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Brief Notes

On consequences of the principle of stationary action for dissipative bodies

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Dedicated to the memory of Professor Małgorzata (Margaret) Woźniak

The aim of this note is to show possible consequences of the principle of stationary action formulated for dissipative bodies. The material structure with internal state variables is considered for those bodies. The appropriate action functional is proposed for a typical dissipative body. Possible variations of fields of dependent state variables are introduced together with a non-commutative rule between operations of taking variations of the field and their partial time derivatives. Assuming vanishing of the first variation of the functional, the balance of linear momentum in differential form is received together with evolution equations for internal state variables and stress boundary condition.

Key words: dissipative bodies, principle of stationary action, Lagrangian, non-commutative rule, internal state variables.

1. Introduction

Historically, at the beginning, the classical Lagrange and Hamilton's formalisms were formulated for the point mechanics problems. Accordingly, if a dynamical system is described by the vector-valued coordinate $\mathbf{q}$ and the Lagrangian $L = T - V$, where T and V are, respectively, the kinetic and potential energy, then one formulates the variational principle of the dynamical system by requiring that between all curves $\mathbf{q} = \mathbf{q}(t)$ in a configuration space $\mathcal{V}$, the actual path (i.e. the solution of the system) is that which makes the action integral stationary

$$I = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt. \tag{1.1}$$

Taking the first variation $\delta\mathbf{q}$ subject to the conditions $\delta\mathbf{q}(t_0) = \delta\mathbf{q}(t_1) = 0$, the stationarity of the action requires $\delta I = 0$, which is equivalent to the Euler–Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0\,. \tag{1.2}$$

It is well known that the general equations of continuum mechanics and dissipative phenomena employed at the present time cannot be derived from Hamilton's variational principle. The case of bodies described by hyper-elastic material structure is exceptional [13]. The interest to derive governing equation in rational mechanics goes back to Levi-Civita [12]. There are several attempts to formulate the stationary action principle for dissipative systems, recently. e.g. Battezzati in [1] proposed to formulate the principle for a system with internal friction. More general review of other attempts can be found in the Habilitation Thesis of R. Kotowski [6] and in [7–9].

For many years it was a known fact that the equations of heat transfer could not be derived from exact variational principle. It was always necessary in describing the governing equations of a physical process as a variational problem (e.g. [2, 4, 7–11, 16, 19, 20, 22, 25, 26]). The interest follows not only from the purely theoretical view-point but also from the practical one: a variational principle can help to form a general and systematic approximative procedure for establishing the solution from direct study of the variational integral. We can refer here to the well known and widely used numerical method, like FEM of BIM, where solutions are searched by minimization techniques, in which weak formulation of governing equations appear with arbitrary variations.

In order to derive the governing equations of irreversible phenomena by the variational technique, some artificial restrictions must be made, concerning the basic rules of variational calculus. A good example is served by the variational principle formulation made in [3,17,19] for the case of heat conduction equation, in which the action functional contains one physical quantity (temperature), but this quantity is represented by two different symbols. The first symbol is subject to variation and the other one is not varied at all. However, by setting the two symbols, the same after the variation process has been finished, the governing differential equation of the process in consideration is obtained.

Different procedure was proposed by VUJANOVIĆ in [22] and applied to governing equation of the generalized (hyperbolic) theory of heat conduction with finite wave speed, in which the new Lagrangian was proposed with an explicit dependence on time through the exponential term $\exp t/\tau$, appearing as a factor. This term has the power t/τ, where τ is the thermal relaxation time. The corresponding transition to the classical case, i.e. with infinite speed of thermal disturbance (parabolic case), is performed by setting the relaxation time equal to zero.

Extended stationary action principle in modelling heterogeneous media, in which constitutive equation appears, has been recently proposed by RYCHLEWSKA [18] in the book dedicated to the memory of the late Professor Margaret Woźniak.

Here the proposed new method of deriving the class of equations appearing in some physical irreversible processes, is based on the variational principle which has a Hamiltonian structure, and in most cases its form does not differ from that known for conservative systems. However, the crucial assumption of the proposed method is in non-commutative rule between operations of taking variations of the field and their partial time and/or spatial derivatives, cf. Eq. (2.2).

In the discrete case, dealing with vector-valued generalized coordinate $\mathbf{q}$, we might try to interpret the non-commutative rule as a kind of discontinuity of the mixed second derivatives of generalized coordinate $\mathbf{q}$, regarded as a function of time t and parameter s, responsible for the variation, for all t at $s = 0$. In the continuous case, the non-commutative rule means discontinuity of the mixed second derivatives of some vector–valued field $\mathbf{u}$, regarded as the function of time t, the space coordinate x and the same parameter s, responsible for the variation. Unfortunately, the mathematics is not so easy, and this interpretation rather fails. It is the existence proof made by the Russian mathematician TOLSTOV [21] about the function of two variables, which may possess mixed second derivatives not equal on the set of positive measure (which we would like to have). This discontinuity set, however, is a fat Cantor set, which means that it cannot be formed of intervals. Hence we need a different interpretation.

For dissipative systems, the loss of energy is a crucial effect which is called irreversibility. In the case of mechanical systems, especially for deformable bodies, dissipation of mechanical energy is described by the second law of thermodynamics, which restricts all mechanical (or thermo-mechanical) processes to those which satisfy the internal dissipation inequality [6, 7, 14, 16]. The irreversibility means that it is impossible to reverse the process, in which dissipation energy occurs without changing the environment. Irreversibility means that variation of quantity (e.g. internal state variable in the case of dissipative bodies) $\delta\mathbf{m}$ and the variation of its time derivative are related by the dissipative mechanism governing the process considered, not the time differentiation.

It means that the time derivative $\frac{\partial}{\partial t}\delta\mathbf{m}$ on one side must be different from the variation of the time derivative of the quantity, i.e. $\delta\frac{\partial}{\partial t}\mathbf{m}$. On the other side, this variation is the dynamic quantity and it should depend on the irreversible forces (nonconservative, according to [23])) acting upon the system. We are not going to tamper with the usual notation of the variation of $\delta\mathbf{m}$ and the velocity of variation $\frac{\partial}{\partial t}\delta\mathbf{m}$. These two vectors are regarded as purely kinematic in nature. The vector $\delta\mathbf{m}$ means that we consider the *infinitesimal transformation* (i.e. the first variation) replacing $\mathbf{m}(t,x)$ by $\mathbf{m}(t,x)+s\mathbf{h}(t,x)$, where $\mathbf{h}(t,x)$ is an arbitrary differentiable function of t and x, and s is a small parameter passing through zero. Then, from the definition,

$$\delta\mathbf{m} = \frac{\partial}{\partial s}(\mathbf{m}(t,x)+s\mathbf{h}(t,x))\delta s = \mathbf{h}(t,x)\delta s. \tag{1.3}$$

Hence in this notation we have

$$\frac{\partial}{\partial t}\delta\mathbf{m} = \frac{\partial}{\partial t}\mathbf{h}(t,x)\delta s. \tag{1.4}$$

Since the vector $\delta\frac{\partial}{\partial t}\mathbf{m}$ has a purely dynamic character and its form depends on the nature of dissipative (nonconservative) phenomena (forces) acting on the body, the infinitesimal transformation replaces $\frac{\partial}{\partial t}\mathbf{m}(t,x)$ by $\frac{\partial}{\partial t}\mathbf{m}(t,x)+s\mathbf{k}(t,x)$, where $\mathbf{k}(t,x)$ is not an arbitrary differentiable function of t and x. This function, however, may differ from $\frac{\partial}{\partial t}\mathbf{h}(t,x)$ by a part that is related to the function $\mathbf{h}(t,x)$ via some relationship depending on the irreversible phenomena of the system. Hence we can write

$$\delta\frac{\partial}{\partial t}\mathbf{m} = \frac{\partial}{\partial s}\left\{\frac{\partial}{\partial t}\mathbf{m}(t,x)+s\mathbf{k}(t,x)\right\}\delta s = \mathbf{k}(t,x)\delta s, \tag{1.5}$$

together with

$$\mathbf{k}(t,x) = \frac{\partial}{\partial t}\mathbf{h}(t,x) + \mathcal{F}\left(\mathbf{m}(t,x),\frac{\partial}{\partial t}\mathbf{m}(t,x),t,x\right)\mathbf{h}(t,x). \tag{1.6}$$

Notice that dependence of the function $\mathcal{F}$ on time t and space variable x may be through some additional state variable, not written explicitly, here.

Comparing (1.3) and (1.5) with (1.6) we end up with the following noncommutative rule

$$\left[\delta,\frac{\partial}{\partial t}\right]\mathbf{m} = \mathcal{F}(\mathbf{m}(t,x),\frac{\partial}{\partial t}\mathbf{m}(t,x),t,x)\delta\mathbf{m}\,. \tag{1.7}$$

We can see that the case when the function $\mathcal{F}=0$ corresponds to a commutative rule. This noncommutative rule will be crucial in developing a new variation principle for deformable body made of the dissipative material with internal state variables.

In the previous paper [5] the variational technique developed here was applied to a long-line (telegraph) equation, and hyperbolic and parabolic models of heat conduction. In the next paper we will generalize the present derivation to the case of thermo-mechanics. Then some thermal initial boundary-value problems of technical interest will be analyzed to reduce them to ordinary differential equations, their solutions being often capable of being solved in analytic closed form.

2. Variational principle

Irreversibility in time means that the energy of the system is not conserved, hence we face a nonconservative case. B. VUJANOVIĆ, from Novi Sad, proposed in 1974–75 [23, 24] to describe the irreversible phenomena by Hamilton's principle with neglecting the commutative rule

$$\left[\delta, \frac{d}{dt}\right]\mathbf{q} = 0 \qquad \text{or} \qquad \left[\delta, \frac{\partial}{\partial t}\right]\mathbf{u} := \delta\frac{\partial}{\partial t}\mathbf{u} - \frac{\partial}{\partial t}\delta\mathbf{u} = 0 \tag{2.1}$$

for discrete or continuous cases, respectively. Here, in the discrete case, by the variation of $\mathbf{q}$ we understand the value of the partial differential of $\mathbf{q}(s,t)$ regarded as a function of two variables s and t calculated at s,

$$\delta\mathbf{q}(t) = \frac{\partial}{\partial s}(\mathbf{q}(t) + s\xi(t))\delta s = \xi(t)\delta s,$$

where s is from some interval $(s_0, s_1) \ni 0$, and $\xi(t)$ is an arbitrary differentiable function of t. For the variation of $\dot{\mathbf{q}}(t)$ we put extra field $\zeta(t)$

$$\delta\dot{\mathbf{q}}(t) = \frac{\partial}{\partial s}(\dot{\mathbf{q}}(t) + s\zeta(t))\delta s = \zeta(t)\delta s \qquad \text{and} \qquad \frac{d}{dt}\delta\mathbf{q} = \dot{\xi}(t)\delta s\,. \tag{2.2}$$

Similarly we define the variation of the field $\mathbf{u}$. Here we can repeat our arguments used in the Introduction and related to the internal state variable vector $\mathbf{m}$ for a dissipative body described by ISV's approach. Namely, two vectors $\delta\mathbf{q}(t)$ and $\frac{d}{dt}\delta\mathbf{q}$ are regarded as purely kinematic in nature, while the vector $\delta\frac{\partial}{\partial t}\mathbf{q}$ has a purely dynamic character and its form depends on the nature of dissipative nonconservative forces acting on the dynamical system. Hence we can expect that there is a difference between the time derivative of $\delta\mathbf{q}(t)$ and this last vector, i.e. the variation of the time derivative of $\dot{\mathbf{q}}$. This difference, due to the dimensional analysis and our assumption, is proportional to $\delta\mathbf{q}$, where its proportionality coefficient may depend on the state of the system as well, i.e.

$$\delta\frac{\partial}{\partial t}\mathbf{q}(t) = \frac{d}{dt}\delta\mathbf{q}(t) + \mathcal{G}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\delta\mathbf{q}(t)\,. \tag{2.3}$$

Here the function $\mathcal{G}$ plays a similar role to that of the function $\mathcal{F}$ in (1.6), and its non-vanishing value manifests the non-commutativity of the both operations δ and d/dt.

B. Vujanović assumes

$$\left[\delta, \frac{d}{dt}\right]\mathbf{q} \neq 0 \qquad \text{or} \qquad \left[\delta, \frac{\partial}{\partial t}\right]\mathbf{u} \neq 0\,, \tag{2.4}$$

in discrete and continuous cases, if $\mathbf{q} = \mathbf{q}(t)$ and $\mathbf{u} = \mathbf{u}(x,t)$, respectively.

2.1. Derivation of non-conservative differential equations

Let a governing equation of a physical system, described by a vector field $\mathbf{u}$, be

$$F_L\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial x_i}, x_i, t\right) + F_D\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial x_i}, x_i, t\right) = 0\,, \tag{2.5}$$

where $i = 1, \ldots, n$ and the part F_L is the *Lagrange part* which is derivable from a Hamilton principle:

$$\delta_C I = 0 \qquad \text{if} \qquad F_L\left(u, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial x_i}, x_i, t\right) = 0, \tag{2.6}$$

with the commutative rule (2.1) (hence the notation δ_C), with suitable initial and boundary conditions, especially $\delta\mathbf{u}(t_0) = \delta\mathbf{u}(t_1) = 0$, and with the action I and the Lagrangian density L

$$I = \int\limits_{t_0}^{t_1} \int\limits_{\Omega} L\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial x_i}, x_i, t\right) d\omega dt. \tag{2.7}$$

The nonconservative (dissipative) part F_D of (2.5) cannot be derived from variational principle of Hamilton's type and is related to dissipative phenomena of the physical system under consideration. We will introduce the dissipative characteristics of our system through noncommutative rules of variation and partial differentiation, in order to be able to derive Eq. (2.5) using the action functional (2.7). Now we pass to a non-commutative case and assume the following *noncommutative rules*: for temporal

$$\left[\delta, \frac{\partial}{\partial t}\right]\mathbf{u} = H\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right)\delta\mathbf{u} \tag{2.8}$$

and for spatial commutator

$$\left[\delta, \frac{\partial}{\partial x_j}\right] \mathbf{u} = G_j \left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial x_i}, x_i, t\right) \delta \mathbf{u}\,, \tag{2.9}$$

where H and G_j are suitably chosen functions, and $j = 1, \ldots, n$. Then performing the variation δI, with suitable initial and boundary conditions, we end up with

$$\delta I = \int\limits_{t_0}^{t_1} \int\limits_{\Omega} \left\{ \{L\}_{E-L} + \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial t})} H + \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial x_j})} G_j \right\} \delta \mathbf{u}\, d\omega\, dt, \tag{2.10}$$

where the summing convention is applied, and

$$\{L\}_{E-L} := \frac{\partial L}{\partial \mathbf{u}} - \frac{\partial}{\partial x_j} \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial x_j})} - \frac{\partial}{\partial t} \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial t})}\,. \tag{2.11}$$

Assume now the functions H and G_j in (2.8) and (2.9) as

$$\frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial t})} H + \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial x_j})} G_j = F_D \tag{2.12}$$

and identifying $\{L\}_{E-L} = F_L$, we obtain from the variational equation

$$\delta I = \int\limits_{t_0}^{t_1} \int\limits_{\Omega} \left\{ \{L\}_{E-L} + \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial t})} H + \frac{\partial L}{\partial(\frac{\partial \mathbf{u}}{\partial x_j})} G_j \right\} \delta \mathbf{u}\, d\omega\, dt = 0, \tag{2.13}$$

the *Euler–Lagrange's equation for dissipative system* as $F_L + F_D = 0$, in view of (2.6), (2.10)–(2.13).

3. Application to deformable dissipative bodies

In this section we will apply the presented method for deriving the governing equations of a deformable body with internal state variable (ISV) material structure [14, 15]. For a dissipative material of the body $\mathcal{B}$ we have to assume the Lagrangian as follows

$$L = T - V + D$$

where D denotes the dissipative energy.

Let us assume that state $\sigma \in \Sigma$ of the body, identified with a regular domain $\mathcal{B} \subset E^3$, is described by the regular function of motion $\phi : \mathcal{B} \times [t_0, t_1] \to E^3$, the vector field of internal state variable $\mathbf{m} \in \mathbf{R}^m$. The material properties of $\mathcal{B}$ are described by the mass density ρ_0 and two constitutive functions: the density of stored energy Ψ, which – in this mechanical case – is identical with the density of free energy function, and the evolution function $R(\sigma)$. Now we define the next fields: particle velocity $\mathbf{v} = \phi_t = \partial\phi/\partial t$, displacement gradient $\mathbf{F} = \nabla\phi$, density of body force $\mathbf{b}$, density of surface force $\mathbf{s}$. Moreover, we assume that the boundary $\mathcal{S}$ of $\mathcal{B}$ can be divided into 2 parts $\mathcal{S} = \mathcal{S}_d \cup \mathcal{S}_s$, and on $\mathcal{S}_d$ we have displacement boundary condition $\phi\,|_{\mathcal{S}_d} = \bar{\phi}$, while on $\mathcal{S}_s$ we have the surface force $\mathbf{s}$ given. Assume the initial distributions for the velocity $\mathbf{v}_0$ through the body as well as for the ISV $\mathbf{m}_0$ given. We will use the notation $\mathbf{m}_t$ for the partial time derivative $\partial\mathbf{m}/\partial t$.

Now the densities of kinetic energy T and of dissipative energy D of the body are defined as

$$T = \rho_0(\mathbf{v})^2/2, \qquad D = \mathbf{a}\cdot\mathbf{m}_t,$$

where $\mathbf{a}$ is a dimensional constant vector: $energy \cdot time/[\mathbf{m}]$, where $[\mathbf{m}]$ means the dimension of $\mathbf{m}$. Notice, that additional to the kinetic energy of the body motion, we have the dissipative energy $D = \mathbf{a}\cdot\mathbf{m}_t$ generated by the intrinsic structure described by the rate of internal state variable vector $\mathbf{m}_t$, To be in agreement with the classical transformation rule for the energy the variable $\mathbf{m}$ should be invariant under Euclidean transformations. It results in the possible choice of the ISV.

To define the Lagrangian density of our deformable body we need the potential energy V. It cannot be written as a density with respect to the volume measure, since on the boundary of $\mathcal{B}$ acts the surface force $\mathbf{s}$. The potential energy V has two contributions and they can be written as a sum of the integrals with respect to the volumetric and surface measure. Hence we pass to the action functional immediately:

$$I = \int_{t_0}^{t_1}\int_{\mathcal{B}}\left(\left(\frac{1}{2}\rho_0(\phi_t)^2 + \sum_{j=1}^{m} a_j\ m_{jt}\right) - \left(\Psi(\nabla\phi,\mathbf{m}) - \rho_0\mathbf{b}\cdot\phi\right)\right)d\omega\,dt + \int_{t_0}^{t_1}\int_{\mathcal{S}_s}\mathbf{s}\cdot\phi\,ds\,dt + \int_{\mathcal{B}}\rho_0\mathbf{v}_0\cdot\phi(t_0)\,dv - \int_{\mathcal{B}}\mathbf{a}\cdot\mathbf{m}(t_1)\,d\omega. \tag{3.1}$$

Hence we admit any variation of ϕ and of $\mathbf{m}$ such that

$$\delta\phi(t_1) = 0 \qquad \text{and} \qquad \delta\phi\,|_{\mathcal{S}_d} = 0 \qquad \text{and} \qquad \delta\mathbf{m}(t_0) = 0. \tag{3.2}$$

Take the 1^{st} variation of I under the non-commutativity rules:

$$\begin{aligned}\delta\left(\frac{\partial}{\partial t}\phi\right) &= \frac{\partial}{\partial t}\delta\phi \qquad \text{and} \qquad \delta(\nabla\phi) = \nabla(\delta\phi),\\ a_j\delta\left(\frac{\partial}{\partial t}m_j\right) &= a_j\frac{\partial}{\partial t}\delta m_j + \frac{a_j}{c_j}(R_j\delta m_j - m_{jt}\delta m_j) \quad \text{for } j = 1,2,\ldots,m,\end{aligned} \tag{3.3}$$

where c_j, $j = 1, 2, \ldots, m$, form a constant vector $\mathbf{c}$ of dimension $[\mathbf{m}]$, numerically equal to 1. Hence

$$\begin{aligned}\delta I = &\int_{t_0}^{t_1}\int_{\mathcal{B}} (\rho_0\phi_t\delta\phi_t + \mathbf{a}\cdot\delta\mathbf{m}_t + \rho_0\mathbf{b}\cdot\delta\phi)\, d\omega\, dt\\ &+\int_{t_0}^{t_1}\int_{\mathcal{S}_s} \mathbf{s}\cdot\delta\phi\, ds\, dt + \int_{\mathcal{B}} \rho_0\mathbf{v}_0\cdot\delta\phi(t_0)\, d\omega\\ &-\int_{t_0}^{t_1}\int_{\mathcal{B}} (\partial_{\nabla\phi}\Psi(\nabla\phi,\mathbf{m})\cdot\delta\nabla\phi + \partial_{\mathbf{m}}\Psi(\nabla\phi,\mathbf{m})\cdot\delta\mathbf{m})\, d\omega\, dt\\ &-\int_{\mathcal{B}} \mathbf{a}\cdot\delta\mathbf{m}(t_1)\, d\omega.\end{aligned}$$

Notice that $\mathbf{a}\cdot\mathbf{m}_t = \sum_{j=1}^{m} a_j\ m_{jt}$ and we are grouping terms that stand in front of the variations δm_j and δm_{jt}, $j = 1, 2, \ldots, m$. Then applying the rule (3.3), performing the integrations of terms that are the partially differentiated with respect to time and the space, respectively, using the unit normal $\mathbf{N}$ to $\mathcal{S}$, we end up with

$$\begin{aligned}\delta I = &\int_{t_0}^{t_1}\int_{\mathcal{B}} [\nabla\cdot(\partial_{\nabla\phi}\Psi(\nabla\phi,\mathbf{m})) - \frac{\partial}{\partial t}\rho_0\phi_t + \rho_0\mathbf{b}]\cdot\delta\phi\, d\omega\, dt\\ &-\int_{t_0}^{t_1}\int_{\mathcal{S}} \partial_{\nabla\phi}\Psi(\nabla\phi,\mathbf{m}))\mathbf{N}\cdot\delta\phi\, ds\, dt + \int_{t_0}^{t_1}\int_{\mathcal{S}_s} \mathbf{s}\cdot\delta\phi\, ds\, dt\\ &-\int_{t_0}^{t_1}\int_{\mathcal{B}}\sum_{j=1}^{m}\left(\partial_{m_j}\Psi(\nabla\phi,\mathbf{m}) - \frac{a_j}{c_j}(R_j - m_{jt})\right)\delta m_j\, d\omega\, dt\\ &+\int_{\mathcal{B}} [\rho_0\phi_t\cdot\delta\phi\mid_{t_0}^{t_1} - \mathbf{a}\cdot\delta\mathbf{m}\mid_{t_0}^{t_1} - \mathbf{a}\cdot\delta\mathbf{m}(t_1) + \rho_0\mathbf{v}_0\cdot\delta\phi(t_0)]\, d\omega.\end{aligned}$$

In view of the constraint (3.2), the first surface integral reduces to the integral over $\mathcal{S}_s$ and the last integral to $-\int_{\mathcal{B}}[\rho_0(\phi_t(t_0) - \mathbf{v}_0)\cdot\delta\phi(t_0)]\, dv$. Now due to the arbitrariness of the variations we obtain from the condition $\delta I = 0$ four local equations: two PDE's and two function equations, namely

$$\nabla\cdot\tau(\mathbf{F},\mathbf{m}) - \frac{\partial}{\partial t}\rho_0\mathbf{v} + \rho_0\mathbf{b} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{3.4}$$

$$\frac{c_j}{a_j}\partial_{m_j}\Psi(\mathbf{F},\mathbf{m}) - R_j + \frac{\partial}{\partial t}m_j = 0 \quad \text{for } j = 1,2,\ldots,m, \text{ in } \mathcal{B}, \tag{3.5}$$

$$\tau \cdot \mathbf{N} = \mathbf{s} \quad \text{on } \mathcal{S}_s \text{ and } \mathbf{v}(t_0) = \mathbf{v}_0, \tag{3.6}$$

where we put

$$\mathbf{F} := \nabla\phi, \qquad \mathbf{v} := \phi_t, \qquad \tau(\mathbf{F},\mathbf{m}) := \partial_{\nabla\phi}\Psi(\nabla\phi,\mathbf{m}).$$

We can see that the first equation is the balance of linear momentum, the second one is the system of evolution equations for internal state variables, while the last two are: the stress boundary condition and the velocity initial condition, respectively.

4. Conclusions

We have presented a new method of deriving the governing field equations for a dissipative deformable body described by the internal state variable model and based on a variational principle of stationary action. The main idea is based on the observation that for dissipative systems, the variation of time derivatives of a field is different from the time derivative of the variation of the field, i.e. *noncommutative rules* exist. Similar observation can be made in referring to spatial differentiation. The present approach deals with smooth fields and the case of discontinuous fields needs an extra elaboration.

The existence of a variation principle for given field equations has several advantages. The main one refers to availability of analytical or approximate solution of the equations (cf. [23]). In many cases the variational solution constitutes good approximation of the true one. This justifies the use of variational methods in treating complicated problems, like those involving irreversible phenomena, which cannot be solved directly.

The choice of the selected variational principle is, to a certain extent, of sentimental nature. Besides, several principles maybe used together to test the accuracy of the solution obtained. Indeed, if two or more principles give approximately the same result, it is reasonable to think that it is reliable. However, for certain classes of problems, some principles lead to more complex numerical calculations than others. It must also be kept in mind that some earlier formulations like those of Biot [2] and Vujanović [22], are less general than those of Glansdorff–Prigogine [16] and Lambermont–Lebon [10, 11]. The former papers are more adapted to describe heat transfer and related situations, the latter embrace a larger field of macroscopic physics including chemical reactions, diffusion processes and fluid flows. In our opinion, the method presented here is less restrictive and more natural than others.

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