Arch. Mech., **64**, 6, pp. 603–615, Warszawa 2012

Self-vibration of thin plate band with non-linear functionally graded material

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The subject of this paper is the analysis of free vibration of a thin plate band made of nonlinear functionally graded material. The considered material has periodic properties in one direction and slow but non-linear functionally graded properties in the other. The main attention is given to description of the effect of the material distribution on the overall response of the composite. The modelling approach is based on the tolerance averaging of the equation of motion. The general results are illustrated by the free vibration analysis of the bracket and the plate band simply supported on both sides.

Key words: self-vibrations, plate band, functionally graded materials.

1. Introduction

1.1. Formulation of the modelling problem

It is known that elastic responses of microheterogeneous media are described at microscopic level by PDEs with non-uniformly oscillating piecewise constant coefficients. As an example of these media we can mention solids made of functionally graded materials (FGM) [1, 2]. The analytical procedure leading to these equations is usually referred to as the mathematical modelling of FGM. At the same time the averaged model equations describe the behavior of FGM on the macroscopic level.

So far, the known general approaches to the averaging PDEs with non-uniformly oscillating coefficients were based on the asymptotic methods of the homogenization theory [3, 4]. The mathematical foundations of the formal asymptotic homogenization are based on the formal limit passage to zero with a certain length parameter describing the size of microstructure [5]. For FGM with a deterministic structure this parameter can be defined as the supreme of diameters of local periodicity cells. Consequently, the concept of locally periodic function was introduced in [6], which makes it possible to accomplish the limit passage to zero with the microstructure size length parameter in the framework of the asymptotic homogenization of FGM [7].

In the last decade a new approach to the mathematical modelling of FGM has been proposed. This approach is referred to as the tolerance modeling of FGM and the overview of results in this field was summarized in [8]. There are two main reasons for applying the tolerance modeling of differential equations as an alternative to the asymptotic non-uniform homogenization. Firstly, for many microheterogeneous media the space distribution of material properties is not uniquely described by means of locally periodic functions. Secondly, the asymptotically homogenized equations are independent of the microstructure size length parameter. Hence, they are unable to describe the effect of the length scale on the overall behaviour of FGM. The tolerance averaging of differential equations overcomes the aforementioned restrictions.

The existing attempts used to model the dynamic behaviour of FGM were performed for materials for which the geometry variation was linear [9]. Although in 2011, the work [10] was published, which allowed non-linear functions of the distribution of the individual materials, but the interfaces between various materials have remained parallel to each other. As a result, the geometry materials proposed in [10] have remained linear at the micro level. In this paper, the author aims to extend the application of the tolerant averaging method to FGM modelling structures in which the interfaces between different materials are described by non-linear functions.

1.2. Numerical approaches

Another type of methods currently used for modelling the dynamic behaviour of materials such as FGM is the numerical approach. The papers describing this approach can be divided according to the methods used in them into the mesh methods [11, 12, 13] and meshless methods [14]. In the case of mesh methods, the most common method is the finite element method. For plates and shells with non-uniform structure a variety of advanced methods is used. For example, they can be based on:

- QUAD-8 shear flexible element based on higher order structural theory [13],
- four-noded quadrilateral plate bending element [12],
- a node-based strain smoothing merged into shear-locking-free triangular plate elements [15],
- field-consistency approach and free forms of shear membrane locking problems [11].

In the case of meshless methods the works worth mentioning is based on *kp-Ritz method* [16] and another one containing a very wide, cross-cutting list of bibliography [14].

1.3. The subject of consideration

The subject of this paper considers the free vibration analysis of a thin plate band made of non-linear functionally graded material. The considered material has periodic properties in one direction and slow but non-linear functionally graded properties in the other (Fig. 1a). The plate band has periodically inhomogeneous microstructure slowly varying in space: the λ-periodic structure along ξ^1 coordinate. It has smoothly graded apparent (averaged) properties in the perpendicular direction of the ξ^1 axis, along ξ^2 axis (Figs. 1a and 1b). This structure is the generalization of linear variables of FGM structures analyzed in papers [9, 17].

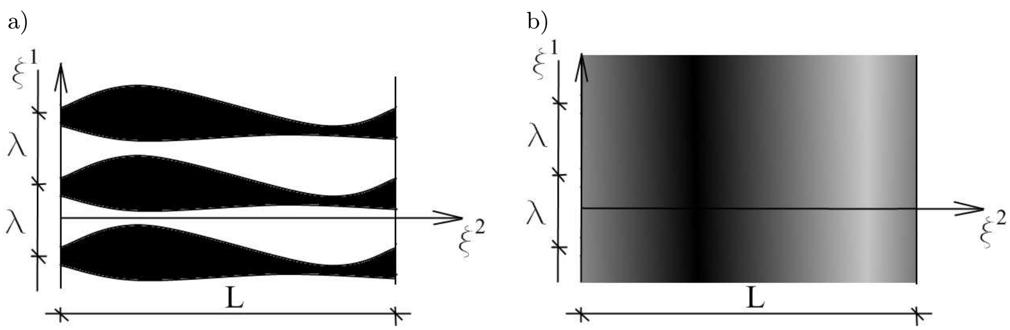

Fig. 1. Fragment of the plate's midplane with non-linear functionally graded microstructure: a) microscopic level, b) macroscopic level.

1.4. The objective and general assumption

The main objective of the research is to derive and apply a deterministic macroscopic model describing the dynamic behaviour of microheterogeneous plate band made of two components. The general assumption of the research is that the generalized period λ is sufficiently small when compared to the width of plate band L (Fig. 1a). The main attention is given to describing the effect of the material distribution on the overall response of the composite.

1.5. Methods of the investigation

The problems of the plates of this kind have been investigated by means of different methods. However, the exact analysis of those plates within solid mechanics is too complicated to constitute the basis for solving most of the engineering problems. Thus, many different approximate modelling methods for functionally graded material plates have been formulated [18, 19].

The proposed modelling approach is the generalization of the tolerance averaging technique (TAT). This technique was presented in detail in [20]. The tolerance averaging technique, in contrast to the homogenization technique for equations with non-uniformly oscillating coefficients, describes the effect of the microstructure size on the overall behaviour of composite elements. This method, by assuming the decomposition of the displacement field (2.9), allows us to calculate not only the fundamental frequency of free vibrations, which are well described by theories based on homogenisation and asymptotic analysis, but also makes it possible to describe the higher frequency of free vibrations which is characteristic for a given microgeometry [21].

2. Modelling

2.1. The direct description

The general way to create equations of motion is the same as in [9, 22], which is discussed in detail. The starting point for modelling are well known equations of linear elasticity theory, which are used to write equations on the micro scale.

The bases of modelling procedure are:

– strain-displacements relations

$$\kappa_{\alpha\beta} = -w_{|\alpha\beta}, \tag{2.1}$$

where $\kappa_{\alpha\beta}$ is curvature, w is displacement field;

– constitutive equations

$$m^{\alpha\beta} = DH^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}, \tag{2.2}$$

where

$$H^{\alpha\beta\gamma\delta} = \frac{1}{2}\big\{g^{\alpha\mu}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\mu} - \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\mu} + \epsilon^{\alpha\mu}\epsilon^{\beta\gamma})\big\}, \tag{2.3}$$

$$D = \frac{E\delta^3}{12(1-\upsilon^2)}, \tag{2.4}$$

E – Young's modulus, δ – thickness of plate, ν – Poisson's number, ϵ^{ij} – component of Ricci's tensor, g – component of contravariant metric tensor.

After applying formulas (2.1)–(2.4) we can write the equation of motion for the band plate under consideration as

$$m^{\alpha\beta}{}_{|\alpha\beta} + p - \rho\ddot{w} = 0, \tag{2.5}$$

where p is the external load, the symbol is the covariant derivative.

Due to the complex and non-linear variable geometry of the composite under consideration in the micro scale, this equation has highly oscillating coefficients, so it is difficult to solve.

2.2. Tolerance averaging technique

Next we use the tolerance averaging technique (TAT) for modelling the dynamic behaviour of thin plates, as it was presented in [23]. The more extensive discussion about TAT and the bibliography containing various examples of applications of this theory can be found in the monographs [8, 36]. This theory has been used successfully by many authors for different applications such as: statics of plates [24], dynamics of plates [25, 26, 27], heat conduction [28, 29, 30], stability analysis of plates and shells [31, 32, 33]. For the purpose of this paper we shall describe only briefly some of the concepts defined by this theory.

The most important operators and lemmas are:

– an averaging operator

$$\langle f\rangle(\xi^{\alpha}) = \frac{1}{\lambda}\int\limits_{\xi^1-\lambda/2}^{\xi^1+\lambda/2} f(y,\xi^2)dy, \tag{2.6}$$

where y is a local coordinate;

– a slowly varying function $F(\cdot) \in SV_{\Delta}(T)$:

$$\forall x,y\in\Pi \quad x-y\in\Delta \;\Rightarrow\; |DF(x)-DF(y)| < \varepsilon_{DF}, \tag{2.7}$$

where ε_{DF} is a tolerance parameter

$$DF \in \{F, \nabla F, F, \ldots\}; \tag{2.8}$$

– the displacement field disjoint

$$w(\cdot,t) = w^0(\cdot,t) + q^A(\cdot)V_A(\cdot,t), \tag{2.9}$$

where

$$w^0(\cdot,t)\in SV_{\Delta}(T), \quad V_A(\cdot,t)\in SV_{\Delta}(T)$$

are the basic unknowns, $q^A(\cdot)$ are the known shape functions;

– the most important theorems

$$\langle fF\rangle(x) \cong \langle f\rangle F(x), \tag{2.10}$$

$$\langle f\nabla(hF)\rangle(x) \cong \langle fF\nabla h\rangle(x), \tag{2.11}$$

$$\langle f\nabla\nabla(hF)\rangle(x) \cong \langle fF\nabla\nabla h\rangle(x). \tag{2.12}$$

These definitions and theorems will be used to build the equations of the averaged model. A wider discussion on the derivation and the proof of these associations has been summarized in the monographs [8].

2.3. The averaging equations

The modelling procedure is based on two steps. At first, we put into the equation of motion (2.5) the assumption of the decomposition of the displacement field (2.9) and we obtain the equation with $N+1$ unknowns w^0 and V_A, where $A = 1, \ldots, N$

$$\langle m^{\alpha\beta}_{|\alpha\beta} + p - \rho(\ddot{w}^0 + q^A \ddot{V}_A) \rangle = 0. \tag{2.13}$$

We obtain the missing equation by orthogonalization method, multiplying the equation of motion by the functions q^A and we get the equation:

$$\langle q^A (m^{\alpha\beta}_{|\alpha\beta} + p - \rho(\ddot{w}^0 + q^A \ddot{V}_A)) \rangle = 0. \tag{2.14}$$

After substituting the constitutive equations, strain-displacements relations, displacement field disjoint and many mathematical transformations, we get the averaging equations:

$$\begin{aligned} \langle BH^{\alpha\beta\gamma\delta} w^0_{|\gamma\delta} \rangle_{|\alpha\beta} &+ (\langle BH^{\alpha\beta 11} q^A_{|11} \rangle V_A)_{|\alpha\beta} \\ &+ (\langle BH^{\alpha\beta 22} q^A \rangle V_{A|22})_{|\alpha\beta} + \langle \rho \ddot{w} \rangle = \langle p \rangle. \end{aligned} \tag{2.15}$$

The coefficients in the above system are continuous and slowly varying functions. It is therefore possible to find a solution to this system of equations using the finite difference method.

3. The free vibrations of the plate band

3.1. The assumptions

Let us consider the following example of the free vibrations of a thin plate band. This plate is shown in Fig. 1 in rectangular coordinates. We make some assumptions:

– the displacement field disjoint

$$w = w^0 + qV, \tag{3.1}$$

where

$$w^0 \in SV_\Delta(T), \qquad V \in SV_\Delta(T);$$

– no external loading $p = 0$,

– the harmonic vibration

$$\begin{aligned} w^0(\xi^\alpha, t) &= \bar{w}(\xi^\alpha)\cos(\omega t), \\ V(\xi^\alpha, t) &= \bar{V}(\xi^\alpha)\cos(\omega t); \end{aligned} \tag{3.2}$$

– the shape function

$$q(\cdot) = \lambda^2\Big(\cos\Big(\frac{2\pi\xi^1}{\lambda}\Big) + C\Big), \tag{3.3}$$

where the constant C is obtained from the equation

$$\langle q\rho\rangle = 0 \tag{3.4}$$

as

$$C = \frac{\lambda\xi^2\Big(-\rho_1 + \rho_1\cos\Big(\frac{2\pi d}{\xi^2\lambda}\Big) - \rho_2 - \rho_2\cos\Big(\frac{2\pi d}{\xi^2\lambda}\Big)\Big)}{2\pi(\rho_1 d + \rho_2\lambda\xi^2 - \rho_2 d)}. \tag{3.5}$$

After many mathematic transformations a fourth row partial differential equations' system with two equations with 2 unknowns is built as

$$\begin{gathered}\langle BH^{2222}w^0_{,22}\rangle_{,22} + (\langle BH^{2211}q^A_{,11}\rangle V_A)_{,22} + (\langle BH^{2222}q^A\rangle V_{A,22})_{,22} + \langle\rho\ddot{w}\rangle = \langle p\rangle,\\[2ex] \begin{aligned}\langle q^A_{,11}BH^{1122}\rangle w^0_{,22} &+ \langle q^A_{,11}BH^{1111}q^B_{,11}\rangle V_B + \langle q^A_{,11}BH^{1122}q^B\rangle V_{B,22}\\ &+ (\langle q^A_{,11}BH^{2222}\rangle w^0_{,22})_{|22} + (\langle q^A BH^{2211}q^B_{,11}\rangle V_B)_{|22}\\ &+ (\langle q^A BH^{2222}q^B\rangle V_{B,22})_{,22} + \langle q^A\rho q^B\rangle\ddot{V}_B = \langle q^A p\rangle.\end{aligned}\end{gathered} \tag{3.6}$$

The coefficients in this system of equations are continuous and they can be found by symbolic calculations. In order to find a particular solution to the system of equations we have the task's defined boundary conditions. In the case of bracket (left bank clamped, right side free), we write it as

$$w|_{\xi^2=0} = 0 \quad \text{and} \quad \Big(\frac{\partial w}{\partial\xi^2}\Big)\Big|_{\xi^2=0} = 0, \tag{3.7}$$

$$\Big(\frac{\partial^3 w}{(\partial\xi^2)^3}\Big)\Big|_{\xi^2=L} = 0 \quad \text{and} \quad \Big(\frac{\partial^2 w}{(\partial\xi^2)^2}\Big)\Big|_{\xi^2=L} = 0, \tag{3.8}$$

because (3.1)

$$\begin{aligned} w^0|_{\xi^2=0_i} &= 0, \quad \Big(\frac{\partial w^0}{\partial\xi^2}\Big)\Big|_{\xi^2=0_i} = 0,\\ V|_{\xi^2=0_i} &= 0, \quad \Big(\frac{\partial V}{\partial\xi^2}\Big)\Big|_{\xi^2=0_i} = 0,\end{aligned} \tag{3.9}$$

$$\left(\frac{\partial^3 w^0}{(\partial\xi^2)^3}\right)\Big|_{\xi^2=L} = 0, \qquad \left(\frac{\partial^2 w^0}{(\partial\xi^2)^2}\right)\Big|_{\xi^2=L} = 0,$$

$$\left(\frac{\partial^3 V}{(\partial\xi^2)^3}\right)\Big|_{\xi^2=L} = 0 \quad \left(\frac{\partial^2 V}{(\partial\xi^2)^2}\right)\Big|_{\xi^2=L} = 0, \tag{3.10}$$

and for the simple support on both sides

$$w|_{\xi^2=r_i} = 0 \qquad \text{and} \qquad \left(\frac{\partial^2 w}{(\partial\xi^2)^2}\right)\Big|_{\xi^2=r_i} = 0, \tag{3.11}$$

because (3.1)

$$\begin{aligned} w^0|_{\xi^2=r_i} &= 0, \qquad \left(\frac{\partial^2 w^0}{(\partial\xi^2)^2}\right)\Big|_{\xi^2=r_i} = 0, \\ V|_{\xi^2=r_i} &= 0, \qquad \left(\frac{\partial^2 V}{(\partial\xi^2)^2}\right)\Big|_{\xi^2=r_i} = 0, \end{aligned} \tag{3.12}$$

where $r_i = 0$ or $r_i = L$, respectively.

3.2. Numerical results

We use the finite difference method to obtain the numerical solution of this equations system, using our own computer program in MS Visual C++. We could change any geometrical and material parameter of a plate and obtain the first and higher frequency of free vibrations and shapes of displacement field corresponding to it. In this way, we can analyze the influence of the material's distribution on the frequency of free vibrations.

For example, we used the following materials:

– matrix: $E_1 = 20$ GPa, $\nu_1 = 0.2$, $\rho_1 = 2800$ kg/m^3,
– walls: $E_2 = 210$ GPa, $\nu_2 = 0.7$, $\rho_2 = 7800$ kg/m^3,

and geometrical data:

– microstructure size $\lambda = 0.1$ m,
– thickness of plate $h = 3$ cm,
– bandwidth $L = 1$ m,
– quantity of material walls – 50 %,
– quantity of material matrix – 50 %.

The border between the materials was determined by polynomial functions of fourth order form $y = ax^4+bx^3+cx^2+dx+e$, where the constants a, b, c, d, e were always determined to ensure the condition of equal volumes of both components of the composite. By varying these factors and controlling the geometry of the composite, we do not change the percentage composition of the material. Below are the numerical results obtained for different geometries. On the horizontal axis we have the position of the gravity centre of the walls' material; on the vertical axis we can see the frequency of vibrations of the composite. In the figures below we see pictorials like this . They indicate that the result in selected point is obtained for the appropriate distribution of walls and matrix.

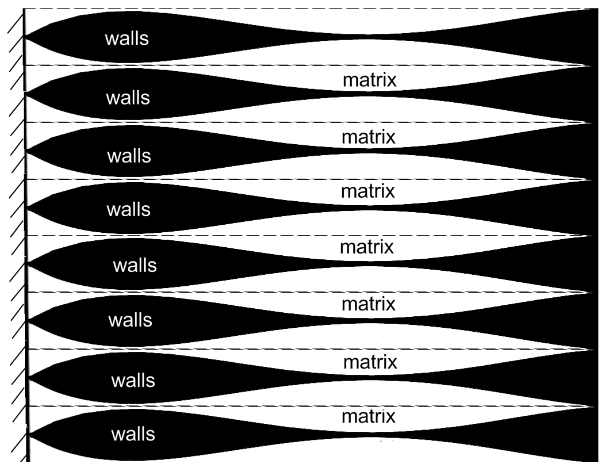

Fig. 2. Example of the considered plate band.

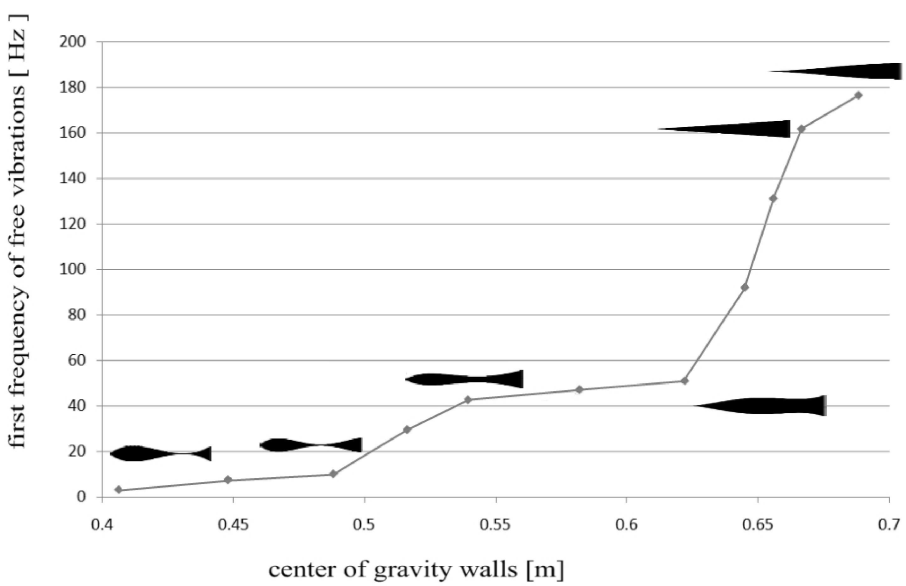

Fig. 3. Dependency of first frequency of free vibrations from location of centre of gravity of walls ($d_1 = 0$, $d_2 = 0.1$ m) left side clamped, right side free.

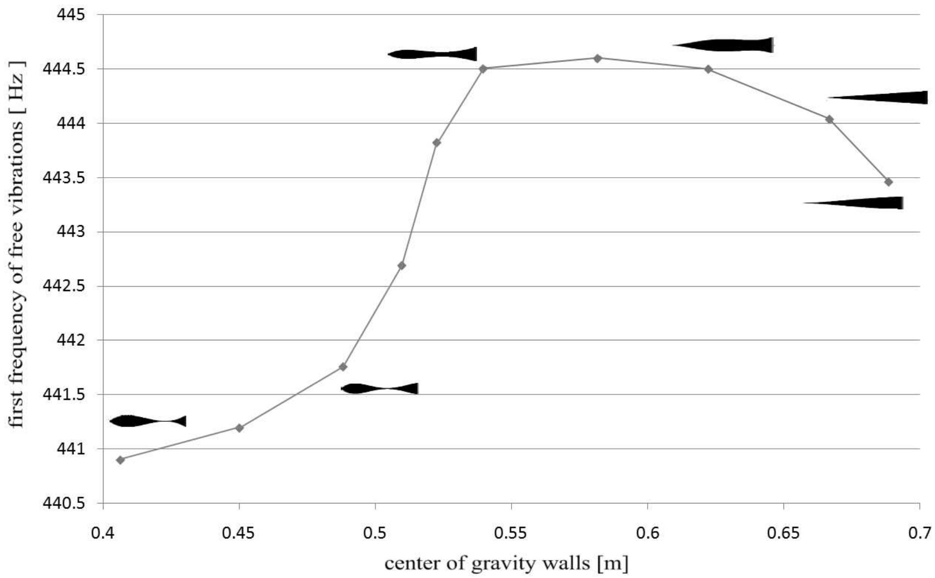

Fig. 4. Dependency of first frequency of free vibrations from location of the centre of gravity of walls ($d_1 = 0$, $d_2 = 0.14$ m) both sides simply supported.

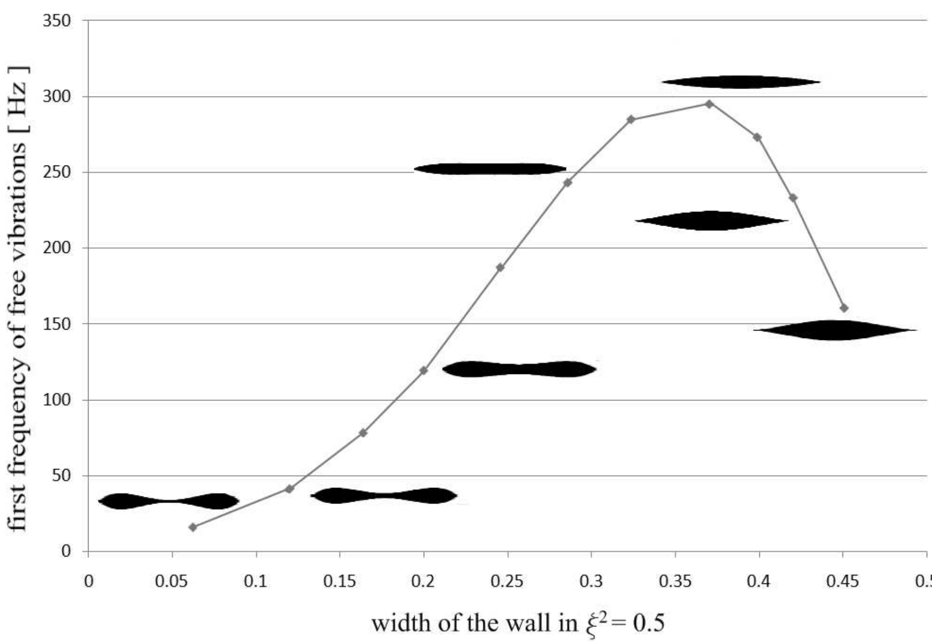

Fig. 5. Dependency of first frequency of free vibrations from width of walls in $\xi^2 = 0.5$ ($d_1 = 0$, $d_2 = 0$) left side clamped, right side free.

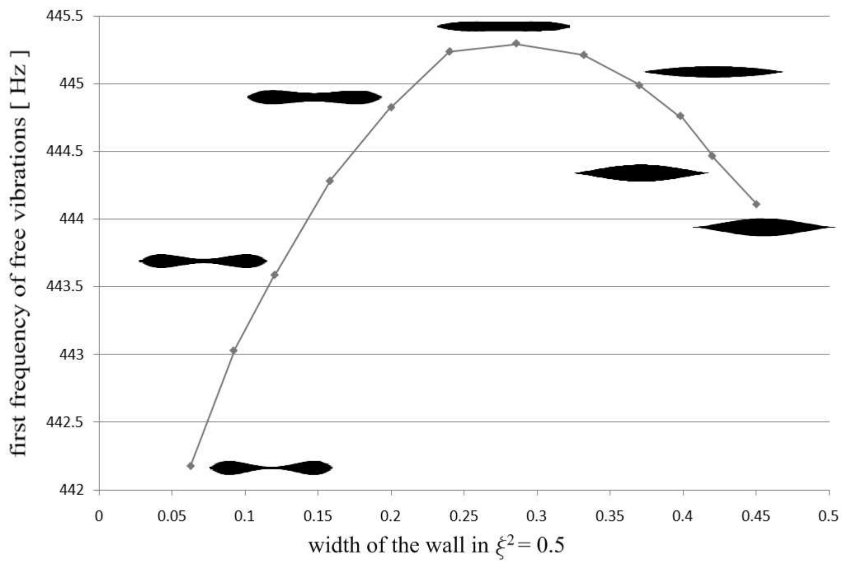

Fig. 6. Dependency of first frequency of free vibrations from width of walls in $\xi^2 = 0.5$ ($d_1 = 0$, $d_2 = 0$) both sides simply supported.

4. Conclusions

1. The tolerance averaging technique can be successfully applied to formulate averaging model of dynamic behaviour of the composite plates with non-linear functionally graded material. Despite the complicated and non-linear structure of the material at the micro level, it was possible to use TAT to obtain solutions to a given problem. By assuming the decomposition of the displacement field it is also possible to find a higher frequency of free vibrations associated with the unknown V and corresponding to the microstructure of the material. This will be the subject of this author's further research.

2. We can see a very strong dependency of frequency of free vibrations of the plate band from the material distribution for the bracket and very weak for the simple support on both sides.

3. The first frequency of free vibrations of the plate band made of non-linear functionally graded material could be completely different for the same material proportion of walls and matrix.

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Received October 27, 2011; re-submitted July 11, 2012.