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Effects of boundary reinforcement on local singular fields in linearly elastic materials

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We consider the local deformation near a point at the interface between free and fixed boundary segments in an elastic half-plane undergoing plane strain deformations. Using asymptotic analysis, we show that the addition of a reinforcement along the free boundary effectively eliminates the well-known oscillatory behavior of the displacement and stress fields in the vicinity of the point leading to a strong square-root stress singularity. In addition, we demonstrate that the reinforcement induces a deformation field which is smooth locally and bounded at the point of interest.

Key words: boundary reinforcement, local singular fields, plane elasticity.

1. Introduction

The incorporation of surface mechanics into mathematical models describing deformation of elastic solids has drawn an increasing amount of attention in the literature recently (see, for example [1]–[21] and the references contained therein). We note, specifically, the seminal papers [1]–[3] which describe a theory for finite deformations of elastic solids with thin elastic films attached to their bounding surfaces. These works are particularly important in that they generalize the well-known Gurtin–Murdoch theory of the mechanics of surface stressed solids [4] which has been used extensively in the literature in continuum models of deformation at the nanoscale where the high surface area to volume ratio means that the separate contributions of the surface can no longer be ignored. Recently, Chhapadia *et al.* [5] have adapted the theory in [1] to incorporate curvature-dependence of surface energy to describe size-effects of physical phenomena at small length scales.

In [6], the authors presented a rigorous analysis of a series of non-standard boundary value problems corresponding to the linearized version of the theory developed in [1]. It was of particular interest in [6] to determine the contribution of the reinforcing thin film attached to the bounding surfaces of the solid. This paper continues that study by investigating the contribution of boundary reinforcement (surface mechanics) to the classical mixed boundary-value problem describing the plane-strain field near a point at the interface between free and fixed boundary segments in an elastic half-plane (Fig. 1). It is well-known that both surface displacements along the free boundary and contact stress distributions exhibit oscillatory behavior in the vicinity of the point [22]. In this paper, we demonstrate that a reinforcement along the free boundary effectively eliminates the oscillatory behavior of the stress field in the vicinity of the point leading to a strong square-root singularity. In addition, we show that the displacement field is smooth locally and bounded at the point of interest.

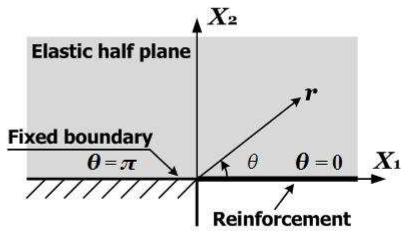

Fig. 1. Schematic of the problem.

2. Notation and prerequisites

Plane-strain deformations of a linearly elastic, homogeneous and isotropic solid are characterized by a displacement vector $\mathbf{u}$ whose components (u, v, w), with respect to the standard basis in $\mathbb{R}^3$, are assumed to satisfy the relations

$$u = u_1(x_1, x_2), \qquad v = u_2(x_1, x_2), \qquad w = 0, \tag{2.1}$$

where (x_1, x_2) represent Cartesian coordinates in $\mathbb{R}^2$. In the absence of body forces, the reduced displacement field and the corresponding stress components $\sigma_{\alpha\beta}$, $\alpha, \beta = 1, 2$, can be described in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + ix_2$ (or $z = re^{i\theta}$ in the polar coordinate system, where $r^2 = x_1^2 + x_2^2$ and $\tan\theta = x_2/x_1$) in the half plane (see Fig. 1) by [23]

$$\sigma_{11} + \sigma_{22} = 2[\phi(z)' + \overline{\phi(z)'}], \tag{2.1a}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\overline{z}\phi(z)'' + \psi(z)'\right], \tag{2.1b}$$

$$2\mu(u_1 + iu_2) = \kappa\phi(z) - z\overline{\phi(z)'} - \overline{\psi(z)}. \tag{2.1c}$$

Here, the constant κ is defined as:

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4v, \tag{2.3}$$

where υ is Poisson's ratio taking values in the range $0 < \upsilon < \frac{1}{2}$. Thus, κ satisfies the following inequality

$$1 < \kappa < 3. \tag{2.4}$$

For the purpose of the present study, Eqs. (2.2) are re-written more conveniently in the form;

$$\sigma_{22} = 2\,\mathrm{Re}[\phi\,(z)'] + \mathrm{Re}[\overline{z}\phi\,(z)''] + \mathrm{Re}[\psi\,(z)'], \tag{2.2a}$$

$$2\mu\,(u_1 + iu_2) = \kappa\phi\,(z) - z\overline{\phi\,(z)'} - \overline{\psi\,(z)}, \tag{2.2b}$$

$$2\mu u_1 = \mathrm{Re}[\kappa\phi\,(z) - \overline{z}\phi\,(z)' - \psi\,(z)]. \tag{2.2c}$$

3. Local singular field near mixed boundary

We consider the local deformation near a point at the interface between an elastic half-plane fixed at $\theta = \pi$ and a load-free boundary along $\theta = 0$ (see Fig. 1). As noted in [22], "A familiar example of a mixed boundary-value problem of this type is furnished by the particular problem of a rigid flat-ended punch that is bonded to the straight edge of a semi-infinite elastic slab and subjected to an axial load". Our particular interest lies in the case when the load-free boundary ($\theta = 0$) is reinforced with a thin solid film whose bending rigidity is negligible. For consistency, we assume that no initial tension is applied at the reinforced boundary ($\theta = 0$).

For convenience, we represent u_α and $\sigma_{\alpha\beta}$, $\alpha, \beta = 1, 2$ by the same functions when referred to the polar coordinate system. The corresponding boundary conditions pertaining to this problem can be summarized as follows:

$$u_1(r, \pi) = u_2(r, \pi) = \sigma_{22}(r, 0) = 0, \tag{3.1a}$$

$$\sigma_{12}(r, 0) = N_{,1}\,(x_1)\,, \qquad 0 < r < \infty \tag{3.1b}$$

where the deformation-induced surface tension $N\,(x_1)$ is given by [6]

$$N\,(x_1) = F\,(x_1)\,u_{1,1}\,(x_1)\,, \qquad F\,(x_1) > 0. \tag{3.2}$$

Here $F = E\mathcal{A}$, E is Young's modulus and $\mathcal{A}$ is the cross-sectional area of the reinforcement at x_1 [6]. Consequently, the boundary condition Eq. (3.1b) on $\theta = 0$ becomes

$$\sigma_{12}(r, 0) = E\mathcal{A}u_{1,11}\,(r, 0)\,. \tag{3.3}$$

We are particularly interested in displacement solutions which admit the asymptotic representation

$$u_\alpha = r^\rho f_\alpha(\theta) + o(r^\rho), \quad \alpha = 1, 2, \qquad \text{as } r \to 0 \tag{3.4}$$

uniformly for $\theta \in [0, \pi]$, where, ρ is a real constant in the range $0 < \rho < 1$ and f_α are smooth functions on $[0, \pi]$. Solutions of the form (9), *to leading order*, take the form

$$u_\alpha = r^\rho f_\alpha(\theta), \quad \alpha = 1, 2 \qquad \text{as } r \to 0. \tag{3.5}$$

It is clear from Eq. (2.2c) that we can achieve the leading order solution (3.5) by assuming that ϕ and ψ take the form

$$\phi(z) = Az^\rho, \quad \psi(z) = Bz^\rho, \qquad 0 < \rho < 1 \tag{3.6}$$

where A and B are complex constants to be determined.

From Eqs. (2.2b), (3.6), it follows that the leading order solution (3.5) corresponds to the singular stress

$$\sigma_{12} = O(r^{\rho-1}) \qquad \text{as } r \to 0. \tag{3.7}$$

Similarly, the leading order solution (3.5) is such that

$$u_{1,11}(r, 0) = \rho(\rho - 1)r^{\rho-2} f_1(0) \qquad \text{as } r \to 0. \tag{3.8}$$

From (3.7) and (3.8), it is clear that the boundary condition (3.3) requires that, *to leading order*,

$$u_{1,11}(r, 0) = 0 \qquad \text{as } r \to 0. \tag{3.9}$$

Consequently, we have the following boundary conditions

$$u_1(r, \pi) = u_2(r, \pi) = 0, \qquad \sigma_{22}(r, 0) = 0, \tag{3.10a}$$

$$u_{1,11}(r, 0) = 0 \qquad \text{as } r \to 0. \tag{3.10b}$$

We note that, from (3.5), (3.8) and (3.9), it is sufficient to replace the condition (3.10b) by the simpler condition

$$u_1(r, 0) = 0 \qquad \text{as } r \to 0.$$

That is, the leading order solution satisfies the boundary condition

$$u_1(r, 0) = 0. \tag{3.11}$$

In what follows, the discussion is confined to the leading order solution from Eqs. (3.5), (3.6) and (3.11).

From Eqs. (2.2c), (3.6), we obtain:

$$\begin{aligned} 4\mu u_1 &= 2\kappa \operatorname{Re}[\phi(z)] - \overline{z}\phi(z)' - z\overline{\phi(z)'} - \overline{\psi(z)} - \psi z \\ &= \kappa[Az^\rho + \overline{A}\overline{z^\rho}] - \overline{z}A\rho z^{\rho-1} - z\overline{A}\rho\overline{z^{\rho-1}} - (Bz^\rho + \overline{B}\overline{z^\rho}). \end{aligned}$$

Substituting $z = re^{i\theta}$ we then have

$$4\mu u_1 = r^\rho[\kappa(Ae^{i\rho\theta} + \overline{A}e^{-i\rho\theta}) - \rho Ae^{i(\rho-2)\theta} - \rho\overline{A}e^{i(2-\rho)\theta} - Be^{i\rho\theta} - \overline{B}e^{-i\rho\theta}]. \tag{3.12}$$

Similarly, Eqs. (2.2a, b) yield

$$\begin{aligned} 2\mu(u_1 + iu_2) &= \kappa Az^\rho - z\overline{A\rho z^{\rho-1}} - \overline{Bz^\rho} \\ &= r^\rho(\kappa Ae^{i\rho\theta} - \rho\overline{A}e^{i(2-\rho)\theta} - e^{-i\rho\theta}\overline{B}), \end{aligned} \tag{3.13}$$

$$\begin{aligned} \sigma_{22} = \rho r^{\rho-1}\Big[&Ae^{i(\rho-1)\theta} + \overline{A}e^{i(1-\rho)\theta} + \frac{1}{2}(\rho-1)(Ae^{i(\rho-3)\theta} + \overline{A}e^{i(3-\rho)\theta}) \\ &+ \frac{1}{2}(Be^{i(\rho-1)\theta} + \overline{B}e^{i(1-\rho)\theta})\Big]. \end{aligned} \tag{3.14}$$

At the reinforced boundary ($\theta = 0$), since $\sigma_{22} = 0$ at $\theta = 0$ we obtain from Eq. (3.14) that

$$\sigma_{22} = \rho r^{\rho-1}\left[\left(1 + \frac{1}{2}(\rho-1)\right)(A + \overline{A}) + \frac{1}{2}(B + \overline{B})\right] = 0. \tag{3.15}$$

Comparing coefficients on both sides of Eq. (3.15) yields

$$(\rho + 1)2\operatorname{Re}A + 2\operatorname{Re}B = 0. \tag{3.16}$$

In addition, Eq. (3.11) yields, from Eq. (3.12),

$$2\mu u_1 = r^\rho\left[(\kappa - \rho)\operatorname{Re}A - \operatorname{Re}B\right] = 0. \tag{3.17}$$

Again, comparing coefficients on both sides of Eq. (3.17), we have that

$$(\kappa - \rho)\operatorname{Re}A - \operatorname{Re}B = 0. \tag{3.18}$$

Therefore, in view of Eqs. (3.16) and (3.18), we conclude that

$$\operatorname{Re}A = 0, \qquad \operatorname{Re}B = 0. \tag{3.19}$$

Now, at the fixed boundary ($\theta = \pi$), since $u_1 = u_2 = 0$ at $\theta = \pi$, Eq. (3.13) becomes

$$0 + 0i = r^\rho(\kappa Ae^{i\rho\pi} - \rho\overline{A}e^{i(2-\rho)\pi} - e^{-i\rho\pi}\overline{B}). \tag{3.20}$$

We thus obtain

$$\kappa Ae^{i\rho\pi} - \rho\overline{A}e^{i(2-\rho)\pi} - e^{-i\rho\pi}\overline{B} = 0, \qquad \because r^\rho \neq 0. \tag{3.21}$$

In view of the results in Eq. (3.19), Eq. (3.21) can be rewritten as

$$\kappa e^{2i\rho\pi} + \rho e^{2i\pi} = -\frac{B}{A} = C \in \mathbb{R}.$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, the above condition becomes

$$\kappa(\cos 2\rho\pi + i\sin 2\rho\pi) + \rho(\cos 2\pi + i\sin 2\pi) = -\frac{B}{A} = C \in \mathbb{R}. \tag{3.22}$$

Consequently, the only possible root (ρ, $0 < \rho < 1$) satisfying Eq. (3.22) is $\rho = \frac{1}{2}$. Therefore, we obtain that

$$\kappa\cos\pi + \frac{1}{2} = -\frac{B}{A}, \qquad B = A\left(\kappa - \frac{1}{2}\right). \tag{3.23}$$

As a result, the unknown complex potentials are given by

$$\phi(z) = Az^{1/2}, \qquad \psi(z) = Bz^{1/2}, \qquad B = A\left(\kappa - \frac{1}{2}\right). \tag{3.24}$$

Remark 1. It should be emphasized that the boundary conditions (3.1) are not sufficient for a complete statement of a plane-strain problem for a half-plane since no restrictions have been placed on the nature of the displacements or stresses at infinity. Consequently, we expect that the parameter A in the solution (3.24) will depend on the applied (remote) loading.

4. Stress and displacement fields

The corresponding stress and displacement fields for the elastic half-plane can now be completely determined from (3.24). In fact, from Eq. (2.2b), the normal and tangential components of the displacement field can be expressed as

$$\begin{aligned} 2\mu(u_1 + iu_2) &= \kappa\phi(z) - z\overline{\phi(z)} - \overline{\psi(z)} \\ &= r^{1/2}\left[(\kappa A + B)\cos\frac{\theta}{2} + \frac{A}{2}\cos\frac{3\theta}{2}\right] \\ &\quad + ir^{1/2}\left[(\kappa A - B)\sin\frac{\theta}{2} + \frac{A}{2}\sin\frac{3\theta}{2}\right]. \end{aligned}$$

Since A and B are imaginary numbers (see Eq. (3.19)), we obtain that

$$\begin{aligned} u_1 &= \frac{1}{2\mu} ir^{1/2}\left[(\kappa A - B)\sin\frac{\theta}{2} + \frac{A}{2}\sin\frac{3\theta}{2}\right], \\ iu_2 &= \frac{1}{2\mu} r^{1/2}\left[(\kappa A + B)\cos\frac{\theta}{2} + \frac{A}{2}\cos\frac{3\theta}{2}\right], \end{aligned} \tag{4.1}$$

where $B = A\left(\kappa - \frac{1}{2}\right)$. The stress fields are obtained similarly. For example, from Eqs. (2.1a), (2.2a) and (3.24), the normal and tangential stress components near the origin are given by

$$
\begin{aligned}
\sigma_{22} &= ir^{-1/2}\left[A\left(\frac{1}{4}\sin\frac{5}{2}\theta - \sin\frac{\theta}{2}\right) - \frac{B}{2}\sin\frac{\theta}{2}\right], \\
\sigma_{11} + \sigma_{22} &= -2ir^{-1/2}A\sin\frac{\theta}{2}.
\end{aligned} \tag{4.2}
$$

Also, Eqs. (2.1b) and (3.24) yield

$$
\begin{aligned}
\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = &-\frac{A}{2}r^{-1/2}\left(\cos\frac{5\theta}{2} - i\sin\frac{5\theta}{2}\right) \\
&+ Br^{-1/2}\left(\cos\frac{\theta}{2} - i\sin\frac{2\theta}{2}\right).
\end{aligned} \tag{4.3}
$$

Therefore, we obtain

$$
i\sigma_{12} = r^{-1/2}\left(-\frac{A}{4}\cos\frac{5\theta}{2} + \frac{B}{2}\cos\frac{\theta}{2}\right).
$$

The corresponding results are plotted through Figs. 2–6.

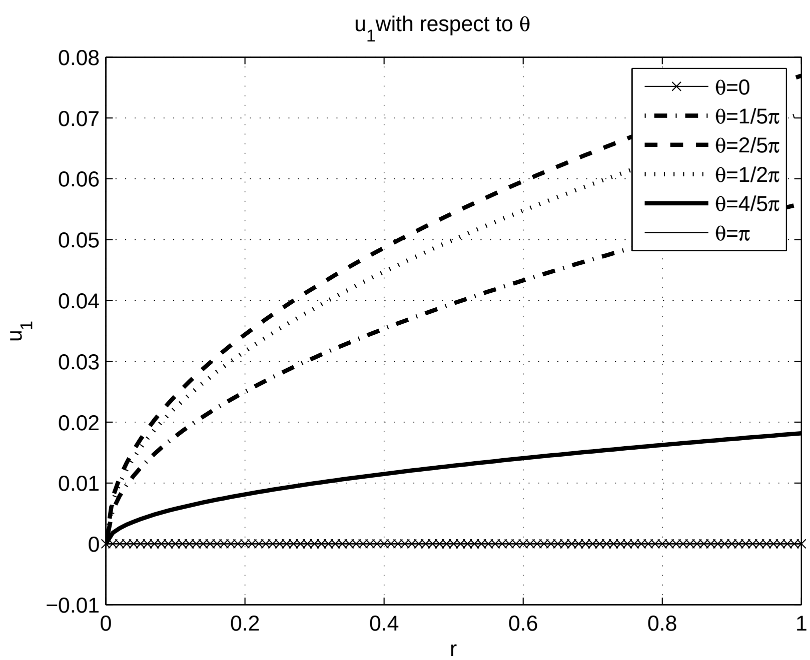

Fig. 2. Normal displacement (u_1), when $A/\mu = 0.1$ and $\kappa = 2.7$.

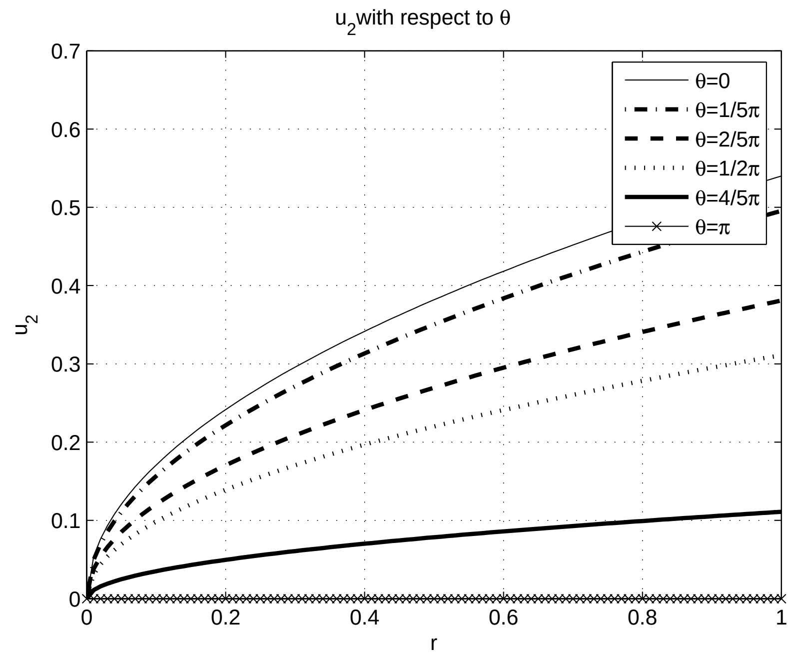

Fig. 3. Tangential displacement (u_2), when $A/\mu = 0.1$ and $\kappa = 2.7$.

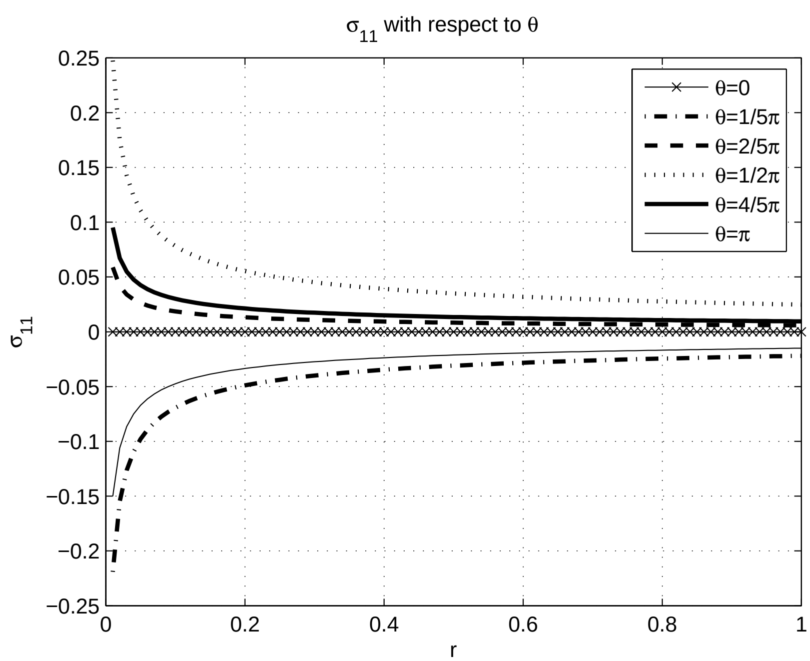

Fig. 4. Stress component (σ_{11}), when $A/\mu = 0.1$ and $\kappa = 2.7$.

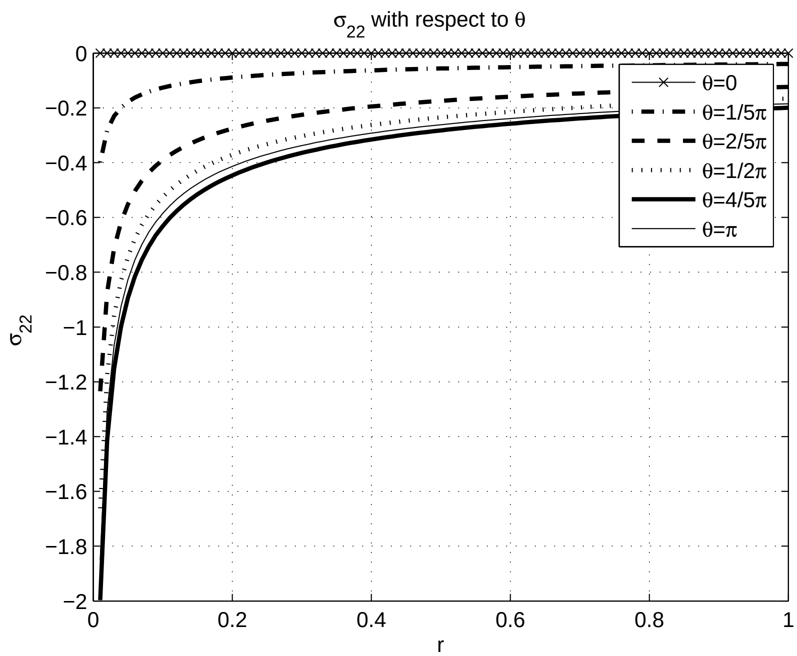

FIG. 5. Stress component (σ_{22}), when $A/\mu = 0.1$ and $\kappa = 2.7$.

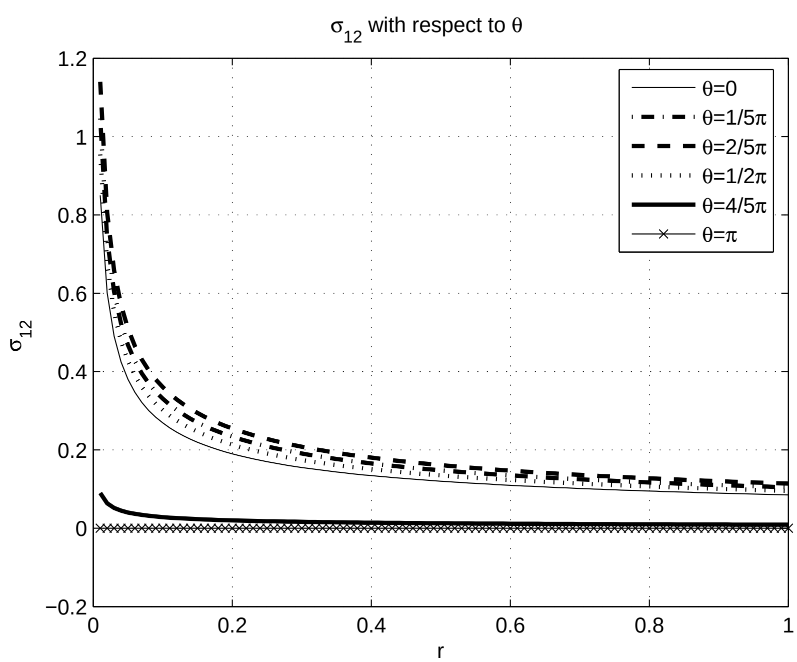

FIG. 6. Stress component (σ_{12}), when $A/\mu = 0.1$ and $\kappa = 2.7$.

Figures 2 and 3 indicate that the introduction of boundary reinforcement removes the anomalous oscillatory behavior of the displacement field (as predicted by the linear theory of elasticity [24]) in the vicinity of the point at the interface between free and fixed boundary segments. In fact, both normal (u_1) and tangential (u_2) displacements are found to be zero (see Figs. 2 and 3) at the fixed boundary ($\theta = 0$) satisfying the imposed boundary conditions (Eqs. 3.10). In addition, the displacement fields are smooth locally and bounded at the origin ($r = 0$). In the case of the corresponding stress distributions, it is clear from Figs. 4, 5 and 6 that the stresses are similarly free of the classical oscillatory behavior in the vicinity of the origin ($r = 0$). where they exhibit a strong square-root singularity. In addition, the stresses (σ_{11} and σ_{22}) along $\theta = 0$ are found to be zero reflecting the imposed boundary conditions (see Eq. (3.1)) while the shear stress (σ_{12}) along $\theta = 0$ is nonzero which can be attributed to the presence of boundary reinforcement. At the fixed boundary ($\theta = \pi$), σ_{11} and σ_{22} have non zero values whereas σ_{12} is found to be identically zero.

Remark 2. We have also investigated the possibility of more general (weakly) singular solutions corresponding to

$$\phi(z) = Az^{\rho} + Cz^{\rho}\ln z, \qquad \psi(z) = Bz^{\rho} + Dz^{\rho}\ln z. \tag{4.4}$$

In these cases, we find that the complex constants C and D in Eq. (4.4) are found to be identically zero so that we again arrive at the results in Eq. (3.24):

$$\phi(z) = Az^{1/2}, \qquad \psi(z) = Bz^{1/2}, \qquad B = A\left(\kappa - \frac{1}{2}\right).$$

5. Conclusions

We consider the local deformation near a point at the interface between free and fixed boundary segments in an elastic half-plane undergoing plane strain deformations. Using asymptotic analysis, we show that the addition of a reinforcement along the free boundary effectively eliminates the well-known oscillatory behavior of the displacement and stress fields in the vicinity of the point leading to a strong square-root singularity in the corresponding stress distributions and a displacement field which is smooth locally and bounded at the point of interest. The explicit form of the stresses near the origin is obtained displaying the anticipated discontinuity across the origin where the boundary data experiences an abrupt change. Finally, we note that, in contrast to the classical case [21], the shear stress (σ_{12}) along $\theta = 0$ is nonzero which can be attributed to the presence of boundary reinforcement.

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