\beta_1 k_2 - \beta_2 k_{21})\Delta - \gamma(\beta_1+\beta_2)]\frac{\partial}{\partial x_l},\\
&L^{(e)}_{l5}(\mathbf{D_x}) = \frac{1}{k\mu_0}[(\beta_2 k_1 - \beta_1 k_{12})\Delta - \gamma(\beta_1+\beta_2)]\frac{\partial}{\partial x_l},
\end{aligned}
\tag{4.5}
$$

$$L_{4l}^{(e)}(\mathbf{D_x}) = L_{5l}^{(e)}(\mathbf{D_x}) = 0, \quad L_{44}^{(e)}(\mathbf{D_x}) = \frac{1}{k}(k_2\Delta - \gamma)\Delta,$$

(4.5)[cont.] $$L_{45}^{(e)}(\mathbf{D_x}) = -\frac{1}{k}(k_{12}\Delta + \gamma)\Delta, \quad L_{54}^{(e)}(\mathbf{D_x}) = -\frac{1}{k}(k_{21}\Delta + \gamma)\Delta,$$

$$L_{55}^{(e)}(\mathbf{D_x}) = \frac{1}{k}(k_1\Delta - \gamma)\Delta, \qquad l,j = 1,2,3.$$

2)

$$\mathbf{Y}^{(e)}(\mathbf{x}) = (Y_{lj}^{(e)}(\mathbf{x}))_{5\times 5}, \qquad Y_{11}^{(e)}(\mathbf{x}) = Y_{22}^{(e)}(\mathbf{x}) = Y_{33}^{(e)}(\mathbf{x}) = \gamma_2(\mathbf{x}),$$

(4.6) $$Y_{44}^{(e)}(\mathbf{x}) = Y_{55}^{(e)}(\mathbf{x}) = \frac{1}{\xi_3^4}\left[\gamma_3(\mathbf{x}) - \gamma_1(\mathbf{x})\right] + \frac{1}{\xi_3^2}\gamma_2(\mathbf{x}),$$

$$Y_{lj}^{(e)}(\mathbf{x}) = 0, \qquad l \neq j, \ l,j = 1,2,\ldots,5,$$

where $\gamma_1(\mathbf{x})$ and $\gamma_2(\mathbf{x})$ are defined by (4.3),

$$\gamma_3(\mathbf{x}) = -\frac{e^{i\xi_3|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}, \qquad \xi_3^2 = -\frac{\gamma}{k}(k_1 + k_2 + k_{12} + k_{21}).$$

We introduce the matrix

(4.7) $$\mathbf{\Gamma}^{(e)}(\mathbf{x}) = \mathbf{L}^{(e)}(\mathbf{D_x})\mathbf{Y}^{(e)}(\mathbf{x}).$$

Theorem 4. *If the condition* (2.13) *is satisfied, then the matrix* $\mathbf{\Gamma}^{(e)}(\mathbf{x})$ *defined by* (4.7) *is the fundamental solution of system of equilibrium equations* (2.12), *where the matrices* $\mathbf{L}^{(e)}(\mathbf{D_x})$ *and* $\mathbf{Y}^{(e)}(\mathbf{x})$ *are given by* (4.5) *and* (4.6), *respectively.*

Remark 2. The matrix $\mathbf{\Gamma}^{(e)}(\mathbf{x})$ is constructed by harmonic (γ_1), biharmonic (γ_2) and metaharmonic (γ_3) functions. Obviously, the matrix $\boldsymbol{L}^{(e)}(\mathbf{D_x})$ is not obtained from $\boldsymbol{L}^{(s)}(\mathbf{D_x})$ by replacing $\omega = 0$.

5. Basic properties of fundamental solutions

Theorems 1–4 lead to the following results.

Theorem 5. *Each column of the matrix* $\mathbf{\Gamma}^{(r)}(\mathbf{x})$ *is a solution of homogeneous equation*

$$\mathbf{A}^{(r)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

at every point $\mathbf{x} \in R^3$ *except the origin, where* $r = s,p,q,e$.

Theorem 6. *If condition* (2.13) *is satisfied, then the fundamental solution of the system*

$$\mathbf{A}^{(o)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

is the matrix $\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lj}(\mathbf{x}))_{5\times 5}$, *where*

$$
\begin{aligned}
&\Psi_{lj}(\mathbf{x}) = \frac{1}{\mu}\left(\Delta\delta_{lj} - \frac{\lambda+\mu}{\mu_0}\frac{\partial^2}{\partial x_l \partial x_j}\right)\gamma_2(\mathbf{x}) \\
&\qquad = \frac{1}{\mu_0}\gamma_{2,lj}(\mathbf{x}) - \frac{1}{\mu}R_{lj}\gamma_2(\mathbf{x}) = \lambda' \frac{\delta_{lj}}{|\mathbf{x}|} + \mu' \frac{x_l x_j}{|\mathbf{x}|^3}, \\
&\Psi_{44}(\mathbf{x}) = \frac{k_2}{k}\gamma_1(\mathbf{x}), \qquad \Psi_{45}(\mathbf{x}) = -\frac{k_{12}}{k}\gamma_1(\mathbf{x}), \qquad \Psi_{54}(\mathbf{x}) = -\frac{k_{21}}{k}\gamma_1(\mathbf{x}), \\
&\Psi_{55}(\mathbf{x}) = \frac{k_1}{k}\gamma_1(\mathbf{x}), \qquad \Psi_{lm} = \Psi_{ml} = 0, \qquad \lambda' = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}, \\
&\mu' = -\frac{\lambda+\mu}{8\pi\mu\mu_0}, \qquad R_{lj}(\mathbf{D_x}) = \frac{\partial^2}{\partial x_l \partial x_j} - \Delta\delta_{lj}, \quad l,j = 1,2,3, \;\; m = 4,5.
\end{aligned}
\tag{5.1}
$$

It is easy to verify that $\mathbf{R}(\mathbf{D_x}) = (R_{lj}(\mathbf{D_x}))_{3\times 3} = \operatorname{curl}\operatorname{curl}$.

Obviously, Theorem 6 leads to the following result.

Corollary 1. *The relations*

$$
\Psi_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Psi_{mn}(\mathbf{x}) = O(|\mathbf{x}|^{-1})
\tag{5.2}
$$

hold in the neighborhood of the origin, where $l, j = 1,2,3$ and $m,n = 4,5$.

We shall use the following lemma.

Lemma 2. *If condition* (2.13) *is satisfied, then*

$$
\begin{aligned}
\Delta n_{11}(\Delta) &= -\frac{1}{\mu}\Lambda_1(\Delta) + \frac{1}{k\mu_0}(\Delta+\lambda_4^2)B^*_{11}, \\
n_{21}(\Delta) &= \frac{i\omega}{k\mu_0}(\Delta+\lambda_4^2)[\beta_1(k_2\Delta+a_2) - \beta_2(k_{12}\Delta+a_{12})], \\
n_{31}(\Delta) &= -\frac{i\omega}{k\mu_0}(\Delta+\lambda_4^2)[\beta_1(k_{21}\Delta+a_{21}) - \beta_2(k_1\Delta+a_1)].
\end{aligned}
\tag{5.3}
$$

P r o o f. Taking into account the equalities (3.10) and

$$
(\mu_0\Delta+\rho\omega^2)B^*_{11}(\Delta) + i\omega\beta_1\Delta B^*_{12}(\Delta) + i\omega\beta_2\Delta B^*_{13}(\Delta) = \det\mathbf{B}
$$

we have

$$
\Delta n_{11}(\Delta) = -\frac{1}{k\mu\mu_0}[\det\mathbf{B} - (\mu\Delta+\rho\omega^2)B^*_{11}(\Delta)] = -\frac{1}{\mu}\Lambda_1(\Delta) + \frac{1}{k\mu_0}(\Delta+\lambda_4^2)B^*_{11}.
$$

The formulae $(5.3)_2$ and $(5.3)_3$ are proven in a quite similar manner. □

We introduce the notations

$$
\begin{aligned}
&d_{11}^{(m)} = -\frac{1}{k\mu_0\lambda_m^2}\eta_{1m}B^*_{11}(-\lambda_m^2), \qquad d_{11}^{(4)} = \frac{1}{\rho\,\omega^2}, \\
&d_{q1}^{(m)} = \eta_{2m}n_{q1}(-\lambda_m^2), \qquad d_{q1}^{(m)} = \eta_{1m}n_{1q}(-\lambda_m^2), \\
&d_{qr}^{(m)} = \eta_{1m}n_{qr}(-\lambda_m^2), \qquad m = 1,2,3, \;\; q,r = 2,3.
\end{aligned}
\tag{5.4}
$$

On the basis of Lemma 2 we can rewrite the fundamental solution $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ in the simple form that (3.19) (or (3.20)) for $\mathbf{x} \neq \mathbf{0}$. We have the following result.

THEOREM 7. *If* $\mathbf{x} \neq \mathbf{0}$, *then*

$$\begin{aligned} \Gamma_{lj}^{(s)}(\mathbf{x}) &= \sum_{m=1}^{3} d_{11}^{(m)} \gamma_{m,lj}^{(s)}(\mathbf{x}) + d_{11}^{(4)} R_{lj} \gamma_4^{(s)}(\mathbf{x}), \\ \Gamma_{l;q+2}^{(s)}(\mathbf{x}) &= \sum_{m=1}^{3} d_{1q}^{(m)} \gamma_{m,l}^{(s)}(\mathbf{x}), \qquad \Gamma_{q+2;l}^{(s)}(\mathbf{x}) = \sum_{m=1}^{3} d_{q1}^{(m)} \gamma_{m,l}^{(s)}(\mathbf{x}), \\ \Gamma_{q+2;r+2}^{(s)}(\mathbf{x}) &= \sum_{m=1}^{3} d_{qr}^{(m)} \gamma_{m}^{(s)}(\mathbf{x}), \qquad l,j = 1,2,3, \;\; q,r = 2,3. \end{aligned} \tag{5.5}$$

P r o o f. Let $\mathbf{x} \neq \mathbf{0}$. It is easy to verify that

$$\Delta \gamma_m^{(s)}(\mathbf{x}) = -\lambda_m^2 \gamma_m^{(s)}(\mathbf{x}), \qquad \delta_{lj} \gamma_m^{(s)}(\mathbf{x}) = -\frac{1}{\lambda_m^2} \left(\frac{\partial^2}{\partial x_l \partial x_j} - R_{lj} \right) \gamma_m^{(s)}(\mathbf{x}), \quad l,j = 1,2,3, \;\; m = 1,2,3,4. \tag{5.6}$$

On the other hand, from $(5.3)_1$ it follows that

$$\begin{aligned} n_{11}(-\lambda_m^2) - \frac{1}{\mu \lambda_m^2} \Lambda_1(-\lambda_m^2) \\ = -\frac{1}{k \mu_0 \lambda_m^2} (\lambda_4^2 - \lambda_m^2) B_{11}^*(-\lambda_m^2), \quad m = 1,2,3,4. \end{aligned} \tag{5.7}$$

By virtue of (3.13), (3.15), (5.6) and (5.7) from (3.20) we have

$$\begin{aligned} \Gamma_{lj}^{(s)}(\mathbf{x}) &= \left[\frac{1}{\mu} \Lambda_1(\Delta) \delta_{lj} + n_{11}(\Delta) \frac{\partial^2}{\partial x_l \partial x_j} \right] \sum_{m=1}^{4} \eta_{2m} \gamma_m^{(s)}(\mathbf{x}) \\ &= \sum_{m=1}^{4} \eta_{2m} \left[\frac{1}{\mu} \Lambda_1(-\lambda_m^2) \delta_{lj} + n_{11}(-\lambda_m^2) \frac{\partial^2}{\partial x_l \partial x_j} \right] \gamma_m^{(s)}(\mathbf{x}) \\ &= \sum_{m=1}^{4} \eta_{2m} \left[-\frac{1}{\mu \lambda_m^2} \Lambda_1(-\lambda_m^2) \left(\frac{\partial^2}{\partial x_l \partial x_j} - R_{lj} \right) + n_{11}(-\lambda_m^2) \frac{\partial^2}{\partial x_l \partial x_j} \right] \gamma_m^{(s)}(\mathbf{x}) \\ &= \sum_{m=1}^{4} \eta_{2m} \left[-\frac{1}{k \mu_0 \lambda_m^2} (\lambda_4^2 - \lambda_m^2) B_{11}^*(-\lambda_m^2) \frac{\partial^2}{\partial x_l \partial x_j} + \frac{1}{\mu \lambda_m^2} \Lambda_1(-\lambda_m^2) R_{lj} \right] \gamma_m^{(s)}(\mathbf{x}). \end{aligned} \tag{5.8}$$

On the basis of (5.4) and identities

$$(\lambda_4^2 - \lambda_j^2)\eta_{2j} = \eta_{1j}, \qquad j = 1, 2, 3,$$

$$\eta_{2m}\Lambda_1(-\lambda_m^2) = \begin{cases} 0 & \text{for } m = 1, 2, 3, \\ 1 & \text{for } m = 4, \end{cases}$$

from (5.8) we obtain

$$\Gamma_{lj}^{(s)}(\mathbf{x}) = \sum_{m=1}^{3} d_{11}^{(m)} \frac{\partial^2}{\partial x_l \partial x_j}\gamma_m^{(s)}(\mathbf{x}) + \sum_{m=1}^{4} \frac{1}{\mu\lambda_m^2}\eta_{2m}\Lambda_1(-\lambda_m^2) R_{lj}\gamma_4^{(s)}(\mathbf{x})$$
$$= \sum_{m=1}^{3} d_{11}^{(m)}\gamma_{m,lj}^{(s)}(\mathbf{x}) + d_{11}^{(4)} R_{lj}\gamma_4^{(s)}(\mathbf{x}).$$

The other formulae of (5.5) can be proven quite similarly. □

Theorem 7 leads to the following result.

Theorem 8. *The relations*

$$\Gamma_{lj}^{(s)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{mq}^{(s)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),$$
$$\Gamma_{mj}^{(s)}(\mathbf{x}) = O(1), \qquad \Gamma_{jm}^{(s)}(\mathbf{x}) = O(1) \tag{5.9}$$

hold the neighborhood of the origin, where $l, j = 1, 2, 3,\ m, q = 4, 5.$

Lemma 3. *If condition* (2.13) *is satisfied, then*

$$\sum_{m=1}^{3} d_{11}^{(m)} = -\frac{1}{\rho\omega^2}, \qquad \sum_{m=1}^{3} \lambda_m^2 d_{11}^{(m)} = -\frac{1}{\mu_0}. \tag{5.10}$$

P r o o f. It is easy to verify that

$$B_{11}^*(-\lambda_m^2) = k\lambda_m^4 + (a_{12}k_{21} + a_{21}k_{12} - a_1k_2 - a_2k_1)\lambda_m^2 + a,$$
$$\lambda_1^2\lambda_2^2\lambda_3^2 = \frac{\rho\omega^2 a}{k\mu_0}, \tag{5.11}$$

where $a = a_1a_2 - a_{12}a_{21}$. By virtue of (5.11) we obtain

$$\sum_{m=1}^{3} \frac{1}{\lambda_m^2}\eta_{1m}B_{11}^*(-\lambda_m^2) = \frac{B_{11}^*(-\lambda_1^2)}{\lambda_1^2(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_1^2)} + \frac{B_{11}^*(-\lambda_2^2)}{\lambda_2^2(\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)}$$
$$+ \frac{B_{11}^*(-\lambda_3^2)}{\lambda_3^2(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}$$
$$= \frac{a}{\lambda_1^2\lambda_2^2\lambda_3^2} = \frac{k\mu_0}{\rho\omega^2}.$$

Hence, from (5.4) we have

$$\sum_{m=1}^{3} d_{11}^{(m)} = -\frac{1}{k\mu_0}\sum_{m=1}^{3}\frac{1}{\lambda_m^2}\eta_{1m}B_{11}^*(-\lambda_m^2) = -\frac{1}{\rho\omega^2}.$$

Similarly, by virtue of (5.11) we obtain

$$\sum_{m=1}^{3}\eta_{1m}B_{11}^*(-\lambda_m^2) = \frac{B_{11}^*(-\lambda_1^2)}{(\lambda_2^2-\lambda_1^2)(\lambda_3^2-\lambda_1^2)} + \frac{B_{11}^*(-\lambda_2^2)}{(\lambda_1^2-\lambda_2^2)(\lambda_3^2-\lambda_2^2)} + \frac{B_{11}^*(-\lambda_3^2)}{(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)} = k.$$

Finally, from (5.4) we get

$$\sum_{m=1}^{3}\lambda_m^2 d_{11}^{(m)} = -\frac{1}{k\mu_0}\sum_{m=1}^{3}\eta_{1m}B_{11}^*(-\lambda_m^2) = -\frac{1}{\mu_0}. \qquad \square$$

Now we can establish the singular part of the matrix $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ in the neighborhood of the origin.

THEOREM 9. *The relations*

$$\Gamma_{lj}^{(s)}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|) \tag{5.12}$$

hold in the neighborhood of the origin, where $l, j = 1, 2, \ldots, 5$.

P r o o f. Let $\mathbf{x} \neq \mathbf{0}$. In view of (5.1) and (5.5) we obtain

$$\begin{aligned}\Gamma_{lj}^{(s)}(\mathbf{x}) &- \Psi_{lj}(\mathbf{x}) \\ &= \frac{\partial^2}{\partial x_l \partial x_j}\left[\sum_{m=1}^{3} d_{11}^{(m)}\gamma_m^{(s)}(\mathbf{x}) - \frac{1}{\mu_0}\gamma_2(\mathbf{x})\right] + R_{lj}\left[\frac{1}{\rho\omega^2}\gamma_4^{(s)}(\mathbf{x}) + \frac{1}{\mu}\gamma_2(\mathbf{x})\right]\end{aligned} \tag{5.13}$$

for $l, j = 1, 2, 3$. In the neighborhood of the origin from (3.16) we have

$$\gamma_m^{(s)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\sum_{n=0}^{\infty}\frac{(i\lambda_m|\mathbf{x}|)^n}{n!} = \gamma_1(\mathbf{x}) - \frac{i\lambda_m}{4\pi} - \lambda_m^2\gamma_2(\mathbf{x}) + \tilde{\gamma}_m(\mathbf{x}), \tag{5.14}$$

where

$$\tilde{\gamma}_m(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\sum_{n=3}^{\infty}\frac{(i\lambda_m|\mathbf{x}|)^n}{n!}, \qquad m = 1, 2, 3, 4.$$

Obviously,

$$\tilde{\gamma}_m(\mathbf{x}) = O(|\mathbf{x}|^2), \qquad \tilde{\gamma}_{m,j}(\mathbf{x}) = O(|\mathbf{x}|), \qquad \tilde{\gamma}_{m,lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \qquad l,j = 1,2,3, \;\; m = 1,2,3,4. \tag{5.15}$$

On the basis of (5.14) from (5.13) we get

$$\begin{aligned}\sum_{m=1}^{3} d_{11}^{(m)} \gamma_m^{(s)}(\mathbf{x}) - \frac{1}{\mu_0}\gamma_2(\mathbf{x}) &= \sum_{m=1}^{3} d_{11}^{(m)} \left[\gamma_1(\mathbf{x}) - \frac{i\lambda_m}{4\pi} + \tilde{\gamma}_m(\mathbf{x})\right] \\ &\quad - \left(\sum_{m=1}^{3} \lambda_m^2 d_{11}^{(m)} + \frac{1}{\mu_0}\right)\gamma_2(\mathbf{x}).\end{aligned} \tag{5.16}$$

By virtue of equalities (5.10) from (5.16) it follows that

$$\begin{aligned}&\sum_{m=1}^{3} d_{11}^{(m)} \gamma_m^{(s)}(\mathbf{x}) - \frac{1}{\mu_0}\gamma_2(\mathbf{x}) \\ &\qquad = -\frac{1}{\rho\omega^2}\gamma_1(\mathbf{x}) - \frac{i}{4\pi}\sum_{m=1}^{3} \lambda_m d_{11}^{(m)} + \sum_{m=1}^{3} d_{11}^{(m)} \tilde{\gamma}_m(\mathbf{x}).\end{aligned} \tag{5.17}$$

Similarly, from (5.14) we have

$$\begin{aligned}\frac{1}{\rho\omega^2}\gamma_4^{(s)}(\mathbf{x}) + \frac{1}{\mu}\gamma_2(\mathbf{x}) &= \frac{1}{\rho\omega^2}\left[\gamma_1(\mathbf{x}) - \frac{i\lambda_4}{4\pi} - \lambda_4^2\gamma_2(\mathbf{x}) + \tilde{\gamma}_4(\mathbf{x})\right] + \frac{1}{\mu}\gamma_2(\mathbf{x}) \\ &= \frac{1}{\rho\omega^2}\gamma_1(\mathbf{x}) - \frac{i\lambda_4}{4\pi\rho\omega^2} + \frac{1}{\rho\omega^2}\tilde{\gamma}_4(\mathbf{x}).\end{aligned} \tag{5.18}$$

Taking into account (5.15), (5.17), (5.18) and $\Delta\gamma_1(\mathbf{x}) = 0$ ($\mathbf{x} \neq \mathbf{0}$) from (5.13) we obtain

$$\begin{aligned}\Gamma_{lj}^{(s)}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) &= -\frac{1}{\rho\omega^2}\left(\frac{\partial^2}{\partial x_l \partial x_j} - R_{lj}\right)\gamma_1(\mathbf{x}) + \sum_{m=1}^{3} d_{11}^{(m)}\tilde{\gamma}_m(\mathbf{x}) + \frac{1}{\rho\omega^2}\tilde{\gamma}_4(\mathbf{x}) \\ &= -\frac{1}{\rho\omega^2}\Delta\gamma_1(\mathbf{x}) + \text{const} + O(|\mathbf{x}|) \\ &= \text{const} + O(|\mathbf{x}|), \quad l,j = 1,2,3.\end{aligned}$$

The other formulae of (5.12) can be proved quite similar manner. □

Thus, on the basis of Corollary 1 and Theorem 9 the matrix $\mathbf{\Psi}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}^{(s)}(\mathbf{x})$ in the neighborhood of the origin (see (5.2), (5.9) and (5.12)).

Corollary 1 and Theorem 9 lead to the following results.

THEOREM 10. *The relations*

$$\Gamma^{(r)}_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma^{(r)}_{mq}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma^{(r)}_{mj}(\mathbf{x}) = O(1),$$
$$\Gamma^{(r)}_{jm}(\mathbf{x}) = O(1), \qquad l,j = 1,2,3, \;\; m,q = 4,5$$

hold in the neighborhood of the origin, where $l,j = 1,2,\dots,5$ *and* $r = p,q,e$.

THEOREM 11. *The relations*

$$\Gamma^{(r)}_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = \mathrm{const} + O(|\mathbf{x}|)$$

hold in the neighborhood of the origin, where $l,j = 1,2,\dots,5$ *and* $r = p,q,e$.

Thus, the matrix $\mathbf{\Psi}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}^{(r)}(\mathbf{x})$ in the neighborhood of the origin, where $r = p,q,e$.

6. Concluding remarks

1. On the basis of fundamental solutions of the systems (9) to (12) it is possible:

(i) to construct the surface (single-layer and double-layer) and volume potentials and to establish their basic properties (for details of the surface and volume potentials of the classical theories of elasticity and thermoelasticity see [37]);

(ii) to investigate 3D boundary value problems of the theory of the full coupled theory of elasticity for solids with double porosity by means of the potential method (boundary integral method) and the theory of singular integral equations (for an extensive review of works and basic results on the potential method in the classical theories of elasticity and thermoelasticity see [37, 40]);

(iii) to obtain numerical solutions of the boundary value problems by using boundary element method;

(iv) to construct the Green's functions for the simple cases of 3D domain (sphere, half-space and etc.).

2. By using the above mentioned method it is possible to construct the fundamental solutions of the systems of equations in the modern linear theories of elasticity and thermoelasticity for homogeneous isotropic elastic materials with microstructure.

3. Recently, the plane harmonic waves and the basic boundary value problems of the full coupled theory of poroelasticity for materials with double porosity were investigated in [41–44].

References

1. M.A. Biot, *General theory of three-dimensional consolidation*, Journal of Applied Physics, **12**, 155–164, 1941.

2. G.I. Barenblatt, I.P. Zheltov, I.N. Kochina, *Basic concept in the theory of seepage of homogeneous liquids in fissured rocks (strata)*, Journal of Applied Mathematics and Mechanics, **24**, 1286–1303, 1960.

3. R.K. Wilson, E.C. Aifantis, *On the theory of consolidation with double porosity, I*, International Journal of Engineering Science, **20**, 1009–1035, 1982.

4. M.Y. Khaled, D.E. Beskos, E.C. Aifantis, *On the theory of consolidation with double porosity, III*, International Journal for Numerical and Analytical Methods in Geomechanics, **8**, 101–123, 1984.

5. D.E. Beskos, E.C. Aifantis, *On the theory of consolidation with double porosity, II*, International Journal of Engineering Science, **24**, 1697–1716, 1986.

6. N. Khalili, M.A. Habte, S. Zargarbashi, *A fully coupled flow deformation model for cyclic analysis of unsaturated soils including hydraulic and mechanical hysteresis*, Computers and Geotechnics, **35**, 872–889, 2008.

7. N. Khalili, S. Valliappan, *Unified theory of flow and deformation in double porous media*, European Journal of Mechanics, A/Solids, **15**, 321– 336, 1996.

8. Y. Zhao, M. Chen, *Fully coupled dual-porosity model for anisotropic formations*, International Journal of Rock Mechanics and Mining Sciences, **43**, 1128–1133, 2006.

9. N. Khalili, *Coupling effects in double porosity media with deformable matrix*, Geophysical Research Letters, **30**, 2153, 2003.

10. J.G. Berryman, H.F. Wang, *The elastic coefficients of double-porosity models for fluid transport in jointed rock*, Journal of Geophysical Research, **100**, 34611–34627, 1995.

11. J.G. Berryman, H.F. Wang, *Elastic wave propagation and attenuation in a double-porosity dual-permiability medium*, International Journal of Rock Mechanics and Mining Sciences, **37**, 63–78, 2000.

12. S.R. Pride, J.G. Berryman, *Linear dynamics of double-porosity dual-permeability materials, I*, Physical Review E, **68**, 036603, 2003.

13. M. Bai, D. Elsworth, J.C. Roegiers, *Multiporosity/multipermeability approach to the simulation of naturally fractured reservoirs,* Water Resources Research, **29**, 1621–1633, 1993.

14. K.N. Moutsopoulos, A.A. Konstantinidis, I. Meladiotis, Ch.D. Tzimopoulos, E.C. Aifantis, *Hydraulic behavior and contaminant transport in multiple porosity media*, Transport in Porous Media, **42**, 265–292, 2001.

15. M. Bai, J.C. Roegiers, *Fluid flow and heat flow in deformable fractured porous media*, International Journal of Engineering Science, **32**, 1615–1633, 1994.

16. N. Khalili, A.P.S. Selvadurai, *A fully coupled constitutive model for thermo-hydro-mechanical analysis in elastic media with double porosity*, Geophysical Research Letters, **30**, 2268, 2003.

17. I. Masters, W.K.S. Pao, R.W. Lewis, *Coupling temperature to a double-porosity model of deformable porous media*, International Journal for Numerical Methods in Engineering, **49**, 421–438, 2000.

18. S.C. Cowin, *Bone poroelasticity*, Journal of Biomechanics, **32**, 217–238, 1999.

19. S.C. Cowin, G. Gailani, M. Benalla, *Hierarchical poroelasticity: movement of interstitial fluid between levels in bones*, Philosophical Transactions of the Royal Society A: mathematical, physical and engineering sciences, **367**, 3401–3444, 2009.

20. E. Rohan, S. Naili, R. Cimrman, T. Lemaire, *Multiscale modeling of a fluid saturated medium with double porosity: Relevance to the compact bone*, Journal of the Mechanics and Physics of Solids, **60**, 857–881, 2012.

21. R. de Boer, *Theory of Porous Media: Highlights in the historical development and current state*, Springer, Berlin, Heidelberg, New York, 2000.

22. M.A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low frequency range, II. Higher frequency range*, Journal of the Acoustical Society of America, **28**, 168–178, 179–191, 1956.

23. M.A. Biot, *Mechanics of deformation and acoustic propagation in porous media*, Journal of Applied Physics, **33**, 1482–1498, 1962.

24. M.A. Biot, *Theory of finite deformations of porous solids*, Indiana University Mathematics Journal, **21**, 597– 620, 1972.

25. R. de Boer, M. Svanadze, *Fundamental solution of the system of equations of steady oscillations in the theory of fluid-saturated porous media*, Transport in Porous Media, **56**, 39–50, 2004.

26. M. Ciarletta, A. Scalia, M. Svanadze, *Fundamental solution in the theory of micropolar thermoelasticity for materials with voids*, Journal of Thermal Stresses, **30**, 213–229, 2007.

27. S. De Cicco, M. Svanadze, *Fundamental solution in the theory of viscoelastic mixtures*, Journal of Mechanics of Materials and Structures, **4**, 139–156, 2009.

28. M. Svanadze, *The fundamental solution of the oscillation equations of the thermoelasticity theory of mixture of two elastic solids*, Journal of Thermal Stresses, **19**, 633–648, 1996.

29. M. Svanadze, *Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures*, Journal of Thermal Stresses, **27**, 151–170, 2004.

30. M. Svanadze, *Fundamental solution in the theory of consolidation with double porosity*, Journal of the Mechanical Behavior of Materials, **16**, 123–130, 2005.

31. M. Svanadze, S. De Cicco, *Fundamental solution of the system of equations of steady oscillations in the theory of thermomicrostretch elastic solids*, International Journal of Engineering Science, **43**, 417–431, 2005.

32. M. Svanadze, V. Tibullo, V. Zampoli, *Fundamental solution in the theory of micropolar thermoelasticity without energy dissipation*, Journal of Thermal Stresses, **29**, 57–66, 2006.

33. R. Burridge and C.A. Vargas, *The fundamental solution in dynamic poroelasticity*, Geophysical Journal of the Royal Astronomical Society, **58**, 61–90, 1979.

34. G.D. Manolis, D.E. Beskos, *Integral formulation and fundamental solutions of dynamic poroelasticity and thermoelasticity*, Acta Mechanica, **76**, 89–104, 1989.

35. G.D. Manolis, D.E. Beskos, *Corrections and additions to the paper "Integral formulation and fundamental solutions of dynamic poroelasticity and thermoelasticity"*, Acta Mechanica, **83**, 223–226, 1990.

36. L. Hörmander, *The analysis of linear partial differential operators II: differential operators with constant coefficients*, Springer, Berlin, Heidelberg, New York, Tokyo, 1983.

37. V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili, T.V. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, New York, Oxford, 1979.

38. A. Scalia, M. Svanadze, *On the representations of solutions in the theory of thermoelasticity with microtemperatures*, Journal of Thermal Stresses, **29**, 849–864, 2006.

39. M. Svanadze, R. Tracinà, *Representations of solutions in the theory of thermoelasticity with microtemperatures for microstretch solids*, Journal of Thermal Stresses, **34**, 161–178, 2011.

40. T. Gegelia, L. Jentsch, *Potential methods in continuum mechanics*, Georgian Mathematical Journal, **1**, 599–640, 1994.

41. M. Svanadze, *The boundary value problems of the full coupled theory of poroelasticity for materials with double porosity*, Proceedings in Applied Mathematics and Mechanics, **12**, 279–282, 2012.

42. M. Svanadze, A. Scalia, *Mathematical problems in the theory of bone poroelasticity*, International Journal on Mathematical Methods and Models in Biosciences, **1**, No 2, 1211225, pp. 1-4, 2012.

43. M. Ciarletta, F. Passarella, M. Svanadze, *Plane waves and uniqueness theorems in the coupled linear theory of elasticity for solids with double porosity*, Journal of Elasticity, DOI 10.1007/s10659-012-9426-x (to appear).

44. M. Svanadze, A. Scalia, *Mathematical problems in the coupled linear theory of bone poroelasticity*, Computers and Mathematics with Applications, DOI: 10.1016/j.camwa.2013.01.046 (to appear).

Received December 1, 2012; revised version March 25, 2013.