Arch. Mech., **65**, 4, pp. 301–311, Warszawa 2013

Deflection and strength of a sandwich beam with thin binding layers between faces and a core

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The subject of the paper is the analysis of deflection of a five-layer sandwich beam under bending. The mechanical and physical properties vary through the thickness of the beam and depend on the material of each layer, which are: the metal face, binding material (the glue) and metal foam core. The main aim of the paper is to present the analytical model of the five-layer beam and to compare the results of bending analysis obtained theoretically and numerically. In the paper, a mathematical model of the field of displacements, which includes a shear effect and a bending moment, is presented. The system of partial differential equations of equilibrium for the five-layer sandwich beam is derived on the basis of the principle of stationary total potential energy. The equations are analytically solved and the formula describing the deflection of the beam is obtained. The influence of the thickness and mechanical properties of the binding layer on the deflection of the beam under bending is analysed. The comparison of the results obtained in the analytical and numerical (FEM) analysis is shown in graphs and figures.

Key words: sandwich structure, metal foam, multi-layered beam, bending.

1. Introduction

Sandwich structures with a metal foam core are the subject of present-day studies. These structures are characterized by the high impact and heat resistance, good acoustic absorption, vibration's reduction and easy assembly. The bases of the theory of sandwich structures were described by Plantema [1] and Allen [2]. Vinson [3] provided a general characteristic of sandwich structures pointing out theirs advantages. A great number of publications concerning sandwich beam, plates and shells are cited. Banhart [4] described in detail the processes of manufacturing of different types of commercial metal foams. The author also presented the ways – the destructive and non-destructive tests, in which the cellular materials can be characterized. The fields of engineering in which the cellular metals can be used are discussed. Noor *et al.* [5] in their review work consider different models of sandwich structures. The researches concerning different types of problems, like vibrations, buckling or thermal stress are presented. Grigolyuk and Chulkov [6] provided the first hypothesis of the cross-section deformation of sandwich structures. Wang *et al.* [7] discussed the higher-order hypotheses which include shearing of beams and plates. Carrera [8] formulated the zigzag hypotheses for multilayered plates. Iaccarino *et al.* [9] described the effect of a thin soft core on the bending behavior of a sandwich beams. Chakrabarti *et al.* [10] developed a new FE model based on higher-order zigzag theory for the static analysis of laminated sandwich beams with a soft core. Steeves *et al.* [11] and Qin *et al.* [12] presented analytical models of collapse mechanisms of sandwich beams under transverse force. Rakow and Wass [13] described the mechanical properties of aluminium foam under shear. Birman [14] presented modeling and analysis of functionally graded materials. Magnucka-Blandzi and Magnucki [15] and Magnucki *et al.* [16] described strength and buckling problems of sandwich beams with a metal foam core and effective design of these structures. Zenkert [17] investigated debondings in foam core sandwich beams assuming that cracks in the interface between the face and core are present.

This paper is devoted to the strength analysis of a simply-supported sandwich beam. The goal is to elaborate a mathematical model of the beam in which the binding layer will be treated as a separate layer. This way the influence of the mechanical properties and the thickness of the glue can be investigated what is usually omitted when sandwich structures are analyzed. For simplicity reasons, a classical broken line hypothesis has been assumed to describe the deformation of the cross-section of the beam. Such an approach allowed toobtain a formula with which the deflection of the five-layer beam can be determined.

The beam consists of five layers: the upper and lower face, the core and the thin binding layers between the faces and the core. The scheme of the beam is shown in Fig. 1.

The beam has the length L, the width b and the depth H. The thickness of particular layers that is the faces, the core and the binding layers are denoted by t_f, t_c and t_b, respectively. The load has the form of a concentrated force F located in the mid-length of the beam.

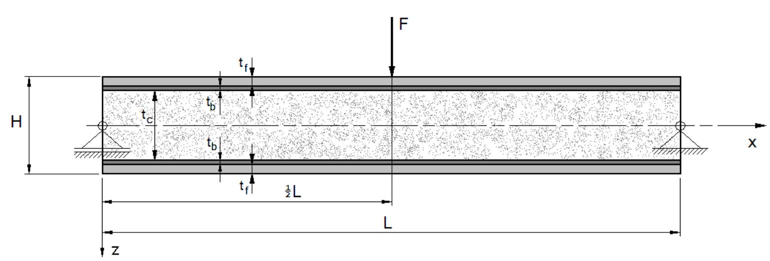

Fig. 1. Scheme of the loaded beam.

2. Analytical analysis

The deformation of the flat cross-section of the five-layer beam is shown in Fig. 2.

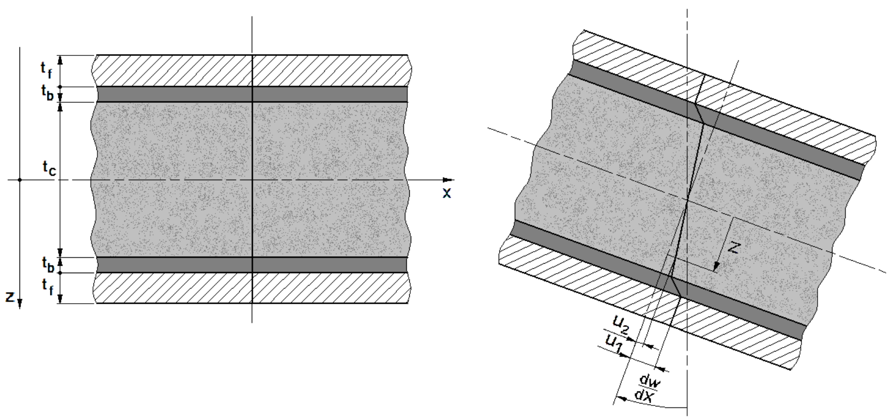

Fig. 2. Scheme of displacements – the hypothesis for the beam.

The field of displacements is formulated as follows:

1. The upper face $-(1/2 + x_1 + x_2) \leq \zeta \leq -(1/2 + x_1)$

$$u(x,\zeta) = -t_c \left[\zeta \frac{dw}{dx} + \psi_1(x) \right]. \tag{2.1}$$

2. The upper binding layer $-(1/2 + x_1) \leq \zeta \leq -1/2$

$$u(x,\zeta) = -t_c \left[\zeta \frac{dw}{dx} + \psi_2(x) - \frac{1}{x_1} \left(\zeta + \frac{1}{2} \right) (\psi_1(x) - \psi_2(x)) \right]. \tag{2.2}$$

3. The core $-1/2 \leq \zeta \leq 1/2$

$$u\left(x,\zeta\right) = -t_c \zeta \left[\frac{dw}{dx} - 2\psi_2\left(x\right) \right]. \tag{2.3}$$

4. The lower binding layer $1/2 \leq \zeta \leq 1/2 + x_1$

$$u\left(x,\zeta\right) = -t_c \left[\zeta \frac{dw}{dx} - \psi_2\left(x\right) - \frac{1}{x_1} \left(\zeta - \frac{1}{2} \right) \left(\psi_1\left(x\right) - \psi_2\left(x\right)\right) \right]. \tag{2.4}$$

5. The lower face $1/2 + x_1 \leq \zeta \leq 1/2 + x_1 + x_2$

$$u\left(x,\zeta\right) = -t_c \left[\zeta \frac{dw}{dx} - \psi_1\left(x\right) \right], \tag{2.5}$$

where

$$x_1 = t_b/t_c, \quad x_2 = t_f/t_c, \quad \zeta = z/t_c, \quad \psi_1(x) = u_1(x)/t_c, \quad \psi_2(x) = u_2(x)/t_c.$$

Strains of the layers of the beam are defined by the geometric relationships in the following form:

1. The upper face

$$\varepsilon_x = -t_c\left[\zeta\frac{d^2w}{dx^2} + \frac{d\psi_1(x)}{dx}\right], \qquad \gamma_{xz} = 0. \tag{2.6}$$

2. The upper binding layer

$$\begin{aligned}\varepsilon_x &= -t_c\left[\zeta\frac{d^2w}{dx^2} + \frac{\psi_2(x)}{dx} - \frac{1}{x_1}\left(\zeta + \frac{1}{2}\right)\left(\frac{d\psi_1(x)}{dx} - \frac{d\psi_2(x)}{dx}\right)\right],\\ \gamma_{xz} &= \frac{1}{x_1}\left[\psi_1(x) - \psi_2(x)\right].\end{aligned} \tag{2.7}$$

2. The core

$$\varepsilon_x = -t_c\zeta\left[\frac{d^2w}{dx^2} - 2\frac{d\psi_2(x)}{dx}\right], \qquad \gamma_{xz} = 2\psi_2(x). \tag{2.8}$$

4. The lower binding layer

$$\begin{aligned}\varepsilon_x &= -t_c\left[\zeta\frac{d^2w}{dx^2} - \frac{\psi_2(x)}{dx} - \frac{1}{x_1}\left(\zeta - \frac{1}{2}\right)\left(\frac{d\psi_1(x)}{dx} - \frac{d\psi_2(x)}{dx}\right)\right],\\ \gamma_{xz} &= \frac{1}{x_1}\left[\psi_1(x) - \psi_2(x)\right].\end{aligned} \tag{2.9}$$

5. The lower face

$$\varepsilon_x = -t_c\left[\zeta\frac{d^2w}{dx^2} - \frac{d\psi_1(x)}{dx}\right], \qquad \gamma_{xz} = 0. \tag{2.10}$$

The physical relationships for individual layers according to Hooke's law are

$$\sigma_x = E\varepsilon_x, \qquad \tau_{xz} = G\gamma_{xz}. \tag{2.11}$$

The bending moment of any cross-section of the beam is

$$\begin{aligned}M_b(x) = \int_A \sigma_x z dA = -bt_c^3\bigg\{&\left(2E_f c_{2f} + 2E_b c_{2b} + \frac{1}{12}E_c\right)\frac{d^2w}{dx^2}\\ &- \left[E_f c_{1f} + E_b\frac{x_1}{6}(3+4x_1)\right]\frac{d\psi_1}{dx} - \left[\frac{1}{6}E_c + \frac{1}{6}E_b x_1(3+2x_1)\right]\frac{d\psi_2}{dx}\bigg\},\end{aligned} \tag{2.12}$$

where

$$c_{1b} = x_1\left(1 + x_1\right), \quad c_{2b} = \frac{1}{12}x_1\left(3 + 6x_1 + 4x_1^2\right), \quad c_{1f} = x_2\left(1 + 2x_1 + x_2\right),$$
$$c_{2f} = \frac{1}{12}x_2\left[12x_1\left(1 + x_1 + x_2\right) + 3 + 6x_2 + 4x_2^2\right],$$

and E_f, E_c, E_b are: Young's modulus of the faces (E_f), Young's modulus of the core (E_c), Young's modulus of the glue layers (E_b).

The transverse force of any cross-section of the beam is

$$Q(x) = \int\limits_A \tau_{xz}dA = 2bt_c\left[G_b\psi_1(x) + \left(G_c - G_b\right)\psi_2(x)\right], \tag{2.13}$$

where

$$G_c = \frac{E_c}{2(1+\nu_c)}, \qquad G_b = \frac{E_b}{2(1+\nu_b)}.$$

2.1. Equations of equilibrium

The potential energy of the elastic strain of the beam is

$$U_\varepsilon = \frac{1}{2}\int\limits_V (\varepsilon_x\sigma_x + \gamma_{xz}\tau_{xz})dV = \frac{1}{2}bt_c\int\limits_0^L (f_{Ef} + f_{Eb} + f_{Ec})dx, \tag{2.14}$$

where

$$f_{Ef} = 2E_f t_c^2\left[c_{2f}\left(\frac{d^2w}{dx^2}\right)^2 - c_{1f}\frac{d^2w}{dx^2}\frac{d\psi_1}{dx} + x_2\left(\frac{d\psi_1}{dx}\right)^2\right],$$
$$\begin{aligned} f_{Eb} = 2E_b t_c^2\Bigg[&c_{2b}\left(\frac{d^2w}{dx^2} - \frac{1}{x_1}\frac{d\psi_1}{dx} + \frac{1}{x_1}\frac{d\psi_2}{dx}\right)^2 \\ &+ \frac{c_{1b}}{2x_1}\left(\frac{d^2w}{dx^2} - \frac{1}{x_1}\frac{d\psi_1}{dx} + \frac{1}{x_1}\frac{d\psi_2}{dx}\right)\left(\frac{d\psi_1}{dx} - (1+2x_1)\frac{d\psi_2}{dx}\right) \\ &+ \frac{1}{4x_1}\left(\frac{d\psi_1}{dx} - (1+2x_1)\frac{d\psi_2}{dx}\right)^2\Bigg] \\ &+ \frac{2}{x_1}G_b[\psi_1(x) - \psi_2(x)]^2, \end{aligned}$$
$$f_{Ec} = \frac{1}{12}E_c t_c^2\left[\frac{d^2w}{dx^2} - 2\frac{d\psi_2}{dx}\right]^2 + 4G_c\psi_2^2(x).$$

The work of the external load is

$$W = \int_0^L qw dx. \tag{2.15}$$

The system of three differential equations obtained from the principle of stationary total potential energy $\delta(U_\varepsilon - W) = 0$ has the following form:

$$\begin{aligned} \delta w) \qquad & bt_c^3\Big\{\Big(2E_f c_{2f} + 2E_b c_{2b} + \frac{1}{12}E_c\Big)\frac{d^4w}{dx^4} \\ & - \Big[E_f c_{1f} + \frac{1}{6}E_b x_1\Big(3 + 4x_1\Big)\Big]\frac{d^3\psi_1}{dx^3} \\ & - \frac{1}{6}[E_c + E_b x_1(3 + 2x_1)]\frac{d^3\psi_2}{dx^3}\Big\} = q - F_0\frac{d^2w}{dx^2}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \delta\psi_1) \qquad & \Big[E_f c_{1f} + \frac{1}{6}E_b x_1(3 + 4x_1)\Big]\frac{d^3w}{dx^3} - 2\Big(E_f x_2 + \frac{1}{3}E_b x_1\Big)\frac{d^2\psi_1}{dx^2} \\ & - \frac{1}{3}E_b x_1\frac{d^2\psi_2}{dx^2} + \frac{2G_b}{x_1 t_c^2}[\psi_1(x) - \psi_2(x)] = 0, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \delta\psi_2) \qquad & \frac{1}{6}[E_c + E_b x_1(3 + 2x_1)]\frac{d^3w}{dx^3} - \frac{1}{3}E_b x_1\frac{d^2\psi_1}{dx^2} \\ & - \frac{1}{3}(E_c + 2E_b x_1)\frac{d^2\psi_2}{dx^2} - \frac{2G_b}{x_1 t_c^2}\psi_1(x) + \frac{1}{t_c^2}\Big(4G_c + \frac{2}{x_1}G_b\Big)\psi_2(x) = 0. \end{aligned} \tag{2.18}$$

The first equation (2.16) of the system is equivalent to the bending moment (2.12). Therefore, for further analysis purpose, the system of three equations that is Eqs. (2.12), (2.17) and (2.18) is used.

2.2. Deflection of a beam

The simply-supported sandwich beam is loaded with the force F. The bending moment for this load case is written in the form $M_b(x) = 0.5Fx$. After simple transformations Eqs. (2.12), (2.17) and (2.18) can be reduced to two equations of the following form:

$$\begin{aligned} & (a_{12}^2 - a_{11}a_{22})\frac{d^2\psi_1}{dx^2} + (a_{12}a_{13} - a_{11}a_{23})\frac{d^2\psi_2}{dx^2} \\ & + a_{11}\frac{2G_b}{x_1 t_c^2}[\psi_1(x) - \psi_2(x)] = a_{12}\frac{F}{2bt_c^3}, \end{aligned} \tag{2.19}$$

$$(a_{12}a_{31}-a_{11}a_{32})\frac{d^2\psi_1}{dx^2}+(a_{13}a_{31}-a_{11}a_{33})\frac{d^2\psi_2}{dx^2} + 4a_{11}\frac{G_c}{t_c^2}\psi_2(x)=a_{31}\frac{F}{2bt_c^3}, \tag{2.20}$$

where:

$$a_{11}=2E_fc_{2f}+2E_bc_{2b}+\frac{1}{12}E_c,\quad a_{12}=E_fc_{1f}+\frac{1}{6}E_bx_1(3+4x_1),$$
$$a_{13}=\frac{1}{6}[E_c+E_bx_1\,(3+2x_1)],\quad a_{22}=2(E_fx_2+\frac{1}{3}E_bx_1),\quad a_{23}=\frac{1}{3}E_bx_1,$$
$$a_{31}=E_fc_{1f}+E_bx_1\,(1+x_1)+\frac{1}{6}E_c,\quad a_{33}=\frac{1}{3}(E_c+3E_bx_1).$$

The system of equations is approximately solved by means of the Bubnov–Galerkin method. The three unknown functions, ψ_1, ψ_2 and w, are assumed in the form of Fourier series

$$\begin{aligned}\psi_1(x)&=\psi_{11}\cos\frac{\pi x}{L}+\psi_{13}\cos\frac{3\pi x}{L}+\dots+\psi_{1k}\cos\frac{k\pi x}{L},\\ \psi_2(x)&=\psi_{21}\cos\frac{\pi x}{L}+\psi_{23}\cos\frac{3\pi x}{L}+\dots+\psi_{2k}\cos\frac{k\pi x}{L},\\ w(x)&=w_1\sin\frac{\pi x}{L}+w_3\sin\frac{3\pi x}{L}+\dots+w_k\sin\frac{k\pi x}{L},\quad k=1,3,5,\dots.\end{aligned} \tag{2.21}$$

As a result of the orthogonalization process the formula describing the maximum deflection of the five-layer beam is obtained in the following form:

$$w=\frac{2FL^3}{\pi^4 D}, \tag{2.22}$$

where:

$$D=\frac{k^2a_{11}}{\frac{1}{k^2}+a_{12}\alpha_k+a_{13}\beta_k}bt_c^3,\qquad \alpha_k=-\frac{a_{12}b_{22}-a_{31}b_{12}}{b_{11}b_{22}-b_{12}b_{21}},$$
$$\beta_k=-\frac{a_{31}b_{11}-a_{12}b_{21}}{b_{11}b_{22}-b_{12}b_{21}},\qquad b_{11}=k^2c_{11}-a_{11}\frac{2}{\pi^2}\frac{G_b}{x_1}\left(\frac{L}{t_c}\right)^2,$$
$$b_{12}=k^2c_{12}+a_{11}\frac{2}{\pi^2}\frac{G_b}{x_1}\left(\frac{L}{t_c}\right)^2,\quad b_{21}=k^2c_{21},$$
$$b_{22}=k^2c_{22}-a_{11}\frac{4}{\pi^2}G_c\left(\frac{L}{t_c}\right)^2,\quad c_{11}=a_{12}^2-a_{11}a_{22},$$
$$c_{12}=a_{12}a_{13}-a_{11}a_{23},\quad c_{21}=a_{12}a_{31}-a_{11}a_{32},\quad c_{22}=a_{13}a_{31}-a_{11}a_{33}.$$

Example calculations have been performed for the beam with the following dimensions: $L = 100$ mm, $H = 20$ mm, $b = 50$ mm, $t_f = 1$ mm. The material properties were: $E_f = 65600$ MPa, $E_c = 1200$ MPa and $\nu_c = \nu_b = 0.3$. Different values of E_b and t_b have been considered according to Table 1, in which the values of deflection determined with the use of equation (2.22) are presented. The beam was loaded with the force $F = 1$ kN.

Table 1. Deflections of a beam w for $k = 3$.

t_b [mm]	E_b [MPa]				
	50	100	500	1000	1500
0.1	0.09341	0.08761	0.08284	0.08228	0.08207
0.2	0.10441	0.09301	0.08361	0.08239	0.08196
0.3	0.11519	0.09837	0.08434	0.08250	0.08185
0.4	0.12580	0.10369	0.08506	0.08261	0.08174
0.5	0.13636	0.10899	0.08578	0.08272	0.08164

3. Numerical analysis

The finite element model of the beam has been built using the ABAQUS code. For modelling of the core and of the binding layers 3D brick elements with eight nodes have been used. The faces of the beam have been modelled with the use of four-node thin shell elements. The tie conditions have been applied between the layers. Because of the symmetry of the problem only a quarter of the beam has been used with proper boundary conditions in the symmetry planes. To obtain the boundary conditions corresponding to the ones assumed in the analytical model all layers have been joined with a rigid plate at the edge of the beam. Similar solution has been applied in the mid-length of the beam. Here, the rigid plate distributes the applied force equally to all layers which prevents from local deformations. The FE model of the beam as well as an example of deformation is shown in Fig. 3.

The static analysis has been performed in which the deflection in the mid-length of the beam has been measured. The dimensions of the beam and the material properties were the same as in the analytical calculations.

The comparison of the results obtained analytically and numerically (FEM) is shown in Fig. 4.

The difference between results obtained numerically and analytically for $k = 1$ is about 10–15%. However, the difference is much less (2–2.5%) for $k = 3$.

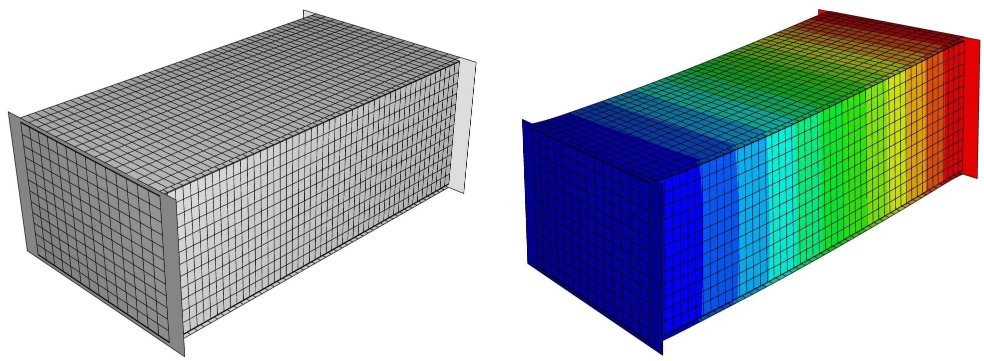

FIG. 3. Numerical model of the beam.

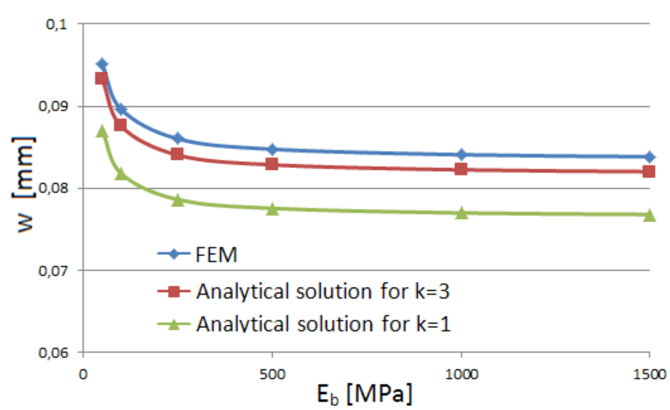

FIG. 4. The comparison of the results obtained analytically and numerically.

4. Conclusions

In the paper, a mathematical model of a five-layer beam was presented. The faces were glued to the core with thin binding layers. The glue was treated as a separate layer. The influence of the thickness and the material properties of the binding layer on the deflection of the beam under bending was analyzed. The results obtained from the FEM analysis have been compared with those given by the analytical model proposed in the paper. A good agreement can be seen between these two approaches – the discrepancy was 2–2.5% at the most.

From the results given in Table 1 it can be observed that for high values of E_b the thickness of the binding layers does not influence considerably the deflection of the beam. Similarly, the thinner the binding layer the smaller its influence on the stiffness of the beam.

Acknowledgements

The studies were supported by the Ministry of Sciences and Higher Education in Poland – Grant No. DS-MK 21-388/2011.

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Received December 4, 2012; revised version May 3, 2013.