Arch. Mech., **65**, 3, pp. 219–236, Warszawa 2013

Receding contact problem for a coated layer and a half-plane loaded by a rigid cylindrical stamp

H. ADIBELLI[1)], I. COMEZ[2)], R. ERDOL[2)]

[1)] *Civil Engineering Department*
Bozok University, 66900, Yozgat, Turkey
e-mail: ahandana@hotmail.com

[2)] *Civil Engineering Department*
Karadeniz Technical University
61080, Trabzon, Turkey

In this study, the frictionless contact problem between a coated layer and a half-plane is considered when they are pressed by a rigid cylindrical stamp. Upon loading, two unknown contact widths and contact pressures occur on the contact areas the advanced contact area between the stamp and the coating; and the receding contact area between the layer and the half-plane. This problem is reduced to two singular integral equations by using the Fourier transform and applying the boundary conditions of the problem. The numerical solution of the system of singular integral equations is obtained by applying the Gauss–Chebyshev integration formulas.

Key words: contact problem, coating, rigid stamp, Fourier transform.

1. Introduction

Due to the transmission of loads through contacting components of many structural and mechanical systems, the problems associated with these loads have garnered considerable interest of researchers. The contact areas and the distribution of contact stresses play fundamental role in engineering structures such as pavements in roads and runways, railway ballast, and foundations.

When two components contact each other without a bond, the applied loads cause the components to deform and the initial contact area decreases to a finite size. This type of contact is termed a receding contact, and the contact area and contact pressures are unknown for this problem. The frictionless receding contact problem between a layer and a half-plane was investigated with the theory of elasticity by [1–6]. The contact problem of an elastic layer lying on two elastic quarter-planes without a bond was solved by [7–9]. The contact problem of a multilayered medium based on integral transforms and matrix analyses is investigated by [10]. The frictionless contact between a rigid stamp and a surface coating-graded interlayer-substrate structure by employing the transfer matrix method and the Fourier transform is examined by [11]. The contact problem between a rigid stamp and a layered composite resting on a simple supports was studied by [12]. The continuous and discontinuous contact problem of two elastic layers resting on a half-plane in the presence of body forces is investigated by [13].

Many studies regarding the receding contact between a layer and a half-plane or a bonded contact of a layered medium have been reported in the literature. In contrast, this paper examines the receding contact problem of a coated layer resting on a half-plane within the framework of linear elasticity theory. A concentrated force is applied to the top surface of a coating through a rigid cylindrical stamp. It is assumed that the effects of friction and the body forces are neglected and that only compressive normal tractions can be transmitted through the contact interfaces. The problem is reduced to a system of singular integral equations for which the contact areas and the contact pressures acting on these areas are unknown given the boundary conditions of the problem and the Fourier integral transform. The system of singular integral equations is solved numerically with the Gauss–Chebyshev integration formulas. Numerical results are given for the contact areas and the contact pressures as a function of various dimensionless quantities.

2. General expressions

The plane strain contact problem under consideration is shown in Fig. 1. A concentrated normal force P is applied to the system by means of a rigid cylindrical stamp of radius R. The elastic layers in the system are dissimilar, the upper one is termed the coating layer and the lower one is simply termed as the layer in this paper. The layers are fully bonded to each other, and the contact

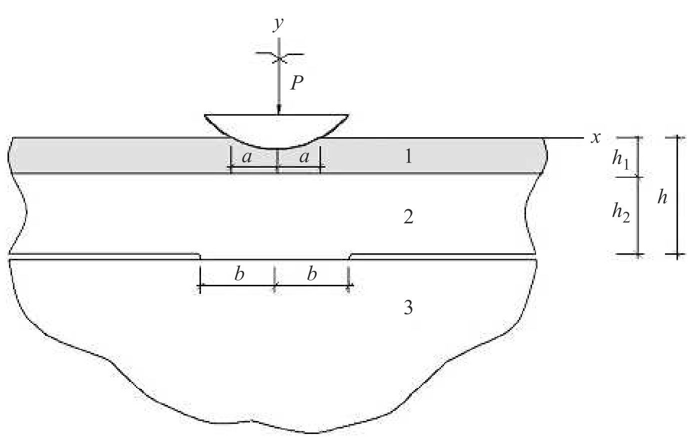

Fig. 1. Geometry of the problem.

between the layer and the half-plane is smooth. Upon loading, the contact half-width between the rigid stamp and the coating increases to a. On the other hand, the contact half-width between the layer and the half-plane decreases to a finite size b.

In the absence of body forces, the two-dimensional Navier equations can be written in terms of the displacement components as follows:

$$
\begin{aligned}
&(\lambda_i+\mu_i)\frac{\partial}{\partial x}\left[\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right]+\mu_i\nabla^2 v_i=0,\\
&(\lambda_i+\mu_i)\frac{\partial}{\partial y}\left[\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right]+\mu_i\nabla^2 v_i=0,
\end{aligned}
\tag{2.1}
$$

where u and v are the x and y components of the displacement vector, respectively, and and μ_i are the Lamé constants. The subscript i $(i = 1, 2, 3)$ refers to the coating layer (upper layer), the layer (lower layer) and the half-plane, respectively.

Observing that $x = 0$ is a plane symmetry, it is sufficient to consider the problem in the region $0 \le x < \infty$ only. Considering the Fourier transformation of the unknown functions ϕ_i and ψ_i, components of the displacement vector $u_i(x, y)$, $v_i(x, y)$ can be expressed as:

$$
\begin{aligned}
u_i(x,y)&=\frac{2}{\pi}\int_0^\infty \phi_i(\alpha,y)\sin(\alpha x)\,d\alpha,\\
v_i(x,y)&=\frac{2}{\pi}\int_0^\infty \psi_i(\alpha,y)\cos(\alpha x)\,d\alpha.
\end{aligned}
\tag{2.2}
$$

Upon substituting (2.2) into (2.1), the Navier equations reduce to two ordinary differential equations. Upon solving the resulting differential equations, the displacements and stresses can be expressed as follows:

$$
\begin{aligned}
u_i(x,y)&=\frac{2}{\pi}\int_0^\infty [(A_i+B_i y)e^{-\alpha y}+(C_i+D_i y)e^{\alpha y}]\sin(\alpha\, x)\,d\alpha,\\
v_i(x,y)&=\frac{2}{\pi}\int_0^\infty \left\{\left[A_i+\left(\frac{\kappa_i}{\alpha}+y\right)B_i\right]e^{-\alpha y}\right.\\
&\qquad\left.+\left[-C_i+\left(\frac{\kappa_i}{\alpha}-y\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha,
\end{aligned}
\tag{2.3}
$$

$$\frac{1}{2\mu_i}\sigma_{xi}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{ \left[\alpha(A_i + B_i y) - \frac{3-\kappa_i}{2}B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) + \frac{3-\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha,$$

$$\frac{1}{2\mu_i}\sigma_{yi}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{ -\left[\alpha(A_i + B_i y) + \frac{1+\kappa_i}{2}B_i\right]e^{-\alpha y} + \left[-\alpha(C_i + D_i y) + \frac{1+\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{2.4}$$

$$\frac{1}{2\mu_i}\tau_{xyi}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{ -\left[\alpha(A_i + B_i y) + \frac{\kappa_i-1}{2}B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) - \frac{\kappa_i-1}{2}D_i\right]e^{\alpha y}\right\}\sin(\alpha x)\,d\alpha,$$

where $\kappa_i = 3 - 4v_i$ for plain strain, and v_i is Poisson's ration. A_i, B_i, C_i and D_i $(i = 1, 2, 3)$ are unknown functions that will be determined by applying the boundary conditions of the problem. Because the stress components vanish at $y \to -\infty$, A_3 and B_3 (which pertain to the expressions of the half-plane) must be zero $(A_3 = B_3 = 0)$.

3. Boundary conditions and singular integral equations

The govering equations of the problem are subjected to the following boundary conditions prescribed at $y = 0$, $y = -h_1$ and $y = -h$:

$$\begin{aligned}
&\sigma_{y1}(x,0) = \begin{cases} -p_1(x), & 0 \le x < a, \\ 0, & a \le x < \infty, \end{cases} \\
&\tau_{xy1}(x,0) = 0, && 0 \le x < \infty, \\
&\sigma_{y1}(x,-h_1) = \sigma_{y2}(x,-h_1), && 0 \le x < \infty, \\
&\tau_{y1}(x,-h_1) = \tau_{xy2}(x,-h_1), && 0 \le x < \infty, \\
&u_1(x,-h_1) = u_2(x,-h_1), && 0 \le x < \infty, \\
&v_1(x,-h_1) = v_2(x,-h_1), && 0 \le x < \infty, \\
&\sigma_{y2}(x,0) = \begin{cases} -p_2(x), & 0 \le x < b, \\ 0, & b \le x < \infty, \end{cases} \\
&\tau_{xy2}(x,-h) = 0, && 0 \le x < \infty, \\
&\tau_{xy3}(x,-h) = 0, && 0 \le x < \infty, \\
&\sigma_{y2}(x,-h) = \sigma_{y3}(x,-h), && 0 \le x < \infty,
\end{aligned} \tag{3.1}$$

$$\frac{\partial v_1(x,0)}{\partial x} = f(x), \qquad 0 \le x < a,$$
$$\frac{\partial v_2(x,-h)}{\partial x} = \frac{\partial v_3(x,-h)}{\partial x}, \qquad 0 \le x < b, \tag{3.2}$$

where a if the half-width of the contact area under the stamp, b is the half-width of the contact area between the layer and the half-plane, and $p_1(x)$ and $p_2(x)$ are the unknown contact pressures om the contact areas a and b, respectively, $f(x)$ is a known function that gives the derivative of the cylindrical stamp profile:

$$f(x) = x/R. \tag{3.3}$$

Upon taking the Fourier transform of the boundary conditions given by (3.1), the unknown constants A_i, B_i, C_i and D_i, (appearing in the displacement and stress expressions) can be determined in terms of the unknown contact pressures $p_1(x)$ and $p_2(x)$. Substituting unknown constants into the mixed boundary conditions (3.2) provides the following system of singular integral equations:

$$\int_0^a p_1(t_1)\left[\frac{1}{t_1+x_1} - \frac{1}{t_1-x_1} + 2K_{11}(x_1,t_1)\right] dt_1 + 2\int_0^b p_2(t_2)[K_{12}(x_1,t_2)]\, dt_2 = f(x), \tag{3.4a}$$

$$\frac{\partial}{\partial x}[v_2(x,y) - v_3(x,y)] = 2\int_0^a p_1(t_1)[K_{21}(x_2,t_1)]\, dt_1 + \int_0^b p_2(t_2)\left[\frac{1}{t_2+x_2} - \frac{1}{t_2-x_2} + 2K_{22}(x_2,t_2)\right] dt_2 = 0. \tag{3.4b}$$

Observing that the contact stress functions are double functions $p_1(t_1) = p_1(-t_1)$ and $p_2(t_2) = p_2(-t_2)$, Eqs. (3.4a) and (3.4b) can be written as Eqs. (3.4c) and (3.4d):

$$\frac{1}{\pi}\int_{-a}^{a} p_1(t_1)\left[\frac{1}{t_1-x_1} + K_{11}(x_1,t_1)\right] dt_1 + \frac{1}{\pi}\int_{-b}^{b} p_2(t_2)[K_{12}(x_1,t_2)]\, dt_2, = \frac{4\mu_1}{\kappa_1+1}\frac{x_1}{R}, \tag{3.4c}$$

$$\frac{1}{\pi}\int_{-a}^{a} p_1(t_1)[K_{21}(x_2,t_1)]\, dt_1 + \frac{1}{\pi}\int_{-b}^{b} p_2(t_2)\left[\frac{1}{t_2-x_2} + K_{22}(x_2,t_2)\right] dt_2 = 0. \tag{3.4d}$$

The expressions for $K_{11}(x_1,t_1)$, $K_{12}(x_1,t_2)$, $K_{21}(x_2,t_1)$ and $K_{22}(x_2,t_2)$ from Eq. (3.4) are given in Appendix A.

The solution of the system of singular integral equations must satisfy the following equilibrium conditions:

$$\int_0^a p_1(t_1)\,dt_1 = \frac{P}{2}, \qquad \int_0^b p_2(t_2)\,dt_2 = \frac{P}{2}. \tag{3.5}$$

4. Numerical solution of the system of singular integral equations

By introducing the following normalizations:

$$\begin{aligned} &x_1 = as_1, \qquad t_1 = ar_1,\\ &x_2 = bs_2, \qquad t_2 = br_2,\\ &G_1(r_1) = \frac{p_1(r_1)}{P/h_2}, \qquad G_2(r_2) = \frac{p_2(r_2)}{P/h_2}, \end{aligned} \tag{4.1}$$

the singular integral equations (3.4c, d) may be expressed in the following form:

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1} G_1(r_1)\left[\frac{1}{r_1-s_1} + K_{11}^*(s_1,r_1)\right] dr_1\\ &\qquad\qquad + \frac{1}{\pi}\int_{-1}^{1} G_2(r_2)[K_{12}^*(s_1,r_2)]\,dr_2 = f(s_1),\\ &\frac{1}{\pi}\int_{-1}^{1} G_1(r_1)[K_{21}^*(s_2,r_1)]\,dr_1\\ &\qquad\qquad + \frac{1}{\pi}\int_{-1}^{1} G_2(r_2)\left[\frac{1}{r_2-s_2} + K_{22}^*(s_2,r_2)\right] dr_2 = 0, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &K_{11}^*(s_1,r_1) = \frac{a}{h_2}K_{11}(x_1,t_1), \qquad K_{12}^*(s_1,r_2) = \frac{b}{h_2}K_{12}(x_1,t_2),\\ &K_{21}^*(s_2,r_1) = \frac{a}{h_2}K_{21}(x_2,t_1), \qquad K_{22}^*(s_2,r_2) = \frac{b}{h_2}K_{22}(x_2,t_2), \end{aligned} \tag{4.3}$$

$$f(s_1) = \frac{4}{\kappa_1+1}\frac{\mu_1}{P/h_2}\frac{1}{R/h_2}\frac{a}{h_2}s_1. \tag{4.4}$$

Similarly, the equilibrium conditions (4.2) become

$$\frac{a}{h_2}\int_{-1}^{1} G_1(r_1)dr_1 = 1, \qquad \frac{b}{h_2}\int_{-1}^{1} G_2(r_2)dr_2 = 1. \tag{4.5}$$

The singular integral equations have an index −1 due to the smooth contact at the end points a and b [14, 15]. The solution of the integral equations may be expressed as

$$G_\eta(r_\eta) = g_\eta(r_\eta)w_\eta(r_\eta), \qquad \eta = 1, 2, \tag{4.6}$$

where $g_\eta(r_\eta)$ is an unknown function that is bounded and continuous in the intervals $-1 \le r_\eta \le 1$ and where $w_\eta(r_\eta)$ is the corresponding weight function:

$$w_\eta(r_\eta) = (1 - r_\eta^2)^{1/2}. \tag{4.7}$$

If one considers the Gauss–Chebyshev integration formulas [14] for the bounded functions $g_\eta(r_\eta)$, then the integral equations and the equilibrium conditions become:

$$\begin{aligned}
&\sum_{j=1}^{N} W_{1j}^N\left[\frac{1}{r_{1j}-s_{1k}} + K_{11}^*(s_{1k}, r_{1j})\right]g_1(r_{1j})\\
&\quad + \sum_{j=1}^{N} W_{1j}^N K_{12}^*(s_{1k}, r_{2j})\, g_2(r_{2j}) = f(s_{1k}), \quad k = 1, 2, \dots, N+1,\\
&\sum_{j=1}^{N} W_{2j}^N K_{21}^*(s_{2k}, r_{1j}) g_1(r_{1j})\\
&\quad + \sum_{j=1}^{N} W_{2j}^N\left[\frac{1}{r_{2j}-s_{2k}} + K_{22}^*(s_{2k}, r_{2j})\right]g_2(r_{2j}) = 0, \quad k = 1, 2, \dots, N+1,
\end{aligned} \tag{4.8}$$

$$\frac{a}{h_2}\sum_{j=1}^{N} W_{1j}^N g_1(r_{1j}) = 1, \qquad \frac{b}{h_2}\sum_{j=1}^{N} W_{2j}^N g_2(r_{2j}) = 1, \tag{4.9}$$

where $r_{\eta j}$ and $s_{\eta k}$ are the collocation points and $W_{\eta j}^N$ is the weighting constant, which can be determined as follows:

$$\begin{aligned}
r_{\eta j} &= \cos\left(\frac{j\pi}{N+1}\right), \qquad k = 1, 2, \dots, N,\\
s_{\eta k} &= \cos\left(\pi\frac{2k-1}{2N+2}\right), \quad k = 1, 2, \dots, N+1,\\
W_{\eta j}^N &= \frac{1 - r_{\eta j}^2}{N+1}.
\end{aligned} \tag{4.10}$$

Because there are $2N + 2$ equations in (4.8) to determine only $2N$ unknowns $g_\eta(r_\eta)$, $(N/2 + 1)$-th equations, (automatically satisfied) are ignored in (4.8).

Thus, (4.8) and (4.9) give $2N + 2$ algebraic equations to determine $2N + 2$ unknowns, which are $g_\eta(r_\eta)$ and the contact widths a and b. Because the system of equations is nonlinear in a and b, an iterative procedure is used to obtain the two unknown contact widths.

5. Numerical results

The calculated results are the contact widths and and the contact pressures $P_1(x)/(P/h_2)$ and $P_2(x)/(P/h_2)$ acting on the contact areas, for various dimensionless quantities, such as h_1/h_2, μ_1/μ_2, μ_3/μ_2, R/h_2, $\mu_2/(P/h_2)$, κ_1, κ_2 and κ_3.

The variation of contact width withh_1/h_2 is given in Table 1. As it is seen in Table 1, with increasing values of h_1/h_2, the contact width between the layer and the half-plane increases but the contact width under the stamp decreases. Along with increasing values of the coating height, this behavior causes the distance between the application point of the load and the half-plane to increase and the stress on the contact area to decrease, which causes separation of the layer from the half-plane.

Table 1. Variation of the contact widths for various values of h_1/h_2 ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 1$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

	Contact widths	
h_1/h_2	a/h_2	b/h_2
Hertzian contact	0.6909883	∞
50	0.6910218	104.06096
10	0.6916101	15.116869
2	0.6990829	4.0611766
1	0.7082785	2.7457192
0.5	0.7194743	2.1071227
0.1	0.7365187	1.6202971
0	0.7424830	1.5051292
Comez [16]	0.7424830	1.5051292
Kahya *et al.* [17]	0.74260	1.50260

As the coating becomes stiffer, the contact width under the stamp a/h_2 decreases, but the contact width between the layer and the half-plane b/h_2 increases (Figs. 2 and 3). Both a/h_2 and b/h_2 increase as the half-plane becomes softer. The variation of μ_1 influences a/h_2 more than it does b/h_2. Contrarily, the variation of μ_3 has a greater effect on b/h_2 than it does on μ_1.

The maximum contact pressures occur in the symmetry plane $x = 0$, and vanish towards the end of the contact areas (Figs. 2–11).

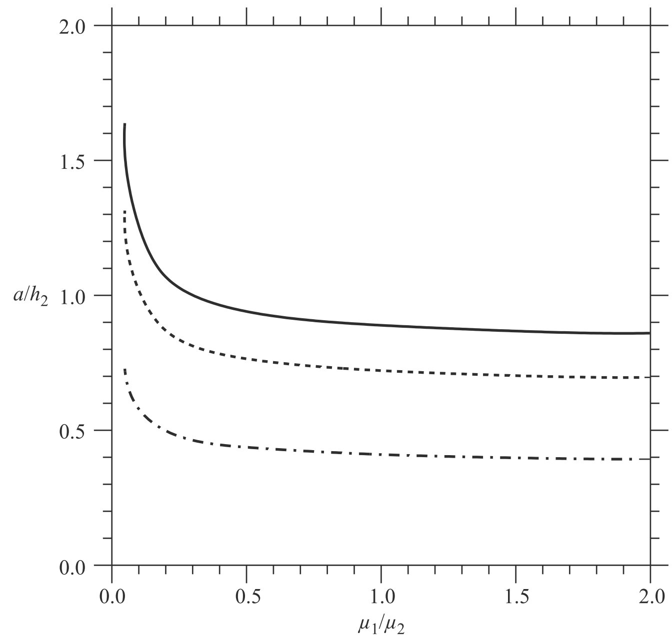

FIG. 2. Variation of the contact widths under the stamp with μ_3/μ_2 ($h_1/h_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

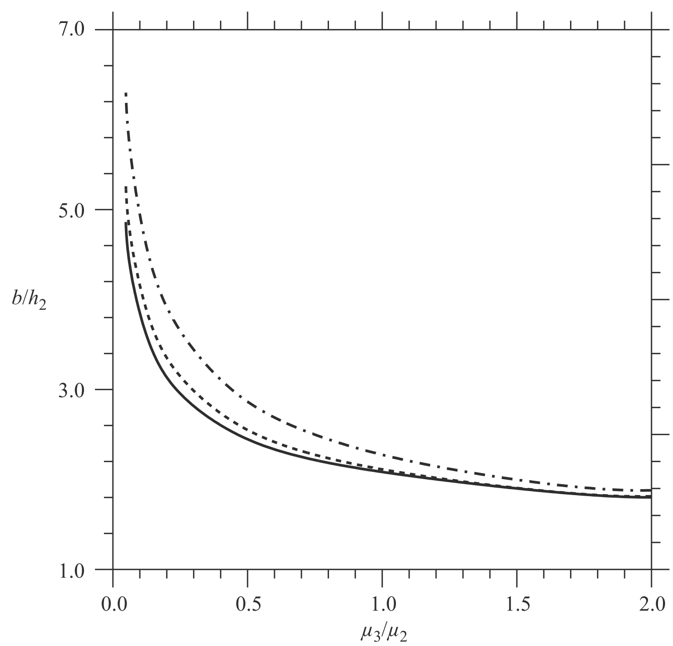

FIG. 3. Variation of the contact widths between the layer and the half-plane with μ_1/μ_2 and μ_3/μ_2 ($h_1/h_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

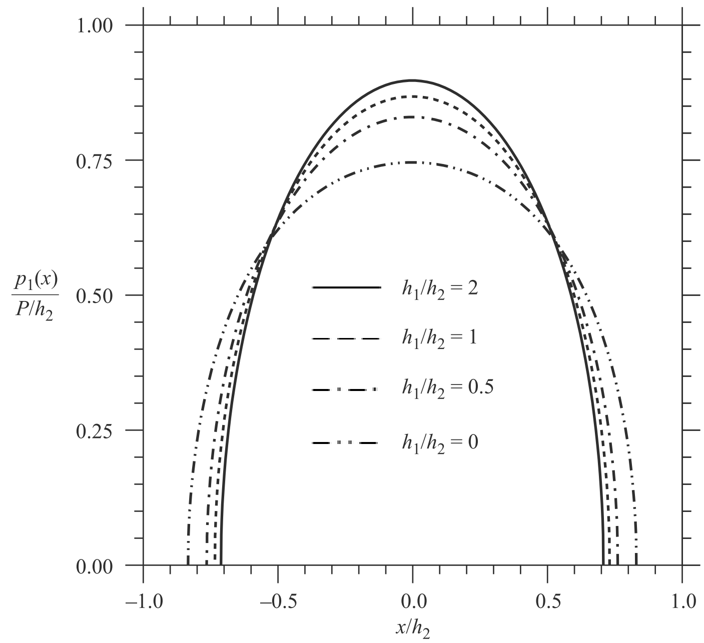

FIG. 4. Contact pressure distribution under the stamp for various values of h_1/h_2 ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

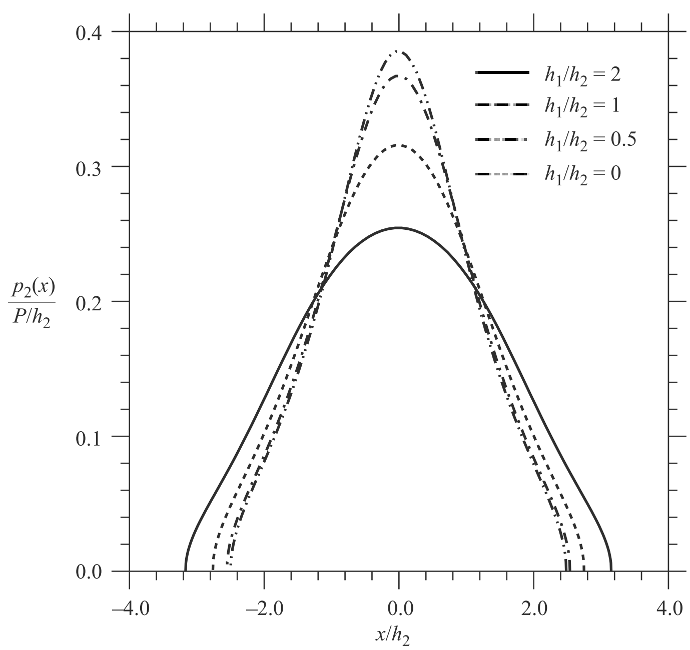

FIG. 5. Contact pressure distribution between the layer and the half-plane for various value of h_1/h_2 ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

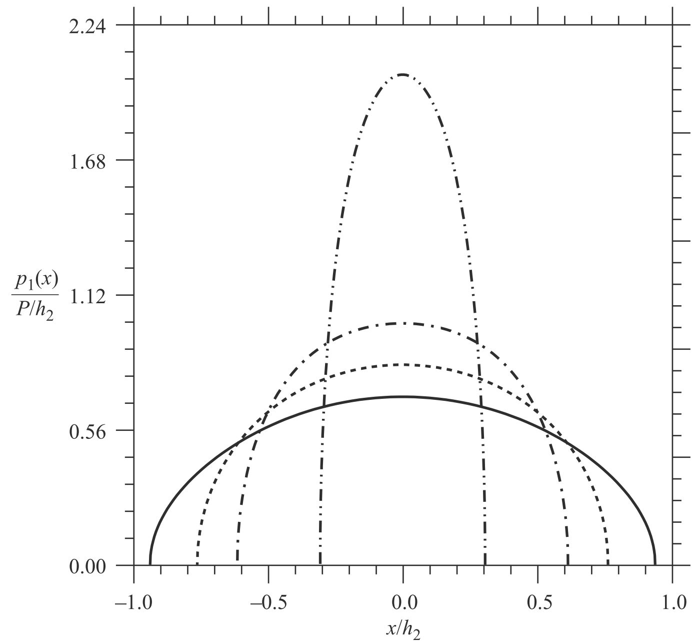

Fig. 6. Contact pressure distribution under the stamp with μ_1/μ_2 ($h_1/h_2 = 0.5$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

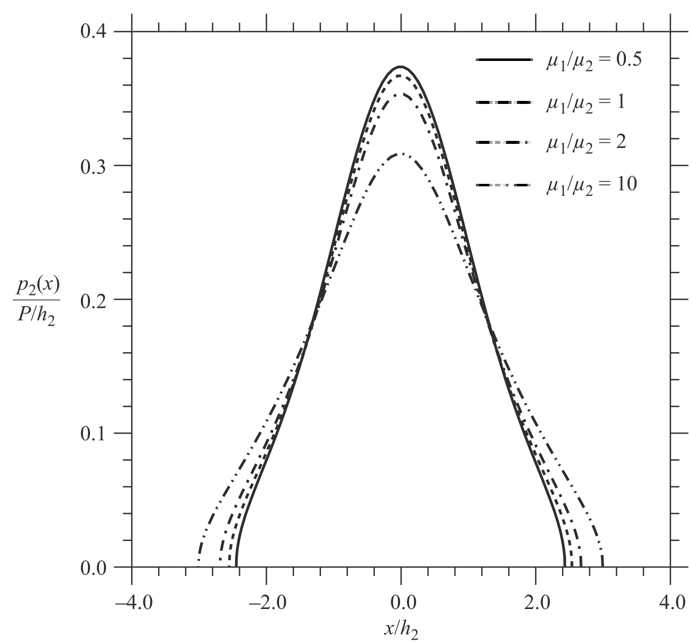

Fig. 7. Contact pressure distribution between the layer and the half-plane with μ_1/μ_2 ($h_1/h_2 = 0.5$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

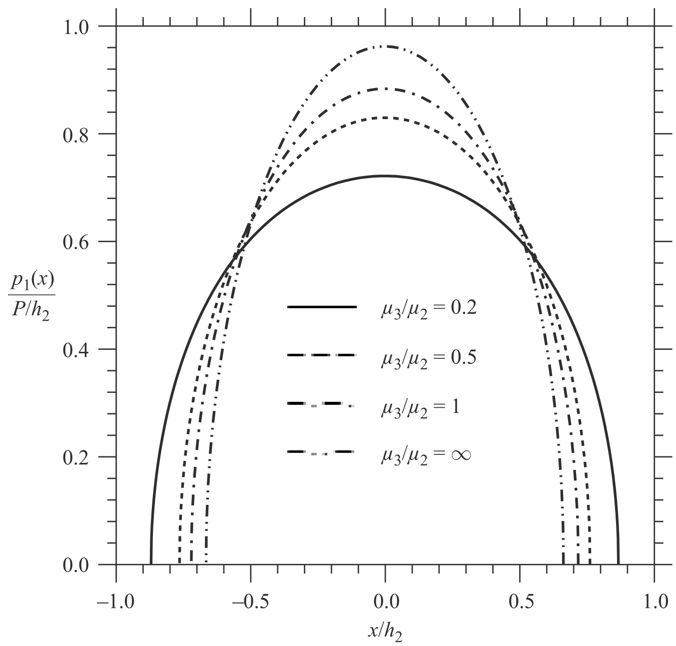

Fig. 8. Effect of the shear ratio, μ_3/μ_2, on the contact pressure distribution under the stamp ($h_1/h_2 = 0.5$, $\mu_1/\mu_2 = 1$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

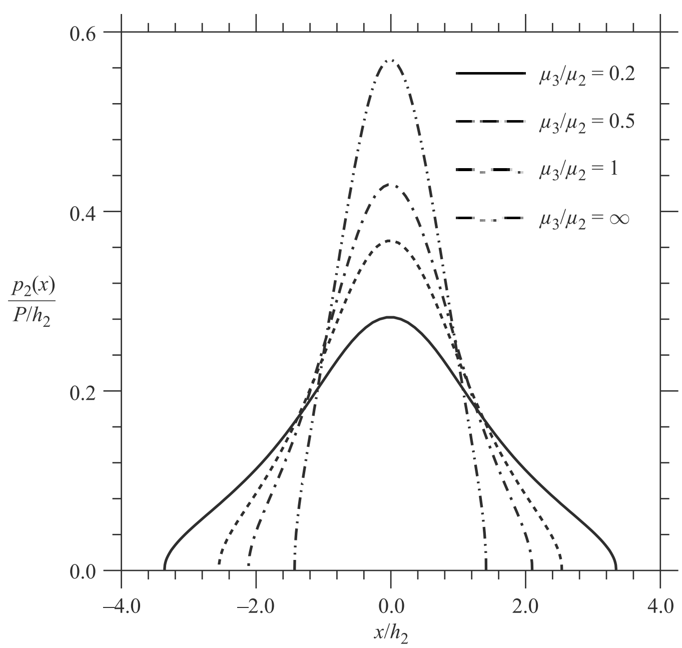

Fig. 9. Effect of the shear ratio, μ_3/μ_2, on the contact pressure distribution between the layer and the half-plane ($h_1/h_2 = 0.5$, $\mu_1/\mu_2 = 1$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

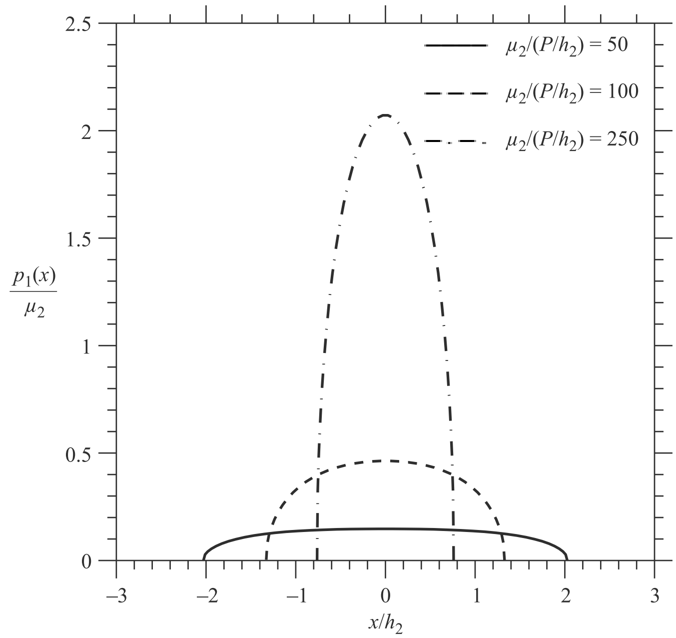

FIG. 10. Effect of the $\mu_2/(P/h_2)$ on the contact pressure distribution under the stamp ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $h_1/h_2 = 0.5$, $\kappa_i = 2$).

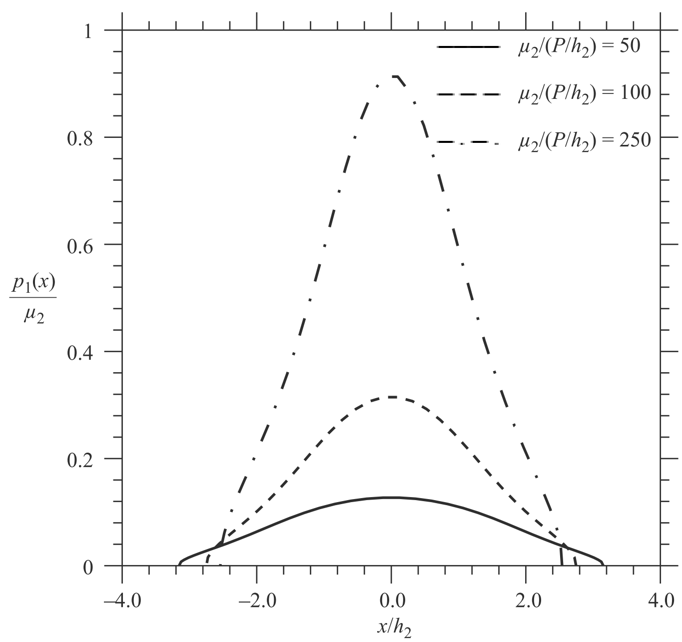

FIG. 11. Effect of the $\mu_2/(P/h_2)$ on the contact pressure distribution between the layer and the half-plane ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 0.5$, $R/h_2 = 250$, $h_1/h_2 = 0.5$, $\kappa_i = 2$).

Figures 4 and 5 show the contact pressure distributions ($P_1(x)/(P/h_2)$ and $P_2(x)/(P/h_2)$) for various values of h_1/h_2, respectively. As the coating height h_1 decreases, the peak of contact pressure under the stamp decreases, but the peak of contact pressure between the layer and the half-plane increases.

Figures 6 and 7 show the effect of μ_1/μ_2 on the contact pressure distributions. Note that the coating stiffens with increasing values of μ_1 and that $P_1(x)/(P/h_2)$ increases while $P_2(x)/(P/h_2)$ decreases. For the case of $\mu_1/\mu_2 = 10$ (the stiffer coating), the maximum occurs at $P_1(x)/(P/h_2)$.

The variation of contact pressure distributions on the contact areas with μ_3/μ_2 are given in Figs. 8 and 9. Contact pressures acting on the contact areas increase with increasing values of μ_3. These figures show that μ_3 has a greater effect on $P_2(x)/(P/h_2)$ than it does on $P_1(x)/(P/h_2)$. In this study, if the shear modulus of the half-plane goes to infinity, ($\mu_3/\mu_2 \to \infty$), then contact pressure distributions associated with contact problems of double layers resting on rigid substrates can be obtained. The maximum values of the contact pressures occur for the case of $\mu_3 \to \infty$.

Figures 10 and 11 show the contact pressure distributions on the contact areas for various values of $\mu_1/(P/h_2)$. When the value of the concentrated load increases, $\mu_1/(P/h_2)$ decreases, and contact pressures acting on the contact areas decrease.

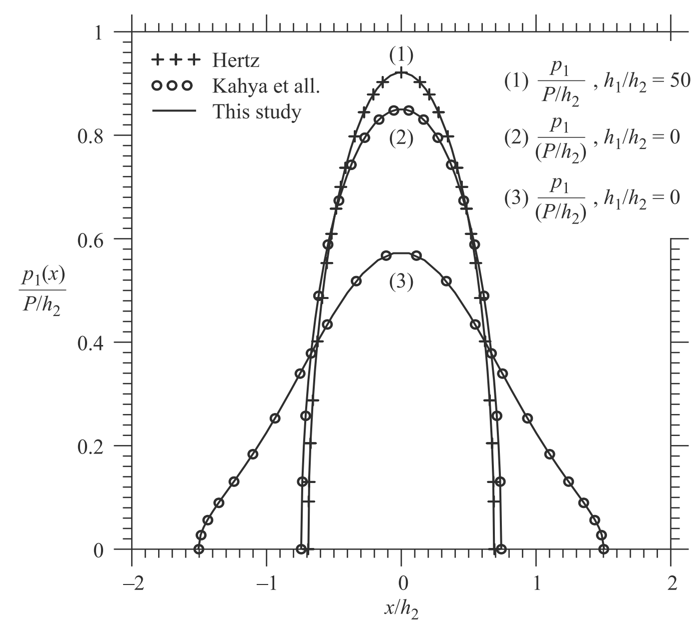

FIG. 12. Comparison of contact pressures for various values of h_1/h_2 ($\mu_1/\mu_2 = 1$, $\mu_3/\mu_2 = 1$, $R/h_2 = 250$, $\mu_2/(P/h_2) = 250$, $\kappa_i = 2$).

In this study, when the material properties of the coating and the layer are selected to be the same ($\mu_1 = \mu_2$, $\kappa_1 = \kappa_2$) or the height of the coating is taken as zero ($h_1 = 0$), the problem reduces the contact problem of an uncoated layer lying on a half-plane. The comparison of different studies for the contact widths and the contact pressure distributions are given in Table 1 and Fig. 12 for various values of h_1/h_2. Our results for contact widths on the contact areas obtained by setting $h_1/h_2 = 0$ are identical to those by COMEZ [16] and KAHYA *et. al.* [17]. Furthermore, with increasing values of h_1/h_2 the contact width under the stamp is close to the Hertzian contact width [18]. This means that if the height of layer is too high ($h_1/h_2 > 10$) the layer behaves as a half-plane (Table 1).

6. Conclusions

The frictionless receding contact problem for a coated layer and a half-plane loaded by a rigid cylindrical stamp is solved using the theory of linear elasticity and Fourier transform.

Maximum values of the contact pressures occur in the symmetry plane $x = 0$. The values gradually decrease in the symmetry plane.

The contact pressure under the stamp can be reduced by selecting thinner and softer coatings. The maximum values of the contact pressures occur for the case of $\mu_3 = \infty$.

As the coating height increases, the contact width between the layer and the half-plane increases, but the contact width under the stamp decreases. As expected, the layer behaves as a half-plane when the height of the layer is too high ($h_1/h_2 > 10$).

Appendix A

Expressions of the Fredholm kernels $K_{11}(x_1, t_1)$, $K_{12}(x_1, t_2)$, $K_{21}(x_2, t_1)$ and $K_{22}(x_2, t_2)$ appearing in Eq. (4.8) are given below:

$$
\begin{aligned}
\text{(A.1)} \qquad K_{11}(x_1, t_1) = \frac{1}{1+\kappa_1} \int\limits_0^\infty & \frac{1}{\Delta}(1+\kappa_1) \\
& \times \{-4\alpha h_1(-1+m)(\kappa_2+m)e^{-2\alpha h_1}(e^{-4\alpha h_2}+1) \\
& + (1+\kappa_1 m)(\kappa_2+m)(1+e^{-4\alpha h_1 - 4\alpha h_2}) \\
& + (-1+m)(-\kappa_2+m\kappa_1)(e^{-4\alpha h_2} - e^{-4\alpha h_1}) \\
& + ((-1+m)(1+\kappa_1 m)(4\alpha^2 h_2^2+1)
\end{aligned}
$$

$$+ (\kappa_2 - \kappa_1 m)(\kappa_2 + m))(e^{-2\alpha h_2} - e^{-4\alpha h_1 - 2\alpha h_2})$$
$$+ (4\alpha h_2(4\alpha^3 h_1 h_2(-1+m)^2 + (1+\kappa_1)(1+\kappa_2)m)$$
$$+ 4\alpha h_1((\kappa_2 + m)^2 + (1-m)^2))$$
$$\times (e^{-2\alpha h_1 - 2\alpha h_2}) - [1+\kappa_1]\Big\} \sin(\alpha x_1)\cos(\alpha t_1)\, d\alpha,$$

(A.2)
$$K_{12}(x_1, t_2) = \frac{1}{1+\kappa_1}\int_0^\infty \frac{G_1}{G_2 \varDelta} e^{-5\alpha h_1 - 3\alpha h_2}(1+\kappa_1)(1+\kappa_2)m$$
$$\times \Big\{e^{-4\alpha h_1}(1 + \kappa_2 + (-2a(h_1 - h_2) + 4a^2 h_1 h_2)$$
$$- m(1+\kappa_1)e^{2\alpha(h_1+h_2)}(-1 - \kappa_2 - 2a(-1+m)$$
$$\times ((h_1 - h_2) + 2ah_1 h) + m(1+\kappa_1))e^{2\alpha h_1}((1+\kappa_1 m)$$
$$\times (-1 + 2ah_2) + (\kappa_2 + m)(-1 + 2ah_1))$$
$$+ e^{2\alpha(2h_1+h_2)}((1 + k_1 m)(1 + 2ah_2)$$
$$+ (\kappa_2 + m)(1 + 2ah_1)))\Big\} \sin(\alpha x_1)\cos(\alpha t_2)\, d\alpha,$$

(A.3)
$$K_{21}(x_2, t_1) = \frac{nm}{n(1+\kappa_2) + 1 + \kappa_3}$$
$$\times \int_0^\infty \frac{1}{\varDelta}\big\{e^{-5\alpha h_1 - 3\alpha h_2}(1+\kappa_1)(1+\kappa_2)(e^{2a(h_1+h_2)}$$
$$\times (1 + \kappa_2 2\alpha(-1+m)(h_1 - h_2 + 2\alpha h_1 h_2) - m(1+\kappa_1))$$
$$+ e^{4\alpha h_1}(-1 - \kappa_2 + m(1+\kappa_1) + 2\alpha(-1+m)$$
$$\times (h_1 - h_2 - 2\alpha h_1 h_2)) + e^{2\alpha h_1}((\kappa_2 + m)(1 - 2\alpha h_1)$$
$$+ (1 + \kappa_1 m)(1 - 2\alpha h_2)) - e^{2\alpha(2h_1+h_2)}((1+\kappa_1 m)(1 + 2\alpha h_2)$$
$$+ (\kappa_2 + m)(1 + 2\alpha h_1)))\big\} \sin(\alpha x_2)\cos(\alpha t_1)\, d\alpha,$$

(A.4)
$$K_{22}(x_2, t_2) = -\frac{n}{n(1+\kappa_2) + 1 + \kappa_3}$$
$$\times \int_0^\infty \frac{1}{\varDelta}\big\{e^{-4\alpha h_1 - 4\alpha h_2}(1+\kappa_2)(16\alpha^3 h_1^2 h_2(-1+m^2)(e^{2\alpha h_1 + 2\alpha h_2})$$
$$+ 4a^2 h_1^2(-1+m)(\kappa_2 + m)(-1 + e^{-4\alpha h_2})e^{2\alpha h_1} + (-1 + e^{2ah_1})\kappa_2$$
$$\times (1 + (e^{2\alpha h_1} + e^{4\alpha h_2})(-1+m) + \kappa_1 m + e^{2\alpha(h_1+2h_2)}(1+\kappa_1 m))$$
$$- m(\kappa_1(-1+m)(e^{4\alpha h_1} - e^{4h_2}) + (1 - e^{4\alpha(h_1+h_2)})(1+\kappa_1 m)$$

$$+(-1+\kappa_1+m(1+\kappa_1^2))e^{2\alpha h_1}(1-e^{4\alpha h_2}))+4\alpha e^{2\alpha h_2}(e^{2\alpha h_1}h_1$$
$$\times(1+\kappa_1)(1+\kappa_2)m+h_2(1+e^{4\alpha h_1})(-1+m)+(1+\kappa_1 m)$$
$$\times e^{2\alpha h_1}(2+2m(-1+\kappa_1)+m(1+\kappa_1^2))))+[1+\kappa_2]\}$$
$$\times\sin(\alpha x_2)\cos(\alpha t_2)\,d\alpha,$$

$$\begin{aligned}(\text{A.5})\quad \Delta = \big\{&(\kappa_2+m)(1+m\kappa_1)(e^{-4\alpha h_1-4\alpha h_2}+1)\\ &+(-1+m)(-\kappa_2+m\kappa_1)(e^{-4\alpha h_1}+e^{-4\alpha h_2})\\ &+e^{-2\alpha h_2}(1+e^{-4\alpha h_1})(1+\kappa_2^2-m+m(\kappa_1+\kappa_2-\kappa_1\kappa_2-2\kappa_1 m)\\ &-4\alpha^2h_2^2(-1+m)(1+m\kappa_1))+e^{-2\alpha h_1}(1+e^{-4\alpha h_2})\\ &\times(\kappa_2(-2+4\alpha^2h_1^2(-1+m)+m(1-\kappa_1))\\ &+m(-1+\kappa_1+4\alpha^2h_1^2(-1+m)+m(1+\kappa_1^2))-2e^{-2\alpha_1-2\alpha h_2}\\ &(1+\kappa_2^2+8\alpha^4h_1^2h_2^2(-1+m)^2-m(1-\kappa_1-\kappa_2-\kappa_1\kappa_2)\\ &+m^2(1+\kappa_1^2)+2\alpha^2((h_1+h_2)^2(2+2(-1+\kappa_1)m\\ &+(1+\kappa_1^2)m^2)-2h_1(h_1+h_2)(2-(3+\kappa_1(-1+\kappa_2)+\kappa_2)m\\ &+(1+\kappa_1^2)m^2)+h_1^2(3+\kappa_2^2-6m-2m\kappa_1\kappa_2+(3+\kappa_1^2)m^2))))\big\}n,\end{aligned}$$

where m and n can be written as $m=\mu_2/\mu_1$ and $n=\mu_3/\mu_2$.

References

1. Y. WEITSMAN, *On the unbounded contact between plates and an elastic half space*, J. Appl. Mech., ASME, **36**, 198–202, 1969.

2. L.M. KEER, J. DUNDURS, K.C. TSAI, *Problems involving a receding contact between an elastic layer and half-space*, J. Appl. Mech., Trans. ASME, **39**, 1115–1120, 1972.

3. M. RATWANI, F. ERDOĞAN, *On the plane contact problem for a frictionless elastic layer*, Int. J. Solid Struct., **9**, 8, 921–936, 1973.

4. M.B. CIVELEK, F. ERDOĞAN, *The axisymmetric double contact problem for a frictionless elastic layer*, Int. J. Solids Struct., **10**, 6, 639–659, 1974.

5. M.R. GEÇIT, *The axisymmetric double contact problem for a frictionless elastic layer indented by an elastic cylinder*, Int. J. Eng. Sci., **24**, 9, 1571–1584, 1986.

6. S.E. EL-BORGI, R. ABDELMOULA, L. KEER, *A receding contact plane problem between functionally graded layer and a homogeneous substrate*, Int. J. Solid Struct., **43**, 3, 658–674, 2006.

7. F. ERDOGAN, M. RATWANI, *The contact problem for an elastic layer supported by two elastic quarter planes*, J. Appl. Mech., Trans. ASME, **41**, 673–677, 1974.

8. H. BUFLER, *Theory of elasticity of a multilayered medium*, J. Elast., **1**, 125–143, 1971.

9. M.J. PINDERA, M.S. LANE, *Frictionless contact of layered half-planes, part II, numerical results*, J. Appl. Mech., ASME, **60**, 640–645, 1993.

10. W.K. BINIENDA, M.J. PINDERA, *Frictionless contact of layered metal matrix and polymer matrix composite half-planes*, Compos. Sci. Technol., **50**, 119–128, 1994.

11. J. YANG, L. KE, *Two-dimensional contact problem for a coating-graded layer-substrate structure under a rigid cylindrical punch*, Int. J. Mech. Sci., **50**, 6, 985–994, 2008.

12. A. BIRINCI, R. ERDÖL, *Frictionless contact between a rigid stamp and an elastic layered composite resting on simple supports*, Math. & Comput. Appl., **4**, 3, 261–272, 1999.

13. F.L. ÇAKIROĞLU, M. ÇAKIROĞLU, R. ERDÖL, *Contact problems for two elastic layers resting on elastic half-plane*, J. Eng. Mech., **127**, 2, 113–118, 2001.

14. F. ERDOGAN, G.D. GUPTA, *On the numerical solution of singular integral equations*, Quaterly Journal of Applied Mathematics, **29**, 525–534, 1972.

15. N.I. MUSKHELISHVILI, *Singular Integral Equations*, Noordhoff, Leyden, The Netherlands, 1958.

16. İ. COMEZ, *Frictional contact problem for a rigid cylindrical stamp and an elastic layer resting on a half-plane*, International Journal of Solids And Structures, **47**, 7–8, 1090–1097, 2010.

17. V. KAHYA, T.S. OZSAHIN, A. BIRINCI, R. ERDOL, *A receding contact problem for an anisotropic elastic medium consisting of a layer and a half-plane*, International Journal of Solids and Structures, **44**, 5695–5710, 2007.

18. K.L. JOHNSON, *Contact Mechanics*, Cambridge University Press, Cambridge, London, 1985.

Received October 12, 2012; revised version April 1, 2013.