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Harmonic vibration of cusped plates in the N-th approximation of Vekua's hierarchical models

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In this paper elastic cusped symmetric prismatic shells (i.e., plates of variable thickness with cusped edges) in the N-th approximation of Vekua's hierarchical models are considered. The well-posedness of the boundary value problems (BVPs) under the reasonable boundary conditions at the cusped edge and given displacements at the non-cusped edge is studied in the case of harmonic vibration. The classical and weak setting of the BVPs in the case of the N-th approximation of hierarchical models is considered. Appropriate weighted functional spaces are introduced. Uniqueness and existence results for the variational problem are proved. The structure of the constructed weighted space is described and its connection with weighted Sobolev spaces is established.

Key words: cusped plates, cusped prismatic shells, degenerate elliptic systems, weighted spaces, Hardy's inequality, Korn's weighted inequality.

1. Introduction

In the 1950's I. Vekua recommended to investigate cusped prismatic shells, i.e., plates whose thickness vanishes on some part or on the whole boundary of the shell projection. One can find a survey concerning cusped prismatic shells in [1, 2].

The first proper work concerning classical bending of cusped elastic plates was conducted by Makhover [3] (see also Mikhlin [4]). Problems for cusped plates have also been investigated by Khvoles, Jaiani, Tsiskarishvili, Khomasuridze, Devdariani, Uzunov, Naguleswaran, Kharibegashvili, Natroshvili, Wendland (see, e.g. [1, 2, 5, 10, 11] and references therein).

Vekua's hierarchical models for elastic prismatic shells are mathematical models (see, e.g., [1, 6, 7]). They were based on expansions of the three-dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier–Legendre series with respect to the variable plate thickness. By taking only the first $r+1$ terms of the expansions, he introduced the so-called r-th approximation. Each of these approximations for $r = 0, 1, \dots$ can be considered as an independent mathematical model of plates. For example, $r = 0$ approximation coincides with plane stress, generalized plane stress; and plane deformation $r = 1$ approximation coincides with Kirchoff–Love plate model see [2].

Bending of the cusped plates in Vekua's hierarchical models is considered, e.g., in [8]–[12]. Non-cusped plates are considered, e.g., by Babuška, Gordeziani, Meunargia, Vashakmadze, etc.

There were some unstudied areas connected with the study of vibration problem of cusped plate in case of Vekua's hierarchical models. Recently, in [13] and [14] harmonic vibrations of plates were considered in case of zeroth and first approximations of Vekua's hierarchical models. In [15], Natroshvili and Kharibegashvili studied the well-posedness of an initial-boundary value problem corresponding to the zeroth approximation of Vekua's hierarchical models for elastic cusped prismatic shells.

In this paper elastic cusped symmetric prismatic shells (i.e., plates of variable thickness with cusped edges) in the N-th approximation of Vekua's hierarchical models are considered. The well-posedness of the boundary value problems (BVPs) under the reasonable boundary conditions at the cusped edge and given displacements at the non-cusped edge is studied in the case of harmonic vibration. The approach works also for non-symmetric prismatic shells. The classical and weak setting of the BVPs in the case of the N-th approximation of hierarchical models is considered. Appropriate weighted functional spaces are introduced. Uniqueness and existence results for the variational problem are proved. The structure of the constructed weighted space is described and its connection with weighted Sobolev spaces is established. Moreover, some sufficient conditions for a linear functional arising on the right-hand side of the variational equation to be bounded are given.

2. Fields equations

Let a $3D$ elastic body occupy a bounded region $\overline{\Omega}$ with boundary $\partial\Omega$:

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x := (x_1, x_2) \in \omega,\ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x) \right\},$$

where $\overline{\omega} = \omega \cup \partial\omega$ is the so-called *projection of the plate* $\overline{\Omega} = \Omega \cup \partial\Omega$.

In what follows, we assume that $\overset{(\pm)}{h}(x) \in C^2(\omega) \cap C(\overline{\omega})$ and the thickness is

$$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) > 0 \quad \text{for } x \in \omega$$

and

$$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) \geq 0 \quad \text{for } x \in \partial\omega,$$

i.e., the thickness may vanish on some part of the boundary.

Further, let $\partial\omega$ be a Lipschitz curve and

$$\begin{aligned} \mathit{\Gamma}_1 &:= \Big\{(x, x_3) \in \mathbb{R}^3 : x \in \partial\omega,\ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x)\Big\}, \\ S^{\pm} &:= \Big\{(x,\ \overset{(\pm)}{h}(x)) \in \mathbb{R}^3 : x \in \omega\Big\}; \end{aligned}$$

denote by γ_1 the projection of $\mathit{\Gamma}_1$ onto $\partial\omega$ and let $\gamma_0 := \partial\omega \backslash \overline{\gamma}_1$.

Obviously,

$$\partial\mathit{\Omega} = \bar{\mathit{\Gamma}}_1 \cup \bar{S}^+ \cup \bar{S}^-,$$

where $\bar{\mathit{\Gamma}}_1$ is a cylindrical *lateral surface*, while S^+ and S^- are *upper* and *lower face surfaces* of the shell. Note that, in general, $\partial\mathit{\Omega}$ is not a Lipschitz surface.

If $\bar{S}^+ \cap \bar{S}^- \neq \emptyset$, then a shell is called a *cusped shell* and the set

$$\mathit{\Gamma}_0 := \bar{S}^+ \cap \bar{S}^- = \Big\{(x, x_3) \in \mathbb{R}^3 : x \in \partial\omega,\ x_3 = \overset{(-)}{h}(x) = \overset{(+)}{h}(x)\Big\}$$

will be referred to as a *cusped edge* of a cusped shell.

In what follows, σ_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, $\mathit{\Phi}_i$ are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta. Moreover, repeated indices imply summation, bar under one of the repeated indices means that we do not sum.

By u_{ir}, σ_{ijr}, e_{ijr}, $\mathit{\Phi}_{jr}$ we denote the r-th order moments of the corresponding quantities u_i, σ_{ij}, e_{ij}, $\mathit{\Phi}_j$ as defined below:

$$(u_{ir}, \sigma_{ijr}, e_{ijr}, \mathit{\Phi}_{jr})(x_1, x_2, t) := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} (u_i, \sigma_{ij},\ e_{ij}, \mathit{\Phi}_j)(x_1, x_2, x_3, t) P_r(a(x_1, x_2)x_3 - b(x_1, x_2))\, dx_3, \quad i, j = \overline{1, 3},$$

$$a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \qquad b(x_1, x_2) := \frac{\overset{(+)}{h} + \overset{(-)}{h}}{2h(x_1, x_2)}.$$

Vekua's hierarchical models for elastic prismatic shells are mathematical models (see [6, 16]). Their constructing is based on the multiplication of the basic equations of linear elasticity:

equations of motion

$$\sigma_{ij,i}+\Phi_j=\rho\ddot{u}_j(x_1,x_2,x_3,t),\quad (x_1,x_2,x_3)\in\Omega\subset\mathbb{R}^3,\quad t>t_0,\quad j=\overline{1,3}; \tag{2.1}$$

generalized Hooke's law (isotropic case)

$$\sigma_{ij}=\lambda\theta\delta_{ij}+2\mu e_{ij},\quad i,j=\overline{1,3},\quad \theta:=e_{ii}; \tag{2.2}$$

kinematic relations

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad i,j=\overline{1,3}, \tag{2.3}$$

by Legendre polynomials $P_r(ax_3-b)$ and then integration with respect to x_3 within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$. By constructing Vekua's hierarchical models in Vekua's first version the stress-vectors are assumed to be known on upper and lower face surfaces. From (2.1)–(2.3), we get, respectively,

$$\sigma_{\alpha jr,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{is}\sigma_{ijs}+\Phi_j^{(r)}=\rho\frac{\partial^2u_{jr}}{\partial t^2},\quad j=\overline{1,3},\ \ r=0,1,\dots, \tag{2.4}$$

$$\sigma_{ijr}(x_1,x_2,t)=\lambda\delta_{ij}\theta_r(x_1,x_2,t)+2\mu e_{ijr}(x_1,x_2,t),\quad i,j=\overline{1,3},\quad r=0,1,\dots, \tag{2.5}$$

$$e_{ijr}=\frac{1}{2}\Big(u_{ir,j}+u_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}u_{js}+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}u_{is},\quad i,j=\overline{1,3},\quad r=0,1,\dots, \tag{2.6}$$

where

$$\theta_r:=e_{iir}=u_{\gamma r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}u_{is},\qquad \overset{r}{b}_{\alpha\underline{r}}:=-(r+1)\frac{h_{,\alpha}}{h},\qquad \overset{r}{b}_{3\underline{r}}=0,$$

$$\overset{r}{b}_{js}:=\begin{cases}0, & s<r,\\ -\overset{r}{a}_{\alpha s}=-(2s+1)\dfrac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+\underline{s}}\overset{(-)}{h}_{,\alpha}}{2h}, & j=\alpha,\ \ s>r,\\ (2s+1)\dfrac{1-(-1)^{\underline{s}+r}}{2h}, & j=3,\ \ s>r,\end{cases}$$
$$\alpha=1,2,\ \ j=\overline{1,3},\quad r,s=0,1,2,\dots;$$

$$\Phi_j^{(r)}:=\overset{(+)}{\sigma}_{3j}-\overset{(+)}{\sigma}_{\alpha j}\overset{(+)}{h}_{,\alpha}+(-1)^r\Big[-\overset{(-)}{\sigma}_{3j}+\overset{(-)}{\sigma}_{\alpha j}\overset{(-)}{h}_{,\alpha}\Big]+\Phi_{jr}$$
$$=Q_{\overset{(+)}{n}j}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2}+(-1)^rQ_{\overset{(-)}{n}j}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2}+\Phi_{jr},$$
$$j=\overline{1,3},\quad r=0,1,2,\dots;$$

$Q_{\overset{(+)}{n}j}$ and $Q_{\overset{(-)}{n}j}$ are components of the stress vectors acting on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively. Thus, we get the infinite system (2.4)–(2.6) with respect to the so-called r-th order moments σ_{ijr}, e_{ijr}, u_{ir}. This system is equivalent to (2.1)–(2.3). Then, substituting (2.6) into (2.5) and then this result into (2.4), we construct an equivalent infinite system with respect to the r-th order moments u_{ir} ([6]). After this, we assume that the moments, for which the subscripts indicating order are greater than N, are equal to zero. Considering only the first $N+1$ equations ($r=\overline{0,N}$) in the obtained infinite system of equations with respect to the r-th order moments u_{ir}, we obtain the N-th order approximation (hierarchical model) governing system consisting of $3N+3$ equations with respect to $3N+3$ unknown functions $\overset{N}{u}_{ir}$, $i=\overline{1,3}$, $r=\overline{0,N}$ (roughly speaking $\overset{N}{u}_{ir}$ is an "approximate value" of u_{ir}, since $\overset{N}{u}_{ir}$ are solutions of the derived finite system). The N-th approximation of the system reads as (see [7], [17])

$$
\begin{aligned}
(2.7)\qquad & \mu\left[\left(h^{2r+1}\overset{N}{v}_{\alpha r,j}\right)_{,\alpha}+\left(h^{2r+1}\overset{N}{v}_{jr,\alpha}\right)_{,\alpha}\right]\\
&+\lambda\delta_{\alpha j}\left(h^{2r+1}\overset{N}{v}_{\gamma r,\gamma}\right)_{,\alpha}+\sum_{s=r+1}^{N}\left(\overset{r}{B}_{\alpha jks}\,h^{r+s+1}\overset{N}{v}_{ks}\right)_{,\alpha}\\
&+\sum_{l=0}^{r-1}\overset{r}{a}_{il}\left[\lambda\delta_{ij}h^{r+l+1}\overset{N}{v}_{\gamma l,\gamma}+\mu h^{r+l+1}\left(\overset{N}{v}_{il,j}+\overset{N}{v}_{jl,i}\right)+\sum_{s=l+1}^{N}\overset{l}{B}_{ijks}\,h^{r+s+1}\overset{N}{v}_{ks}\right]+h^{r}\overset{N}{\Phi}{}^{r}_{j}\\
&=\rho h^{r}\frac{\partial^{2}h^{r+1}\overset{N}{v}_{jr}}{\partial t^{2}},\quad r=\overline{0,N},\quad j=\overline{1,3},\qquad \sum_{q}^{q-1}(\cdots)\equiv 0,
\end{aligned}
$$

where

$$\overset{N}{v}_{kr}:=\frac{\overset{N}{u}_{kr}}{h^{r+1}},\qquad k=\overline{1,3},\ r=\overline{0,N}$$

are unknown so-called weighted "moments" of displacements (in what follows, we omit superscripts like N when it does not lead to misunderstanding). Having $\overset{N}{u}_{kr}$, we can calculate $\overset{N}{e}_{ijr}$ and $\overset{N}{\sigma}_{ijr}$ by means of (2.6), (2.5).

3. Harmonic vibration of the cusped plate

We will consider the case of harmonic vibration, i.e.,

$$
\begin{aligned}
v_{ir}(x,t) &:= e^{-\imath\nu t}\overset{\circ}{v}_{ir}(x), & \Phi_{i}^{(r)}(x,t) &:= e^{-\imath\nu t}\overset{\circ}{\Phi}{}_{i}^{(r)}(x),\\
\nu &= \text{const}>0, & i &= \overline{1,3},\ r=0,1.
\end{aligned}
$$

For $\overset{\circ}{v}_{ir}(x)$, taking into account (2.7), we get the following system:

$$
\begin{aligned}
(3.1)\qquad & -(2r+1)\Big\{\rho\nu^2 h^{2r+1}\overset{\circ}{v}_{jr} + \mu\Big[\Big(h^{2r+1}\overset{\circ}{v}_{\alpha r,j}\Big)_{,\alpha} + \Big(h^{2r+1}\overset{\circ}{v}_{jr,\alpha}\Big)_{,\alpha}\Big] \\
& + \lambda\delta_{\alpha j}\Big(h^{2r+1}\overset{\circ}{v}_{\gamma r,\gamma}\Big)_{,\alpha} + \sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha jks}\, h^{r+s+1}\,\overset{0}{v}_{ks}\Big)_{,\alpha} \\
& + \sum_{l=0}^{r-1}\overset{r}{a}_{il}\Big[\lambda\delta_{ij}h^{r+l+1}\overset{\circ}{v}_{\gamma l,\gamma} + \mu h^{r+l+1}\Big(\overset{\circ}{v}_{il,j} + \overset{\circ}{v}_{jl,i}\Big) + \sum_{s=l+1}^{N}\overset{l}{B}_{ijks}\, h^{r+s+1}\overset{\circ}{v}_{ks}\Big]\Big\} \\
& = (2r+1)h^r\overset{\circ}{\Phi}{}_j^{(r)}, \quad r=\overline{0,N}, \quad j=\overline{1,3}, \qquad \sum_{q}^{q-1}(\cdots)\equiv 0.
\end{aligned}
$$

Denoting by $L^{(N)}(x,\partial)$ the $(3N+3)\times(3N+3)$ matrix differential operator generated by the left-hand side expressions of system (3.1), we can rewrite (3.1) in the following vector form:

$$
(3.2)\qquad L^{(N)}(x,\partial)v(x) = F(x), \qquad x\in\omega,
$$

where

$$
\begin{aligned}
v &:= (\overset{\circ}{v}_{10}, \overset{\circ}{v}_{20}, \overset{\circ}{v}_{30}, \dots, \overset{\circ}{v}_{1N}, \overset{\circ}{v}_{2N}, \overset{\circ}{v}_{3N})^\top, \\
F &:= (\overset{\circ}{\Phi}{}_1^{(0)}, \overset{\circ}{\Phi}{}_2^{(0)}, \overset{\circ}{\Phi}{}_3^{(0)}, 3h\overset{\circ}{\Phi}{}_1^{(1)}, 3h\overset{\circ}{\Phi}{}_2^{(1)}, 3h\overset{\circ}{\Phi}{}_3^{(1)}, \dots, \\
&\qquad (2N+1)h^N\overset{\circ}{\Phi}{}_1^{(N)}, (2N+1)h^N\overset{\circ}{\Phi}{}_2^{(N)}, (2N+1)h^N\overset{\circ}{\Phi}{}_3^{(N)}),
\end{aligned}
$$

the symbol $(\cdot)^\top$ means transposition.

Let

$$
v, v^* \in c^2(\omega)\cap c^1(\overline{\omega}), \qquad v^* := (\overset{\circ}{v}{}_{10}^*, \overset{\circ}{v}{}_{20}^*, \overset{\circ}{v}{}_{30}^*, \dots, \overset{\circ}{v}{}_{1N}^*, \overset{\circ}{v}{}_{2N}^*, \overset{\circ}{v}{}_{3N}^*)^\top,
$$

where v and v^* are arbitrary vectors of the above class. After multiplication (3.2) by v^* and integration by parts we obtain the following Green's formula:

$$
(3.3)\qquad \int_\omega L^{(N)}v\cdot v^* d\omega = B^{(N)}(v,v^*) - \int_{\partial\omega} T_n v\cdot v^* d\partial\omega = \int_\omega F\cdot v^* d\omega.
$$

Here and in what follows, the $\cdot$ denotes the scalar product of two vectors, $n := (n_1, n_2)$ is the inward normal to $\partial\omega$,

$$B^{(N)}(v,v^*) := \int\limits_{\omega} \sum_{r=0}^{N}(2r+1)\Big\{\rho\nu^2 h^{2r+1}\overset{\circ}{v}_{jr}\overset{\circ}{v}{}^*_{jr} \tag{3.4}$$

$$+ h^{2r+1}(\overset{\circ}{v}_{\alpha r,\beta}\overset{\circ}{v}{}^*_{\beta r,\alpha} + \mu\overset{\circ}{v}_{jr,\alpha}\overset{\circ}{v}{}^*_{jr,\alpha} + \lambda\overset{\circ}{v}_{\alpha r,\alpha}\overset{\circ}{v}{}^*_{\beta r,\beta} - \sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha jks}\, h^{r+s+1}\,\overset{\circ}{v}_{ks}\Big)_{,\alpha}\overset{\circ}{v}{}^*_{jr}$$

$$-\sum_{s=0}^{r+1}\overset{r}{a}_{is}\Big[\lambda\delta_{ij}h^{r+s+1}\overset{\circ}{v}_{\alpha s,\alpha} + \mu h^{r+s+1}\Big(\overset{\circ}{v}_{is,j}+\overset{\circ}{v}_{js,i}\Big) + \sum_{l=s+1}^{N}\overset{s}{B}_{ijkl}\,h^{r+l+1}\overset{\circ}{v}_{kl}\Big]\overset{\circ}{v}{}^*_{jr}\Big\}d\omega,$$

$$T_n := \{\sigma_{n10},\sigma_{n20},\sigma_{n30},3h\sigma_{n11},3h\sigma_{n21},3h\sigma_{n31},\ldots,$$
$$(2N+1)h^N\sigma_{n1N},(2N+1)h^N\sigma_{n2N},(2N+1)h^N\sigma_{n3N}\},$$

with

$$\sigma_{nir} = \sigma_{ijr}n_j = \{\lambda\delta_{ij}(\sum_{s=r}^{N}\overset{r}{b}_{ks}\overset{\circ}{v}_{ks} + \overset{\circ}{v}_{kr,k})\}n_j$$
$$+\Big\{\mu\Big[\sum_{s=r}^{1}h^{s+1}(\overset{r}{b}_{ir}\overset{\circ}{v}_{js} + \overset{r}{b}_{js}\overset{\circ}{v}_{is}) + \overset{\circ}{v}_{ir,j} + \overset{\circ}{v}_{jr,i}\Big]\Big\}n_j, \qquad i=1,2,3, \quad r=0,1,$$

where σ_{nir}, $i=1,2,3$, denote the rth moments of the corresponding components of the 3D stresses σ_{ni}, $i=1,2,3$.

From now on, throughout the paper we assume that the plate is symmetric, i.e.,

$$\overset{(-)}{h} = -\overset{(+)}{h}, \quad 2h = h_0 x_2^{\kappa}, \quad h_0 = \text{const} > 0, \quad \kappa = \text{const} \geq 0, \quad x_2 \geq 0.$$

If we consider BVPs for system (3.2) with homogeneous boundary conditions for which the curvilinear integral along $\partial\omega$ in (3.3) disappears, we arrive at the equation

$$B^{(N)}(v,v^*) = \int\limits_{\omega} F\cdot v^* d\omega.$$

Let us consider the following Dirichlet problem in the classical setting: find a $3N+3$-dimensional vector

$$v = (\overset{\circ}{v}_{10},\overset{\circ}{v}_{20},\overset{\circ}{v}_{30},\ldots,\overset{\circ}{v}_{1N},\overset{\circ}{v}_{2N},\overset{\circ}{v}_{3N})^\top,$$

in ω satisfying the system of differential equations (3.2) in ω and the homogeneous Dirichlet boundary condition on

$$[v(x)]^+ = 0, \qquad x \in \partial\omega. \tag{3.5}$$

Note that throughout the paper, for smooth classical solutions, equation (3.2) and boundary condition (3.5) are understood in the classical pointwise sense, while for generalized weak solutions of equation (3.2) is understood in the distributional sense and boundary condition (3.5), understood in the usual trace sense. To derive the weak setting of the above problem, we have to apply Green‘s formulas (3.3). We arrive at the variational equation:

$$B^{(N)}(v,v^*) = \langle F, v^*\rangle, \tag{3.6}$$

where the bilinear form $B^{(N)}(v,v^*)$ is defined by (3.4) and

$$\langle F, v^*\rangle = \int_\omega \sum_{r=0}^{N} (2r+1) h^r \overset{\circ}{\Phi}{}_j^{(r)} \overset{\circ}{v}{}^*_{jr} d\omega. \tag{3.7}$$

Note that the bilinear form (3.4) can be represented as follows:

$$\begin{aligned} B^{(N)}(v,v^*) :=& \int_\omega \sum_{r=0}^{N} (2r+1)\rho\nu^2 h^{2r+1} \overset{\circ}{v}_{jr} \overset{\circ}{v}{}^*_{jr} d\omega \\ &+ \int_\omega \sum_{r=0}^{N} \left(r+\frac{1}{2}\right) a[\lambda\delta_{ij} e_{kkr}(v) e_{ijr}(v^*) + 2\mu e_{ijr}(v) e_{ijr}(v^*)] d\omega \\ =& \int_\omega \sum_{r=0}^{N} (2r+1)\rho\nu^2 h^{2r+1} \overset{\circ}{v}_{jr} \overset{\circ}{v}{}^*_{jr} d\omega \\ &+ \sum_{r=0}^{N} \left(r+\frac{1}{2}\right) \int_\omega a[\lambda\, e_{kkr}(v) e_{iir}(v^*) + 2\mu e_{ijr}(v) e_{ijr}(v^*)]\, d\omega, \end{aligned} \tag{3.8}$$

where e_{ijr} and σ_{ijr} are given by (2.6) and (2.5).

Further, we construct the vectors in $\Omega := \{(x,x_3) : x\in\omega,\ -h(x)<x_3<h(x)\}$:

$$w_i(x,x_3) = \sum_{r=0}^{N} \left(r+\frac{1}{2}\right) h^r \overset{\circ}{v}_{ir}(x)\, P_r(ax_3), \qquad i=1,2,3, \tag{3.9}$$

$$w_i^*(x,x_3) = \sum_{r=0}^{N} \left(r+\frac{1}{2}\right) h^r \overset{\circ}{v}{}^*_{jr}(x)\, P_r(ax_3), \qquad i=1,2,3. \tag{3.10}$$

It can be shown that

$$B(w,w^*) := \int_\Omega (2\rho\nu^2 w_i w_i^* + \sigma_{ij}(w)\, e_{ij}(w^*))\, d\Omega = B^{(N)}(v,v^*), \tag{3.11}$$

where $w(x,x_3) := (w_1,w_2,w_3)$ and $w^*(x,x_3) := (w_1^*,w_2^*,w_3^*)$ are vectors. Indeed,

$$
\begin{aligned}
&B(w,w^*) := \int\limits_{\Omega} (2\rho\nu^2 w_i w_i^* + \sigma_{ij}(w)\, e_{ij}(w^*))\, d\Omega \\
&= \int\limits_{\omega} 2\rho\nu^2 d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sum_{r=0}^{N}\sum_{s=0}^{N}\left(r+\frac{1}{2}\right)\left(s+\frac{1}{2}\right) h^{r+1}h^{s+1}a^2 \overset{\circ}{v}{}^*_{i\underline{r}} \overset{\circ}{v}{}^*_{j\underline{s}} P_r(ax_3)P_s(ax_3)dx_3 \\
&+ \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sum_{r=0}^{N}\sum_{s=0}^{N}\left(r+\frac{1}{2}\right)\left(s+\frac{1}{2}\right) a^2\sigma_{ij\underline{r}}(v)e_{ij\underline{s}}(v^*)P_r(ax_3)P_s(ax_3)dx_3 \\
&= \int\limits_{\omega} \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\Big(2\rho\nu^2 h^{2r+2} a \overset{\circ}{v}{}^*_{i\underline{r}} \overset{\circ}{v}{}^*_{j\underline{r}} + a\sigma_{ij\underline{r}}(v)e_{ij\underline{r}}(v^*)\Big)\, d\omega \\
&= \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int\limits_{\omega}\Big(\rho\nu^2 h^{2r+1} \overset{\circ}{v}{}^*_{i\underline{r}} \overset{\circ}{v}{}^*_{j\underline{r}} + a[\lambda e_{kk\underline{r}}(v)e_{ii\underline{r}}(v^*) + 2\mu e_{ij\underline{r}}(v)e_{ij\underline{r}}(v^*)]\Big)\, d\omega \\
&= B^{(N)}(v,v^*),
\end{aligned}
$$

$B(w,w^*)$ is bilinear form corresponding to the $3D$ potential energy for the displacement vector w.

Due to positive definiteness of the potential energy for $2\lambda+3\mu>0$ and $\mu>0$ we get (see, e.g., [11], proof of Theorem 3.8; [18], proof of Theorem 3.8)

$$
\begin{aligned}
B(w,w) &\geq 2\rho\nu^2 \sum_{i=1}^{3}\int\limits_{\Omega} w_i^2 d\Omega + C_1 \sum_{i,j=1}^{3}\int\limits_{\Omega}[e_{ij}(w)]^2 d\Omega \\
&= 2\rho\nu^2 \sum_{i=1}^{3}\int\limits_{\Omega} w_i^2 d\Omega + C_1 \int\limits_{\omega} d\omega \int\limits_{-h}^{h}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a e_{ijr}(v)\, P_r(ax_3) \\
&\quad \times \sum_{s=0}^{N}\left(s+\frac{1}{2}\right) a\, e_{ijs}(v)\, P_s(ax_3)\, dx_3 \\
&= \rho\nu^2 \int\limits_{\omega}\sum_{i=1}^{3}\sum_{r=0}^{N}(2r+1)h^{2r+1}\overset{\circ}{v}{}^2_{ir} d\omega \\
&\quad + C_1 \int\limits_{\omega}\sum_{i,j=1}^{3}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right) e^2_{ijr}(v)\,\frac{d\omega}{h},
\end{aligned}
\tag{3.12}
$$

where the positive constant C_1 depends only on the material parameters λ and μ. Here we applied the orthogonality of the Legendre polynomials

$$\int_{-h}^{h} P_k(ax_3)\, P_l(ax_3) a\, dx_3 = \frac{2\delta_{kl}}{2k+1}, \qquad k,l = \overline{0,\infty}.$$

After denoting by $C_0 := \min\{1, C_1\}$, (3.11) and (3.12) imply

$$\text{(3.13)} \qquad B^{(N)}(v,v) \geq C_0 \int_\omega \left\{ \sum_{i=1}^{3}\sum_{r=0}^{N}(2r+1)h^{2r+1}\overset{\circ}{v}{}_{ir}^{2} + \sum_{i,j=1}^{3}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)e_{ijr}^2(v)\right\}\frac{d\omega}{h}.$$

REMARK 3.1. In view of relations (3.11)–(3.13) we conclude that $B^{(N)}(v,v) = 0$ yields $v=0$. Indeed, if $B^{(N)}(v,v)=0$ then $B(w,w)=0$ by (3.11). In turn, the latter equality for the strain tensor e_{ij} corresponding to the displacement vector w implies that $e_{ij}(w)=0$, $i,j=1,2,3$, i.e., w is a rigid displacement. Since w vanishes on the part of the lateral boundary Γ_1 of Ω (which contains at least three points not belonging to a straight line) it follows that $w(x,x_3)=0$ in Ω. Therefore, $\overset{\circ}{v}_{ir}(x)=0$, due to formulas (3.9) and (3.13).

Denote by $\mathcal{D}(\omega)$ the space of infinitely differentiable functions with compact support in ω and introduce the linear form in $[\mathcal{D}(\omega)]^{3N+3}$ by the formula

$$\begin{aligned}
\text{(3.14)} \quad (v,v^*)_{X^k_{N,\nu}} &= \sum_{r=0}^{N}(2r+1)\int_\omega h^r\rho\nu^2\overset{\circ}{v}_{ir}\overset{\circ}{v}{}^*_{ir}\,d\omega \\
&\quad + \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_\omega e_{ijr}(v)e_{ijr}(v^*)\frac{d\omega}{h} \\
&= \sum_{r=0}^{N}(2r+1)\int_\omega h^r\rho\nu^2\overset{\circ}{v}_{ir}\overset{\circ}{v}{}^*_{ir}\,d\omega \\
&\quad + \frac{1}{4}\sum_{i,j=1}^{3}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_\omega \Big[h^{r+1}(\overset{\circ}{v}_{ir,j}+\overset{\circ}{v}_{jr,i}) \\
&\quad + \sum_{s=r+1}^{N} h^{s+1}(b^r_{js}\overset{\circ}{v}_{is}+b^r_{is}\overset{\circ}{v}_{js})\Big] \\
&\quad \times \Big[h^{r+1}(\overset{\circ}{v}{}^*_{ir,j}+\overset{\circ}{v}{}^*_{jr,i}) + \sum_{s=r+1}^{N} h^{s+1}(b^r_{js}\overset{\circ}{v}{}^*_{is}+b^r_{is}\overset{\circ}{v}{}^*_{js})\Big]\frac{d\omega}{h}.
\end{aligned}$$

Owing to Remark 3.1 it is easy to verify that (3.14) is an inner product defined on the set of vector-functions $[\mathcal{D}(\omega)]^{3N+3}$.

Denote by $X^{\kappa}_{N,\nu} := X^{\kappa}_{N,\nu}(\omega)$ the completion of the space $[\mathcal{D}(\omega)]^{3N+3}$ with respect to the norm

$$
\begin{aligned}
(3.15)\quad \|v\|^2_{X^{\kappa}_{N,\nu}} &= (v,v)_{X^{\kappa}_{N,\nu}} \\
&= \sum_{r=0}^{N}(2r+1)\rho\nu^2\int\limits_{\omega} h^{2r+1}\sum_{i=1}^{3}\overset{\circ}{v}{}^2_{ir}d\omega \\
&\quad + \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int\limits_{\omega}\sum_{i,j=1}^{3} e^2_{ijr}(v)\frac{d\omega}{h} \\
&= \sum_{r=0}^{N}(2r+1)\rho\nu^2\int\limits_{\omega} h^{2r+1}\sum_{i=1}^{3}\overset{\circ}{v}{}^2_{ir}d\omega \\
&\quad + \frac{1}{4}\sum_{r=0}^{N}\sum_{i,j=1}^{3}\left(r+\frac{1}{2}\right) \\
&\quad \times \int\limits_{\omega}\Big[h^{r+1}(\overset{\circ}{v}_{ir,j}+\overset{\circ}{v}_{jr,i}) + \sum_{s=r+1}^{N} h^{s+1}(b^r_{js}\overset{\circ}{v}_{is}+b^r_{is}\overset{\circ}{v}_{js})\Big]^2\frac{d\omega}{h}
\end{aligned}
$$

corresponding to the inner product (3.14). It is evident that $X^{\kappa}_{N,\nu}$ is a Hilbert space.

Now, we can formulate the weak setting of the homogeneous Dirichlet problem (3.2), (3.5): find a vector $v = (\overset{\circ}{v}_{10},\overset{\circ}{v}_{20},\overset{\circ}{v}_{30},\dots,\overset{\circ}{v}_{1N},\overset{\circ}{v}_{2N},\overset{\circ}{v}_{3N})^{\top} \in X^{\kappa}_{N,\nu}$ satisfying the equality

$$
(3.16)\qquad B^{(N)}(v,v^*) = \langle F, v^*\rangle \qquad \text{for all } v^* \in X^{\kappa}_{N,\nu}.
$$

Here, the vector F belongs to the adjoint space $[X^{\kappa}_{N,\nu}]^*$, in general, and $\langle\cdot,\cdot\rangle$ denotes duality brackets between the spaces $[X^{\kappa}_{N,\nu}]^*$ and $X^{\kappa}_{N,\nu}$.

Now, we have to establish well-posedness of the weak formulation to (3.16) for the Dirichlet homogeneous problem. We start with the following proposition which is very essential for our further analysis.

LEMMA 3.2. *The bilinear form $B^{(N)}(\cdot,\cdot)$ is bounded and strictly coercive in the space $X^{\kappa}_{N,\nu}(\omega)$, i.e., there are positive constants C_0 and C_1 such that*

$$
(3.17)\qquad |B^{(N)}(v,v^*)| \le C_1\|v\|_{X^{\kappa}_{N,\nu}}\|v^*\|_{X^{\kappa}_{N,\nu}},
$$

$$
(3.18)\qquad B^{(N)}(v,v) \ge C_0\|v\|^2_{X^{\kappa}_{N,\nu}},
$$

for all $v, v^* \in X^{\kappa}_{N,\nu}$.

P r o o f. Since $[\mathcal{D}(\omega)]^{3N+3}$ is dense in $X^{\kappa}_{N,\nu}$ it suffices to show the inequalities (3.17) and (3.18) for $v, v^* \in [\mathcal{D}(\omega)]^{3N+3}$. By the $3N+3$ dimensional vectors v and v^* defined in ω, we construct the three-dimensional vectors (3.9) and (3.10) defined in Ω. Due to (3.11) and (3.12), and Hooke's law (2.2) we have

$$\begin{aligned}
|B^{(N)}(v,v^*)|^2 &= |B(w,w^*)|^2 \\
&= \Big[\int_{\Omega} 2\rho\nu^2 w_i w_i^* d\Omega + \int_{\Omega} \big(2\mu e_{ij}(w) + \lambda\delta_{ij}e_{kk}(w)\big)e_{ij}(w^*)\, d\Omega\Big]^2 \\
&\le \int_{\Omega} 2\rho\nu^2 w_i^2 d\Omega \int_{\Omega} 2\rho\nu^2 w_i^{*2} d\Omega + C_2 \sum_{i,j=1}^{3} \int_{\Omega} e_{ij}^2(w) d\Omega \sum_{i,j=1}^{3} \int_{\Omega} e_{ij}^2(w^*) d\Omega \\
&= \int_{\omega} \sum_{i=1}^{3}\sum_{r=0}^{N} 2(2r+1)h^{2r+1}\rho\nu^2 \overset{\circ}{v}{}^2_{ir} d\omega \int_{\omega} \sum_{i=1}^{3}\sum_{r=0}^{N} 2(2r+1)h^{2r+1}\rho\nu^2 (\overset{\circ}{v}{}^*_{ir})^2 d\omega \\
&\quad + C_2 \int_{\omega} \sum_{i,j=1}^{3}\sum_{r=0}^{N} \Big(r+\frac{1}{2}\Big) e_{ijr}^2(v) \frac{d\omega}{h} \int_{\omega} \sum_{i,j=1}^{3}\sum_{r=0}^{N} \Big(r+\frac{1}{2}\Big) e_{ijr}^2(v^*) \,\frac{d\omega}{h} \\
&= C_3 \|v\|^2_{X^{\kappa}_{N,\nu}} \|v^*\|^2_{X^{\kappa}_{N,\nu}},
\end{aligned}$$

where

$$C_3 := \max\{2; C_2\}.$$

Whence (3.17) follows. The inequality (3.18) immediately follows from (3.11) and (3.12). □

Now, we can formulate the following existence and uniqueness results.

THEOREM 3.3. *Let* $F \in [X^{\kappa}_{N,\nu}]^*$. *Then the variational problem* (3.16) *has a unique solution* $v \in X^{\kappa}_{N,\nu}$ *for arbitrary value of the parameter* κ *and*

$$\|v\|_{X^{\kappa}_{N,\nu}} \le \frac{1}{C_0} \|F\|_{[X^{\kappa}_{N,\nu}]^*}.$$

P r o o f. It is direct consequence of Lemma 3.2 and the Lax–Milgram theorem. □

It can be easily shown that if $F \in [L(\omega)]^{3N+3}$ and $\operatorname{supp} F \cap \overline{\gamma}_0 = \emptyset$, then $F \in [X^{\kappa}_{N,\nu}]^*$ and

$$\langle F, v^* \rangle = \int_{\omega} F(x) v^*(x)\, d\omega,$$

since $v^* \in [H^1(\omega_\varepsilon)]^{3N+3}$, where ε is sufficiently small positive number such that

$\operatorname{supp} F \subset \omega_\varepsilon = \omega \cap \{x_2 > \varepsilon\}$. Therefore,

$$|\langle F, v^* \rangle| = \left| \int\limits_\omega F(x) v^*(x)\, d\omega \right| \le \|F\|_{[L_2(\omega)]^{3N+3}} \|v^*\|_{[L_2(\omega_\varepsilon)]^{3N+3}}$$
$$\le \|F\|_{[L_2(\omega)]^{3N+3}} \|v^*\|_{[H^1(\omega_\varepsilon)]^{3N+3}} \le C_\varepsilon \|F\|_{[L_2(\omega)]^{3N+3}} \|v^*\|_{X_N^\kappa}.$$

In this case we obtain the estimate

$$\|v\|_{X_{N,\nu}^\kappa} \le \frac{C_\varepsilon}{C_0} \|F\|_{[L_2(\omega)]^{3N+3}}.$$

Now we establish a representation of the space $X_{N,\nu}^\kappa$ as a weighted Sobolev space. To this end, we introduce the following space:

$$Y_N^\kappa := [\overset{\circ}{W}{}^1_{2,\kappa}(\omega)]^3 \times [\overset{\circ}{W}{}^1_{2,3\kappa}(\omega)]^3 \times \cdots \times [\overset{\circ}{W}{}^1_{2,(2N+1)\kappa}(\omega)]^3, \tag{3.19}$$

where $\overset{\circ}{W}{}^1_{2,\nu}(\omega)$ is a completion of $\mathcal{D}(\omega)$ by means of the norm

$$\|f\|^2_{\overset{\circ}{W}{}^1_{2,\nu}(\omega)} := \int\limits_\omega x_2^\nu |\nabla f(x)|^2\, d\omega, \qquad \nabla f = (f_{,1}, f_{,2}). \tag{3.20}$$

The norm in the space Y_N^κ for a vector $(\overset{\circ}{v}_{10}, \overset{\circ}{v}_{20}, \overset{\circ}{v}_{30}, \cdots, \overset{\circ}{v}_{1N}, \overset{\circ}{v}_{2N}, \overset{\circ}{v}_{3N})$ reads as

$$\|v\|^2_{Y_N^\kappa} := \sum_{r=0}^N \int\limits_\omega x_2^{(2r+1)\kappa} \sum_{j=1}^3 |\nabla \overset{\circ}{v}_{jr}|^2\, d\omega. \tag{3.21}$$

THEOREM 3.4. *Let*

$$\begin{aligned} &\kappa < 1, \qquad \kappa \ne \frac{1}{2r+1}, \qquad \nu^2 \le \frac{1}{h_2 \rho l^2}, \\ &h_2 := \max\{(2r+1)(h_0/2)^{2r+1}\}, \qquad r = \overline{1, N}. \end{aligned} \tag{3.22}$$

Then the linear spaces $X_{N,\nu}^\kappa$ and Y_N^κ as sets of vector functions coincide and the norms $\|\cdot\|_{X_{N,\nu}^\kappa}$ and $\|\cdot\|_{Y_N^\kappa}$ are equivalent.

P r o o f. Rewrite formula (3.15) in the form

$$\|v\|^2_{X_{N,\nu}^\kappa} = \sum_{r=0}^N (2r+1)\rho\nu^2 \int\limits_\omega h_1^{2r+1} x_2^{(2r+1)\kappa} \sum_{i=1}^3 \overset{\circ}{v}{}^2_{ir}\, d\omega$$
$$+ \frac{1}{4} \sum_{r=0}^N \Big(r + \frac{1}{2}\Big) \int\limits_\omega \Big\{ h_1^{2r+1} x_2^{(2r+1)\kappa} 4 \overset{\circ}{v}{}^2_{1r,1}$$

$$+ 2h_1^{2r+1}x_2^{(2r+1)\kappa}\Big[\overset{\circ}{v}_{1r,2}+\overset{\circ}{v}_{2r,1}+\sum_{s=r+1}^{N} h_1^{s-r}\, x_2^{(s-r)\kappa-1}\frac{(2s+1)\kappa}{2}((-1)^{s+r+1}-1)\overset{\circ}{v}_{1s}\Big]^2$$

$$+ h_1^{2r+1}x_2^{(2r+1)\kappa}\Big[2\overset{\circ}{v}_{2r,2} + \sum_{s=r+1}^{N} h_1^{s-r}x_2^{(s-r)\kappa-1}(2s+1)\kappa((-1)^{s+r+1}-1)\overset{\circ}{v}_{2s}\Big]^2$$

$$+ 2h_1^{2r+1}x_2^{(2r+1)\kappa}\Big[\overset{\circ}{v}_{3r,1}+ \sum_{s=r+1}^{N} h_1^{s-r-1}x_2^{(s-r-1)\kappa}\frac{(2s+1)}{2}(1-(-1)^{s+r})\overset{\circ}{v}_{1s}\Big]^2$$

$$+ 2h_1^{2r+1}x_2^{(2r+1)\kappa}\Big[\overset{\circ}{v}_{3r,2} + \sum_{s=r+1}^{N} h_1^{s-r-1}x_2^{(s-r-1)\kappa}\frac{2s+1}{2}(1-(-1)^{s+r})\overset{\circ}{v}_{2s}$$

$$+ \sum_{s=r+1}^{N} h_1^{s-r}x_2^{(s-r)\kappa-1}\frac{(2s+1)\kappa}{2}((-1)^{s+r+1}-1)\,\overset{\circ}{v}_{3s}\Big]^2$$

$$+ \frac{1}{h_1\, x_2^{\kappa}}\Big[\sum_{s=r+1}^{N} h_1^s x_2^{s\kappa} 2s+1(1-(-1)^{s+r})\overset{\circ}{v}_{3s}\Big]^2\Big\}d\omega =: I^1 + I^2,$$

where $h_1 = h_0/2 > 0$,

$$I^1 := \sum_{r=0}^{N}(2r+1)\rho\nu^2\int_{\omega} h_1^{2r+1}x_2^{(2r+1)\kappa}\sum_{i=1}^{3}\overset{\circ}{v}{}_{ir}^2\, d\omega.$$

Let $v \in X_{N,\nu}^{\kappa}$ and show that $v \in Y_N^{\kappa}$. We have to prove that

$$\|v\|^2_{Y_N^{\kappa}} \le C^{(N)}\|v\|^2_{X_{N,\nu}^{\kappa}}. \tag{3.23}$$

Denote by $X_N^{\kappa} := X_N^{\kappa}(\omega)$ the completion of the space of infinity differentiable functions with compact support in ω, and with the help of the norm:

$$\begin{aligned}\|v\|^2_{X_N^{\kappa}} &= (v,v)_{X_N^{\kappa}} \\ &= \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_{\omega}\sum_{i,j=1}^{3} e^2_{ijr}(v)\frac{d\omega}{h} \\ &= \frac{1}{4}\sum_{r=0}^{N}\sum_{i,j=1}^{3}\left(r+\frac{1}{2}\right) \\ &\quad\times\int_{\omega}\Big[h^{r+1}(\overset{\circ}{v}_{ir,j}+\overset{\circ}{v}_{jr,i}) + \sum_{s=r+1}^{N} h^{s+1}(b^r_{js}\overset{\circ}{v}_{is}+b^r_{is}\overset{\circ}{v}_{js})\Big]^2\frac{d\omega}{h} = I^2\end{aligned} \tag{3.24}$$

corresponding to the inner product

$$(v,v^*)_{X_N^\kappa} = \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int\limits_\omega e_{ijr}(v)e_{ijr}(v^*)\,\frac{d\omega}{h}$$
$$= \frac{1}{4}\sum_{i,j=1}^{3}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int\limits_\omega\Big[h^{r+1}(\overset{\circ}{v}_{ir,j}+\overset{\circ}{v}_{jr,i})+\sum_{s=r+1}^{N}h^{s+1}(b_{js}^r\overset{\circ}{v}_{is}+b_{is}^r\overset{\circ}{v}_{js})\Big]$$
$$\times\Big[h^{r+1}(\overset{\circ}{v}{}^*_{ir,j}+\overset{\circ}{v}{}^*_{jr,i})+\sum_{s=r+1}^{N}h^{s+1}(b_{js}^r\overset{\circ}{v}{}^*_{is}+b_{is}^r\overset{\circ}{v}{}^*_{js})\Big]\frac{d\omega}{h}.$$

X_N^κ is a Hilbert space (see [9]).

Evidently,

$$\|v\|^2_{X_N^\kappa}\le C_9\|v\|^2_{X_{N,\nu}^\kappa}. \tag{3.25}$$

Now we will prove

$$\|v\|^2_{Y_N^\kappa}\le C_{10}\|v\|^2_{X_N^\kappa}.$$

We prove this inequality in several steps.

Step 1. Denote by I_r the summand corresponding to r in the norm expression (3.24):

$$I_r := \frac{1}{4}\sum_{i,j=1}^{3}\left(r+\frac{1}{2}\right)\int\limits_\omega\Big[h^{r+1}(v_{ir,j}+v_{jr,i})+\sum_{s=r+1}^{N}h_1^{s+1}(b_{js}^r v_{is}+b_{is}^r v_{js})\Big]^2\frac{d\omega}{h}.$$

Therefore, the summand corresponding to $r=N$ in (3.24) reads as

$$I_N=\frac{2N+1}{8}\times\int\limits_\omega 2\,h_1^{2N+1}\,x_2^{(2N+1)\kappa}\Big\{2v_{1N,1}^2+(v_{1N,2}+v_{2N,1})^2+2v_{2N,2}^2+v_{3N,1}^2+v_{3N,2}^2\Big\}\,d\omega. \tag{3.26}$$

Evidently, $I_N\le\|v\|^2_{X_N^\kappa}$.

Note that the summands I_r, $r=\overline{0,N-1}$, do not contain the derivatives of the weighted moments v_{jN}, $j=1,2,3$. From (3.26) by Korn's inequality (see Appendix) we derive

$$\int\limits_\omega x_2^{(2N+1)\kappa}\sum_{j=1}^{3}|\nabla v_{jN}|^2\,d\omega\le C_0^{(N)}\,I_N\le C_0^{(N)}\|v\|^2_{X_N^\kappa}, \tag{3.27}$$

which estimates the term corresponding to $r = N$ in the norm (3.21); here and in the sequel $C_p^{(k)}$ denote positive constants independent of v.

Step 2. Due to Hardy's and Korn's inequalities (see Appendix) we get

$$\begin{aligned}(3.28)\qquad &\sum_{j=1}^{3}\int_{\omega} x_2^{(2N+1)\kappa-2} v_{jN}^2\, d\omega \\ &\le C_1^{(N)}\int_{\omega} x_2^{(2N+1)\kappa}|\nabla v_{jN}|^2\, d\omega \\ &\le C_2^{(N)}\int_{\omega} x_2^{(2N+1)\kappa}[2v_{1N,1}^2 + 2\, v_{2N,2}^2 + (v_{1N,2}+v_{2N,1})^2 + v_{3N,1}^2 + v_{3N,2}^2]^2\, d\omega \\ &\le C_3^{(N)}\|v\|^2_{X_N^\kappa} \qquad \text{for } (2N+1)\kappa \ne 1.\end{aligned}$$

Step 3. Here we prove that

$$(3.29)\qquad \sum_{j=1}^{3}\int_{\omega} x_2^{(2N-1)\kappa-2}|v_{j\,N-1}|^2\, d\omega \le C_1^{(N-1)}\|v\|^2_{X_N^\kappa} \qquad \text{for } (2N-1)\kappa \ne 1,\ \kappa < 1.$$

Indeed, we have

$$\begin{aligned}I_{N-1} = \frac{2N-1}{8}\int_{\omega} 2\, h_1^{2N-1}\, x_2^{(2N-1)\kappa}\Big\{&2v_{1\,N-1,1}^2 + (v_{1\,N-1,2}+v_{2\,N-1,1})^2 \\ &+2v_{2\,N-1,2}^2 + [\, v_{3\,N-1,1} + (2N+1)v_{1N}]^2 + [\, v_{3\,N-1,2} + (2N+1)v_{2N}]^2 \\ &+4h_1^{2N-1}\, x_2^{(2N-1)\kappa}(2N+1)^2\, v_{3N}^2\Big\}\, d\omega \le \|v\|^2_{X_N^\kappa}\, .\end{aligned}$$

Hence,

$$(3.30)\quad \int_{\omega} x_2^{(2N-1)\kappa}[2v_{1\,N-1,1}^2 + (v_{1\,N-1,2}+v_{2\,N-1,1})^2 + 2v_{2\,N-1,2}^2]\, d\omega \le C_2^{(N-1)}\|v\|^2_{X_N^\kappa},$$

$$(3.31)\quad \int_{\omega} x_2^{(2N-1)\kappa}[v_{3\,N-1,\alpha} + (2N+1)v_{\alpha N}]^2\, d\omega \le C_3^{(N-1)}\|v\|^2_{X_N^\kappa}, \qquad \alpha = 1, 2.$$

From (3.31), in view of (3.28), we get

$$(3.32)\qquad \int_{\omega} x_2^{(2N-1)\kappa} v_{3\,N-1,\alpha}^2\, d\omega \le C_4^{(N-1)}\|v\|^2_{X_N^\kappa}$$

since $(2N-1)\kappa > (2N+1)\kappa - 2$ for $\kappa < 1$.

Now, applying again Hardy's and Korn's inequalities with the help of (3.30) and (3.32) we arrive at the relation:

$$
\begin{aligned}
(3.33)\qquad & \sum_{j=1}^{3}\int_{\omega} x_2^{(2N-1)\kappa-2} v_{j\,N-1}^2\, d\omega \\
&\le C_5^{(N-1)}\int_{\omega} x_2^{(2N-1)\kappa}|\nabla v_{j\,N-1}|^2\, d\omega \\
&\le C_6^{(N-1)}\int_{\omega} x_2^{(2N-1)\kappa}[2v_{1\,N-1,1}^2+2\,v_{2\,N-1,2}^2+(v_{1\,N-1,2}+v_{2\,N-1,1})^2 \\
&\quad + v_{3\,N-1,1}^2+v_{3\,N-1,2}^2]^2\, d\omega \le C_7^{(N-1)}\|v\|^2_{X_N^\kappa}.
\end{aligned}
$$

Step 4. Taking into account that I_r does not contain the derivatives of the moments $v_{j\,r+1},\dots,v_{jN}$, $j=1,2,3$, and employing arguments similar to Step 3, we derive

$$
(3.34)\qquad \sum_{j=1}^{3}\int_{\omega} x_2^{(2r+1)\kappa-2} v_{jr}^2\, d\omega \le C_1^{(r)}\|v\|^2_{X_N^\kappa},\qquad r=N-2,\dots,0,
$$

$$
\text{for } \kappa<1,\ (2l+1)\kappa\neq 1,\ l=r,r+1,\dots,N.
$$

Therefore, by (3.28), (3.29) and (3.34) we obtain

$$
(3.35)\qquad \sum_{r=0}^{N}\sum_{j=1}^{3}\int_{\omega} x_2^{(2r+1)\kappa-2} v_{jr}^2\, d\omega \le C_4^{(N)}\|v\|^2_{X_N^\kappa}
$$

for κ satisfying the conditions (3.22).

Step 5. With the help of Korn's inequality we get

$$
\begin{aligned}
(3.36)\quad \|v\|^2_{Y_N^\kappa} &= \sum_{r=0}^{N}\int_{\omega} x_2^{(2r+1)\kappa}\Big[\sum_{\alpha=1}^{2}|\nabla v_{\alpha r}|^2+|\nabla v_{3r}|^2\Big]\, d\omega \\
&\le C_5^{(N)}\sum_{r=0}^{N}\int_{\omega} x_2^{(2r+1)\kappa}[2v_{1r,1}^2+2v_{2r,2}^2+(v_{1r,2}+v_{2r,1})^2+v_{3r,1}^2+v_{3r,2}^2]\, d\omega \\
&\le C_6^{(N)}\frac{1}{4}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_{\omega} h_1^{2r+1}x_2^{(2r+1)\kappa}[4v_{1r,1}^2+4v_{2r,2}^2 \\
&\quad +2\,(v_{1r,2}+v_{2r,1})^2+v_{3r,1}^2+v_{3r,2}^2]\, d\omega := I^*.
\end{aligned}
$$

As we can easily check $I^* - C_6^{(N)}\,\|v\|^2_{X_N^\kappa}$ contains only the moments v_{jr} without derivatives and can be estimated as

$$(3.37)\quad |\, I^* - C_6^{(N)}\,\|v\|^2_{X_N^\kappa}\,| \le C_7^{(N)} \sum_{r=0}^{N}\sum_{j=1}^{3}\int\limits_{\omega}\left(x_2^{(2r+1)\kappa-2}+x_2^{(2r-1)\kappa}\right)v_{jr}^2\,d\omega.$$

Since $(2r+1)\kappa-2<(2r-1)\kappa$ for $\kappa<1$, from (3.37) and (3.35) we conclude

$$I^* \le C_8^{(N)}\|v\|^2_{X_N^\kappa},$$

which by (3.36) and (3.25) leads to the inequality (3.23).

Now we should show the inverse inequality

$$(3.38)\qquad \|v\|_{X_{N,\nu}^\kappa} \le C_0\,\|v\|_{Y_N^\kappa},$$

where the positive constant C_0 does not depend on v.

The inequality,

$$(3.39)\qquad I^2 \le C_4\|v\|^2_{Y_N^\kappa},\qquad \kappa<1,\qquad \kappa\neq\frac{1}{2r+1},\qquad r=\overline{1,N}$$

is a trivial consequence of Hardy's inequality (see [9]).

Let us now consider

$$\begin{aligned}
|I^1| &\le \sum_{r=0}^{N}(2r+1)h_1^{2r+1}\rho\nu^2\Big|\int\limits_{\omega}x_2^2x_2^{(2r+1)\kappa-2}(\overset{\circ}{v}{}^2_{1r}+\overset{\circ}{v}{}^2_{2r}+\overset{\circ}{v}{}^2_{3r})d\omega\Big|\\
&\le \sum_{r=0}^{N}(2r+1)h_1^{2r+1}\rho\nu^2l^2\int\limits_{\omega}x_2^{(2r+1)\kappa-2}(\overset{\circ}{v}{}^2_{1r}+\overset{\circ}{v}{}^2_{2r}+\overset{\circ}{v}{}^2_{3r})d\omega\\
&\le \sum_{r=0}^{N}c_r(2r+1)h_1^{2r+1}\rho\nu^2l^2\int\limits_{\omega}x_2^{(2r+1)\kappa}(|\nabla\overset{\circ}{v}_{1r}|^2+|\nabla\overset{\circ}{v}_{2r}|^2+|\nabla\overset{\circ}{v}_{3r}|^2)d\omega\\
&\le C_5\sum_{r=0}^{N}\int\limits_{\omega}x_2^{(2r+1)\kappa}\sum_{j=1}^{3}|\nabla\overset{\circ}{v}_{jr}|^2\,d\omega = C_5\,\|v\|_{Y_N^\kappa},
\end{aligned}$$

if $\nu^2\le 1/h_2\rho l^2$, $h_2:=\max\{(2r+1)(h_0/2)^{2r+1}\}$, $C_5:=\max\{c_r\}$, $r=0,\dots,N$, which by (3.39) leads to inequality (3.38). □

Corollary 3.5. *Let the conditions* (3.22) *be satisfied. Then the components* $\overset{\circ}{v}_{jr}$ *of the vector* $v\in X_{N,\nu}^\kappa$ *have zero traces on* $\partial\omega$ *if* $(2r+1)\kappa<1$.

P r o o f. It follows directly from the trace theorems in the Appendix. □

Remark 3.6. From Theorem 3.4 by Hardy's inequality it follows that for $\kappa<1$ and $\kappa\neq\frac{1}{2r+1}$, $r=\overline{1,N}$, the linear functional defined by (3.7) is bounded if

$$x_2^{1-\frac{\kappa}{2}}\varPhi_j^{(r)}\in L_2(\omega),\qquad j=1,2,3,\ r=\overline{0,N}\,.$$

4. Appendix

Let ω be as in Section 1 and let $\mathcal{D}(\omega)$ be the space of infinitely differentiable functions with compact support in ω.

A.1. Hardy's inequality. For every $f \in \mathcal{D}(\omega)$ and $\nu \neq 1$ there holds the inequality

$$\int\limits_{\omega} x_2^{\nu-2} f^2(x)\, d\omega \leq C_\nu \int\limits_{\omega} x_2^{\nu} f_{,2}^2(x)\, d\omega \leq \int\limits_{\omega} x_2^{\nu} |\nabla f(x)|^2\, d\omega, \tag{A.1}$$

where the positive constant C_ν is independent of f.

Using a completion argument with respect to the norm (see (3.20))

$$\|f\|^2_{\overset{\circ}{W}{}^1_{2,\nu}(\omega)} := \int\limits_{\omega} x_2^{\nu} |\nabla f(x)|^2\, d\omega,$$

we conclude that the inequality (A.1) holds for arbitrary $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$.

For proof see [19].

A.2. Weighted Korn's inequality. Let $\varphi = (\varphi_1, \varphi_2) \in [\overset{\circ}{W}{}^1_{2,\nu}(\omega)]^2$ and $\nu \neq 1$. Then

$$\int\limits_{\omega} x_2^{\nu}[|\nabla \varphi_1(x)|^2 + |\nabla \varphi_2(x)|^2]\, d\omega$$
$$\leq C_\nu \int\limits_{\omega} x_2^{\nu}[\varphi_{1,1}^2(x) + \varphi_{2,2}^2(x) + (\varphi_{1,2}(x) + \varphi_{2,1}(x))^2]\, d\omega,$$

where the positive constant C_ν is independent of φ.

The proof can be found in [19] and [20].

A.3. Trace theorem. *Let $0 < \nu < 1$ and $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$. Then the trace of the function f equals to zero on $\partial\omega$.*

For proof see [19] and [21].

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