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Thermal vibration analysis of carbon nanotubes embedded in two-parameter elastic foundation based on nonlocal Timoshenko's beam theory

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This paper presents a new model to consider the thermal effects, Pasternak's shear foundation, transverse shear deformation and rotary inertia on vibration analysis of a single-walled carbon nanotube. Nonlocal elasticity theory is implemented to investigate the small-size effect on thermal vibration response of an embedded carbon nanotube. Based on Hamilton's principle, the governing equations are derived and then solved analytically, in order to determine the nonlocal natural frequencies. Results show that unlike the Pasternak foundation, the influence of Winkler's constant on nonlocal frequency is negligible for low temperature changes. Moreover, the nonlocal frequencies are always smaller as compared to their local counterparts. In addition, in high shear modulus along with an increase in aspect ratio, the nonlocal frequency decreases.

Key words: carbon nanotube, Pasternak's foundation, temperature change, nonlocal elasticity, Timoshenko's beam theory.

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1. Introduction

Since carbon nanotubes (CNTs) were discovered by Iijima [1], they have aroused great interest in the scientific community because of their exceptional mechanical, electronic, electrochemical, and thermal properties [2–4]. These outstanding properties of CNTs led to their usage in the emerging field of nanoelectronics, nanodevices, nanocomposites, etc. It has been shown that the CNTs with extremely high elastic modulus and low mass density can serve as terahertz (THz) nanoresonators [5–8] in nanoelectromechanical systems (NEMS). For example, the stiffness of CNTs is 100 times as that of steel, but the weight is one- sixth as that of the steel [9]. They are thermally stable up to 2800°C in vacuum, with a thermal conductivity which is twice as large as diamond, and having an electric-current – carrying capacity about 1000 times greater than copper wire [10].

There are three major categories for simulating the properties of CNTs: experiments, molecular dynamic simulations (MD) and continuum mechanics. Considering the limited applications of MD and difficulties in conducting controlled experiments for such small scales, continuum modeling is considered to be an appropriate method of investigating the mechanical and dynamical properties of CNTs [11]. In addition, due to difficulties in experimental characterization of nanotubes as well as time-consuming and computationally expensive atomistic simulation, elastic continuum models have been widely used to study the vibration behavior of CNTs [12]. Extensive studies have been conducted on the mechanical properties of CNTs such as static bending [13, 14], free vibration and dynamic response [15–19], buckling [20, 21] and post buckling [22]. Some works are cited herein. Yoon *et al.* [15] studied the vibration behavior of multi-walled carbon nanotubes embedded in an elastic medium using multiple-elastic beam model. Lu *et al.* [23] investigated the wave propagation and vibration properties of single- or multi-walled CNTs based on nonlocal beam model. Wang *et al.* [24] presented analytical solutions for the free vibration of nonlocal Timoshenko's beam models. Wang *et al.* [16] used Timoshenko's beam model and differential quadrature (DQ) method for free vibration analysis of multi-walled carbon nanotubes. Zhang *et al.* [25] developed a double-elastic beam model for transverse vibration of double-walled carbon nanotubes under axial compressive axial load using Euler–Bernoulli's beam theory. Reddy [26] has provided a comprehensive overview of the use of nonlocal theories for modeling beam bending, buckling and vibration. The vibration analysis of SWCNT is mainly focused on the fundamental mode [5]. Ghorbanpour Arani *et al.* [27] presented a nonlocal elastic shell model for buckling analysis of a DWCNT with using the two-parameter foundation. However, only a limited portion of literature is concerned with the vibration analysis of carbon nanotubes considering the thermal effect and Pasternak's foundation.

The present study is concerned with the derivation of the governing equations for vibration of single-walled carbon nanotubes modeled as nonlocal Timoshenko's beam theory, using separation of variables approach. The effects of transverse shear deformation and rotary inertia are considered within the framework of this model. The surrounding elastic medium is described as the Winkler and Pasternak's type foundation. The Hamilton principle is employed to derive the governing equation and boundary conditions. A direct technique is then used to obtain the natural frequency of nonlocal SWCNT with immoveable support. The influences of nonlocal parameter, Winkler and Pasternak's parameters, temperature change, aspect ratio and vibration mode on the free vibration of the SWCNT are discussed in detail.

2. Nonlocal nanobeam model for linear analysis of SWCNT

Classical continuum models such as beam and shell models do not admit intrinsic size dependence on the elastic solutions of inclusion and nonhomogeneities. For small-scale length, the applicability of classic continuum models has become questionable. Furthermore, size-dependent continuum mechanic is used because at small scale length, the material microstructures such as lattice spacing between individual atoms become increasingly significant and thus their effect can no longer be ignored [28]. The weakness of the classical continuum models such as small-scale effect is modified by nonlocal elasticity. Nonlocal continuum field theories are concerned with the physics of material bodies. The nonlocal theory generalizes the classical field theory in two respects: first, the energy balance law is considered valid for the entire body and second, the state of the body at a material point is described by response functional [29].

The nonlocal elasticity theory is developed by ERINGEN [30, 31] and ERINGEN and EDELEN [32]. According to theory of nonlocal elasticity, the stress at point x in a body depends not only on the strain at point x (hyper elastic case) but also on those at all other points of the body. Thus, the nonlocal stress tensor σ at point x is expressed as

$$\sigma_{ij} = \int_V \lambda(|x' - x|, \tau)\varepsilon_{kl}(x')C_{ijkl}\, dV(x'), \tag{2.1}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.2}$$

The terms σ_{ij}, ε_{kl} and C_{ijkl} are the stress, strain and fourth order elasticity tensor, respectively. $\lambda(|x' - x|, \tau)$ is the Kernel function or nonlocal modulus or attenuation function incorporating into constitutive equations, $|x'-x|$ represents the distance in Euclidean form and τ is the material constant that depends on the internal (e.g., lattice parameter, granular size, distance between C-C bonds) and external characteristic length (e.g., wave length). Material constant τ is defined as e_0a/l where e_0 is a constant for adjusting the model in matching with experimental results. Also, a, l denotes the internal and external length parameters, respectively. The Kernel function is given by Eringen as

$$\lambda(|x|, \tau) = (2\pi l^2\tau^2)^{-1}\kappa\left(\frac{\sqrt{x_0 x}}{l\tau}\right), \tag{2.3}$$

where κ is the modified Bessel function. By the combination of Eqs. (2.1) and (2.3) we obtain

$$(1 - \tau^2 l^2\nabla^2)\sigma = t, \qquad \tau = \frac{e_0 a}{l}, \tag{2.4}$$

where $t = C : \varepsilon$ and ':' represents the double dot product. For a beam type structure, the thickness and width are much smaller than its length. Therefore, for beams with transverse motion in $x-z$ plane, the nonlocal constitutive relation can be approximated to one-dimensional form as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{2.5}$$

$$\sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\gamma_{xz}, \tag{2.6}$$

$$\mu = (e_0 a)^2, \tag{2.7}$$

where E and G are Young's and shear modulus, respectively, γ_{xz} is the shear strain, σ_{xx} and σ_{xz} are normal and shear stresses and μ is the scale coefficient or nonlocal parameter.

3. Governing differential equations

The displacement field equation based on Timoshenko's beam theory is given as

$$u_1 = u(x,t) + z\varphi(x,t), \tag{3.1}$$

$$u_2 = 0, \tag{3.2}$$

$$u_3 = w(x,t), \tag{3.3}$$

where u_1 and u_3 are the axial and transverse displacement of the point $(x, 0)$ on the mid-plane (i.e., $z = 0$) of the beam and $\varphi(x,t)$ denotes the rotation of the cross-section beam. The nonzero strains according to Timoshenko's beam theory are expressed as

$$\varepsilon_{xx} = \frac{\partial u(x,t)}{\partial x} + z\frac{\partial \varphi(x,t)}{\partial x}, \tag{3.4}$$

$$\varepsilon_{xz} = \frac{\partial w(x,t)}{\partial x} + \varphi(x,t), \tag{3.5}$$

where ε_{xx} is the axial strain. In all theories, the axial force-strain relation is the same and it is given by

$$N - \mu \frac{\partial^2 N}{\partial x^2} = EA\frac{\partial u(x,t)}{\partial x}. \tag{3.6}$$

In the Timoshenko beam theory, the constitutive relations based on (2.5) and (2.7) are given by

$$M - \mu \frac{\partial^2 M}{\partial x^2} = EI\frac{\partial \varphi(x,t)}{\partial x}, \tag{3.7}$$

$$Q - \mu \frac{\partial^2 Q}{\partial x^2} = K_S GA\gamma_{xz}, \tag{3.8}$$

where K_S is the shear correction factor to compensate for the error due to this constant shear stress assumption. A value of 0.877 was used for K_S by REDDY and PANG [33]. The strain energy of beam, U is given by

$$U = \int_V U_0 \, dV = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} \, dV = \frac{1}{2} \int_0^l \int_A (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz}) \, dA \, dx, \tag{3.9}$$

where U_0 is the strain energy per unit volume or simply the strain energy density and A is the cross sectional area of the beam. By introducing Eqs. (3.4) and (3.5) into Eq. (3.9), the strain energy U can be represented as

$$U = \frac{1}{2} \int_0^l \int_A \{\sigma_{xx}(\frac{\partial u(x,t)}{\partial x} + z\frac{\partial \varphi(x,t)}{\partial x}\Big) + \sigma_{xz}\Big(\frac{\partial w(x,t)}{\partial x} + \varphi(x,t)\Big)\Big\} \, dA \, dx. \tag{3.10}$$

With the integration of Eq. (3.10) in range of A, the strain energy is written as follows:

$$U = \frac{1}{2} \int_0^l \Big\{ N\frac{\partial u(x,t)}{\partial x} + M\frac{\partial \varphi(x,t)}{\partial x} + Q\Big(\frac{\partial w(x,t)}{\partial x} + \varphi(x,t)\Big)\Big\} dx, \tag{3.11}$$

where the normal resultant force N, bending moment M and transverse shear force Q are calculated from

$$N = \int_A \sigma_{xx} \, dA, \qquad M = \int_A \sigma_{xx} z \, dA, \qquad Q = \int_A \sigma_{xz} \, dA. \tag{3.12}$$

The kinetic energy calculated from

$$\begin{aligned} K &= \int_V \frac{\rho}{2}\Big(\frac{\partial u}{\partial t}\Big)^2 dV \\ &= \frac{1}{2} \int_0^l \int_A \rho \Big\{ \Big(\frac{\partial u(x,t)}{\partial t} + z\frac{\partial \varphi(x,t)}{\partial t}\Big)^2 + \Big(\frac{\partial w(x,t)}{\partial t}\Big)^2 \Big\} dA \, dx. \end{aligned} \tag{3.13}$$

The general form of kinetic energy comes in the form below:

$$K = \int_0^l \Big\{ \frac{\rho A}{2}\Big(\frac{\partial u(x,t)}{\partial t}\Big)^2 + \frac{\rho I}{2}\Big(\frac{\partial \varphi(x,t)}{\partial t}\Big)^2 + \frac{\rho A}{2}\Big(\frac{\partial w(x,t)}{\partial t}\Big)^2 \Big\} dx \tag{3.14}$$

In Eq. (3.14), the following relations are considered:

$$\int_A z\,dA = 0, \qquad \int_A z^2\,dA = I, \tag{3.15}$$

where I is the second moment of area about y-axis and ρ is the mass density of beam material. Thus, the result is that the x-axis is taken along the geometric center of the beam. The potential energy is equal to work done by external forces and is given by

$$\begin{aligned} V &= W_E \\ &= -\frac{1}{2}\int_0^l \left\{ f(x,t)u(x,t) + q(x,t)w(x,t) + \bar{N}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 + f_e w(x,t) \right\} dx. \end{aligned} \tag{3.16}$$

Indeed, Eq. (3.16) is an extended form of potential energy in [34]. In this equation, the effects of thermal field and two-parameter elastic medium are considered. In the above equation, negative sign indicates that the work is done on the body; $f(x,t)$ and $q(x,t)$ are the axial and transverse distributed forces (measured per unit length), $\bar{N}$ is applied compressive force and f_e is the density of reaction force of elastic foundation expressed as

$$f_e = K_W w(x,t) - K_G \nabla^2 w(x,t). \tag{3.17}$$

The terms K_W and K_G represent the Winkler and shear modulus (shear layer foundation stiffness) of the elastic medium, respectively. It is assumed that the shearing layer stiffness of the foundation is one-tenth of the value of Winkler's modulus [35]. The equations of motion of the nonlocal SWCNTs embedded in an elastic medium can be derived from the Hamilton principle

$$\delta \int_0^{t_1} [K - (U+V)]\,dt = 0. \tag{3.18}$$

By using calculus of variation and substituting Eqs. (3.11), (3.14) and (3.16) into Eq. (3.18), the Hamilton principle can be represented as

$$\begin{aligned} 0 = \int_0^t \int_0^l \bigg\{ & m_0\left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right) + m_2 \frac{\partial \varphi}{\partial t}\frac{\partial \delta \varphi}{\partial t} - N\frac{\partial \delta u}{\partial x} - M\frac{\partial \delta \varphi}{\partial x} \\ & - Q\left(\frac{\partial \delta w}{\partial x} + \delta\varphi\right) + f(x,t)\delta u + q(x,t)\delta w + \bar{N}\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + f_e \delta w \bigg\}\,dx\,dt. \end{aligned} \tag{3.19}$$

The mass inertia m_0 and m_2 are defined by

$$m_0 = \int_A \rho\,dA = \rho A, \tag{3.20}$$

$$m_2 = \int\limits_A \rho z^2 \, dA = \rho I. \tag{3.21}$$

When integrating by parts of Eq. (3.19) and setting the coefficient of δu, $\delta\varphi$ and δw to zero

$$\delta u = 0; \quad \frac{\partial N}{\partial x} + f(x,t) = m_0 \frac{\partial^2 u}{\partial t^2}, \tag{3.22}$$

$$\delta w = 0; \quad \frac{\partial Q}{\partial x} + q(x,t) - K_W w + K_G \frac{\partial^2 w}{\partial x^2} - \bar{N}\frac{\partial^2 w}{\partial x^2} = m_0 \frac{\partial^2 w}{\partial t^2}, \tag{3.23}$$

$$\delta\varphi = 0; \quad \frac{\partial M}{\partial x} - Q = m_2 \frac{\partial^2 \varphi}{\partial t^2}. \tag{3.24}$$

It is assumed that the axial and transverse distributed forces are equal to zero

$$f(x,t) = q(x,t) = 0. \tag{3.25}$$

Differentiating Eq. (3.24) once related to x and substituting it into Eq. (3.23), according to Eqs. (3.7) and (3.8) we obtain the nonlocal bending moment, M, and shear force Q in the Timoshenko beam theory

$$M = EI\frac{\partial \varphi}{\partial x} + \mu\Big[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2} + m_2\frac{\partial^3 \varphi}{\partial x \partial t^2}\Big], \tag{3.26}$$

$$Q = K_S GA\Big(\frac{\partial w}{\partial x} + \varphi\Big) + \mu\frac{\partial}{\partial x}\Big[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2}\Big]. \tag{3.27}$$

By substituting Eqs. (3.26) and (3.27) into Eq. (3.19), we obtain the complete form of Hamilton's principle

$$\begin{aligned} 0 = \int\limits_0^t \int\limits_0^L \Big\{ & m_0\Big(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\Big) + m_2\frac{\partial \varphi}{\partial t}\frac{\partial \delta\varphi}{\partial t} - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} \\ & - \mu m_0\Big(\frac{\partial^3 u}{\partial x \partial t^2}\Big)\frac{\partial \delta u}{\partial x} - EI\frac{\partial \varphi}{\partial x}\frac{\partial \delta\varphi}{\partial x} - \mu\Big[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2} \\ & + m_2\frac{\partial^3 \varphi}{\partial x \partial t^2}\Big]\frac{\partial \delta\varphi}{\partial x} - K_S GA\Big(\frac{\partial w}{\partial x} + \varphi\Big)\Big(\frac{\partial \delta w}{\partial x} + \delta\varphi\Big) + \bar{N}\Big(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\Big) \\ & - \mu\frac{\partial}{\partial x}\Big[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2}\Big]\Big(\frac{\partial \delta w}{\partial x} + \delta\varphi\Big) \\ & + \Big(K_W w - K_G\frac{\partial^2 w}{\partial x^2}\Big)\delta w \Big\} \, dx \, dt. \end{aligned} \tag{3.28}$$

In the same way, the complete form of equations of motion and natural boundary conditions are obtained as follows:

$$(3.29)\qquad \delta u = 0;\quad EA\frac{\partial^2 u}{\partial x^2} + \mu m_0\left(\frac{\partial^4 u}{\partial x^2 \partial t^2}\right) = m_0\frac{\partial^2 u}{\partial t^2},$$

$$(3.30)\qquad \delta\varphi = 0;\quad EI\frac{\partial^2 \varphi}{\partial x^2} - K_S GA\left(\frac{\partial w}{\partial x} + \varphi\right) + \mu m_2\frac{\partial^4 \varphi}{\partial x^2 \partial t^2} = m_2\frac{\partial^2 \varphi}{\partial t^2},$$

$$(3.31)\qquad \delta w = 0;\quad \frac{\partial}{\partial x}\left[K_S GA\left(\frac{\partial w}{\partial x} + \varphi\right)\right] + K_W w - K_G\frac{\partial^2 w}{\partial x^2} - \bar{N}\frac{\partial^2 w}{\partial x^2}, + \mu\frac{\partial^2}{\partial x^2}\left[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2}\right] = m_0\frac{\partial^2 w}{\partial t^2}.$$

The natural boundary conditions of Eqs. (3-29), (3-30) and (3-31) are, respectively

$$(3.32)\qquad \left(EA\frac{\partial u}{\partial x} + \mu m_0\frac{\partial^3 u}{\partial x \partial t^2}\right)\delta u\bigg|_0^L = 0,$$

$$(3.33)\qquad \left\{K_S GA\left(\frac{\partial w}{\partial x} + \varphi\right) - \bar{N}\frac{\partial w}{\partial x} + \mu\frac{\partial}{\partial x}\left[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2}\right]\right\}\delta\varphi\bigg|_0^L = 0,$$

$$(3.34)\qquad \left\{EI\frac{\partial \varphi}{\partial x} + \mu\left[\bar{N}\frac{\partial^2 w}{\partial x^2} + K_W w - K_G\frac{\partial^2 w}{\partial x^2} + m_0\frac{\partial^2 w}{\partial t^2} + m_2\frac{\partial^3 \varphi}{\partial x \partial t^2}\right]\right\}\delta w\bigg|_0^L = 0.$$

Here $\bar{N}$ represents the axial force on the CNTs and it is expressed as

$$(3.35)\qquad \bar{N} = N_m + N_\theta,$$

where N_m is the axial force due to the mechanical loading prior to buckling and N_θ is the axial force due to the influence of temperature change. Here, the theory of thermal elasticity mechanics is adopted because Young's modulus of SWCNT is insensitive to temperature change in the tube. In fact, Young's modulus has a constant value when the temperature is lower than approximately 1100 K, but decreases rapidly at higher temperature [36]. In addition, the change of Poisson's ratio with temperature is studied in [37]. It is obvious that Poisson's ratio increases with temperature, although the augmentation is limited and can be considered constant at temperatures between 300–1200 K. On the other hand, according to the relation $G = E/(2(1+\upsilon))$ and constant values of E and υ, it is clear that the shear modulus G is constant. Moreover, the temperature dependence of radial and axial coefficient of thermal expansion is investigated in [38] and concludes that all coefficients vary nonlinearly with temperature and are negative at low or room temperature and become positive at high temperature. Thus, researchers usually use the constant value of -1.6×10^{-6} for coefficient of thermal expansions at low or room temperatures [12, 39–43], based on the work [44].

Therefore, the high thermal conductivity of CNTs leads to the uniform and constant axial force N_θ as below [45]

$$N_\theta = -\frac{EA}{1-2\upsilon}\alpha_x\theta, \tag{3.36}$$

where α_x is the coefficient of thermal expansion in the direction of x-axis, υ is Poisson's ratio and θ denotes the change in temperature. Here, changes for low temperature environment will be considered.

For calculating N_m, we have

$$N_m = \int_A \sigma_m dA = \int_A E\left(\frac{\partial u}{\partial x} + z\frac{\partial \varphi}{\partial x}\right) dA = EA(u)|_0^l. \tag{3.37}$$

Therefore, by considering the boundary conditions for immoveable supports, $u(0,t) = u(l,t) = 0$, the axial force due to mechanical loading will be zero.

4. Analytical solution

By the application of the separation of variables, we can assume trigonometric solutions of the form $\varphi(x,t) = \phi(x)e^{i\omega t}$ and $w(x,t) = W(x)e^{i\omega t}$ for vibration analysis of SWCNTs, where $\varphi(x)$ and $W(x)$ are the mode shapes and ω is the frequency of natural vibration [33]. Therefore, with substituting $\varphi(x,t)$ and $w(x,t)$ into Eqs. (3.30) and (3.31) we obtain

$$\frac{d}{dx}\left(EI\frac{d\varphi}{dx}\right) - K_S GA\left(\frac{dW}{dx} + \phi\right) + m_2\omega^2\left(\varphi - \mu\frac{d^2\varphi}{dx^2}\right) = 0, \tag{4.1}$$

$$\begin{aligned}\frac{d}{dx}\left[K_S GA\left(\frac{dW}{dx} + \varphi\right)\right] + K_W W - K_G\frac{d^2W}{dx^2} - \bar{N}\frac{d^2W}{dx^2} + \mu\bar{N}\frac{d^4W}{dx^4} \\ + \mu K_W\frac{d^2W}{dx^2} - \mu K_G\frac{d^4W}{dx^4} - \mu m_0\omega^2\frac{d^2W}{dx^2} + m_0\omega^2 W = 0.\end{aligned} \tag{4.2}$$

To eliminate ϕ from Eqs. (4.1) and (4.2), we solve Eq. (4.1) for $d\varphi/dx$ as follows:

$$\begin{aligned}\frac{d\varphi}{dx} = -\left(\frac{m_0\omega^2 + K_W}{K_S GA}\right)W + \mu\left(\frac{K_G - \bar{N}}{K_S GA}\right)\frac{d^4W}{dx^4} \\ - \left(1 - \frac{\bar{N} + K_G - \mu K_W + \mu m_0\omega^2}{K_S GA}\right)\frac{d^2W}{dx^2}.\end{aligned} \tag{4.3}$$

Differentiating Eq. (4.1) once, and substituting for $\frac{d\phi}{dx}$ from Eq. (4.3) and by some simplifications, we obtain

$$A\frac{d^6W}{dx^6} + B\frac{d^4W}{dx^4} + C\frac{d^2W}{dx^2} + DW = 0, \tag{4.4}$$

where

$$\begin{aligned}
A &= \mu(EI - \mu m_2\omega^2)\left(\frac{\bar{N} - K_G}{K_S GA}\right),\\
B &= (EI - \mu m_2\omega^2)\left(1 - \frac{\bar{N} + K_G - \mu K_W + \mu m_0\omega^2}{K_S GA}\right)\\
&\quad + \mu\left(\frac{\bar{N} - K_G}{K_S GA}\right)(m_2\omega^2 - K_S GA),\\
C &= (m_2\omega^2 - K_S GA)\left(1 - \frac{\bar{N} + K_G - \mu K_W + \mu m_0\omega^2}{K_S GA}\right) + K_S GA\\
&\quad + (EI - \mu m_2\omega^2)\left(\frac{m_0\omega^2 + K_W}{K_S GA}\right),\\
D &= (m_2\omega^2 - K_S GA)\left(\frac{m_0\omega^2 + K_W}{K_S GA}\right).
\end{aligned} \tag{4.5}$$

This is sixth-order governing differential equation which requires six boundary conditions. It is noticeable that by ignoring the thermal and elastic medium parameters, the constitutive differential equation in [33] is derived. Considering the boundary conditions for simply-supported SWCNT with immoveable ends as

$$w(0,t) = w(l,t) = 0, \tag{4.6}$$

$$M = 0 \quad \text{at } x = 0 \qquad \text{and} \qquad x = l. \tag{4.7}$$

By substituting $\varphi(x,t)$ and $w(x,t)$ into Eqs. (3.33) and (3.34), the natural boundary conditions for linear vibration are derived as below:

$$\left\{K_S GA\left(\frac{dW}{dx} + \varphi\right) - \bar{N}\frac{dW}{dx} + \mu\bar{N}\frac{d^3W}{dx^3} - \mu m_0\omega^2\frac{dW}{dx}\right\}\Bigg|_0^l = 0, \tag{4.8}$$

$$\left\{EI\frac{d\phi}{dx} + \mu\left[\bar{N}\frac{d^2W}{dx^2} + K_W W - K_G\frac{d^2W}{dx^2} - m_0\omega^2 W - m_2\omega^2\frac{d\phi}{dx}\right]\right\}\Bigg|_0^l = 0. \tag{4.9}$$

The general solution can be considered as follows:

$$W(x) = \sum_{m=1}^{\infty}\sin\frac{m\pi}{l}x. \tag{4.10}$$

The above solution can satisfy all boundary conditions. With substituting this solution into Eq. (38) and by some simplifications, one can obtain

$$\sum_{m=1}^{\infty}\left\{-A\left(\frac{m\pi}{l}\right)^6+B\left(\frac{m\pi}{l}\right)^4-C\left(\frac{m\pi}{l}\right)^2+D\right\}\sin\frac{m\pi}{l}x=0. \tag{4.11}$$

With substituting Eq. (4.5) into Eq. (4.11), we calculate the natural frequencies for different cases. The constitutive equation in general form is written as follows:

$$\begin{aligned}
&-\left[\mu(EI-\mu m_2\omega^2)\left(\frac{\bar{N}-K_G}{K_SGA}\right)\right]\left(\frac{m\pi}{l}\right)^6\\
&+\left[(EI-\mu m_2\omega^2)\left(1-\frac{\bar{N}+K_G-\mu K_W+\mu m_0\omega^2}{K_SGA}\right)\right.\\
&\left.+\mu\left(\frac{\bar{N}-K_G}{K_SGA}\right)(m_2\omega^2-K_SGA)\right]\left(\frac{m\pi}{l}\right)^4\\
&-\left[(EI-\mu m_2\omega^2)\left(\frac{m_0\omega^2+K_W}{K_SGA}\right)+K_SGA+(m_2\omega^2-K_SGA)\right.\\
&\left.+\left(1-\frac{\bar{N}+K_G-\mu K_W+\mu m_0\omega^2}{K_SGA}\right)\right]\left(\frac{m\pi}{l}\right)^2\\
&+(m_2\omega^2-K_SGA)\left(\frac{m_0\omega^2+K_W}{K_SGA}\right)=0.
\end{aligned} \tag{4.12}$$

5. Results and discussions

Here, we present numerical solutions for the vibration of SWCNTs, considering the effects of thermal field and Pasternak's elastic medium. The effective properties of SWCNTs are taken as in [33]. Young's modulus $E = 1000$ Gpa, Poisson's ratio $\upsilon = 0.19$, mass density $\rho = 2300$ kg/m^3, $K_S = 0.877$ and $G = 420$ Gpa are applied in the analysis. The diameter of the SWCNT is assumed as 1.0 nm. In the following, six different cases are studied in order to consider the different parameters.

Case 1. $\mu = 0$, $K_G = 0$, $K_W = 0$, $\bar{N} = 0$. In this case, the natural frequency for classic problem is calculated. The Winkler and shear moduli are equal to zero. The axial compressive force is not considered in this case. The changes of natural frequencies with aspect ratios for different mode numbers are shown in Fig. 1.

Figure 1 depicts that with increasing in aspect ratio (L/d), the natural frequencies decrease, but with increasing in mode number (m), the natural frequencies increase. This is the most basic case that has been considered.

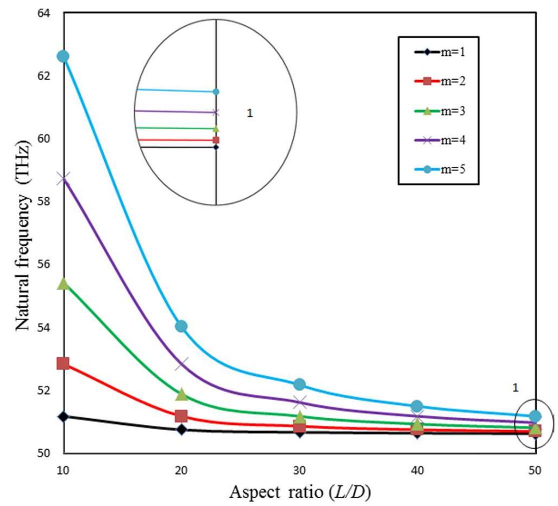

Fig. 1. Variation of natural frequency with aspect ratio L/d for different change in mode number m for SWCNT ($K_W = K_G = \bar{N} = 0$).

Case 2. $K_G = 0$, $K_W = 0$, $\bar{N} = 0$. In the nonlocal cases, with increasing in aspect ratio, the values of natural frequencies increase. However, they are always lower than local frequencies (Fig. 1). The effects of nonlocal parameter for different aspect ratios on natural frequencies are illustrated in Fig. 2.

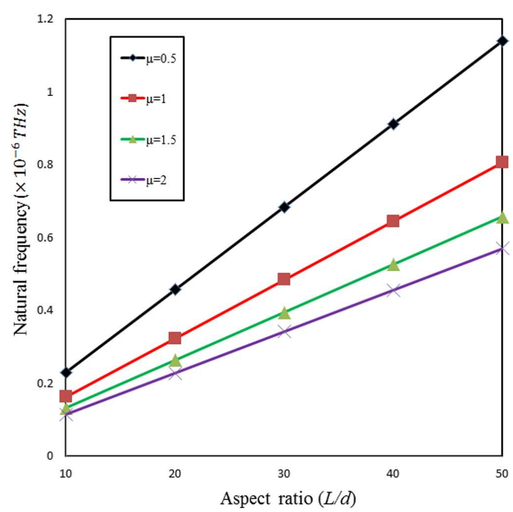

Fig. 2. Change of natural frequency with aspect ratio L/d for different change in nonlocal parameter.

It is concluded that by increasing the nonlocal parameter (μ), in different aspect ratios, the natural frequencies decrease. The variation of natural frequencies with aspect ratios for different mode number is presented in Fig. 3.

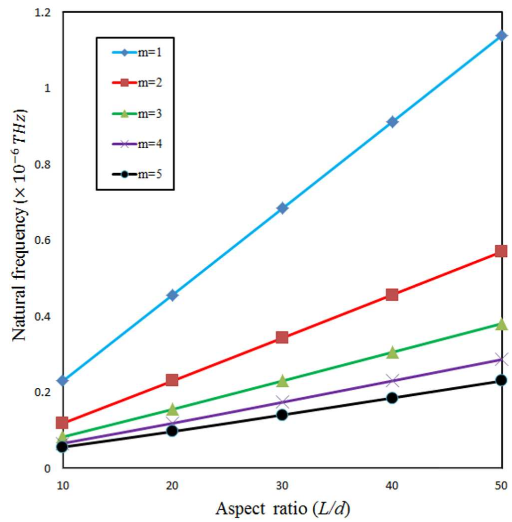

FIG. 3. Mode number effect on natural frequency of a SWCNT for different aspect ratio L/d and $\mu = 0.5$.

It can be shown that by increasing in mode number, unlike the case 1, the natural frequencies significantly reduce.

Case 3. $K_G = 0$, $\bar{N} = 0$. In the absence of Pasternak's foundation and thermal effect, we consider the effect of Winkler's modulus in constitutive equation. The scale coefficients are taken as $e_0 a \leq 2$ [40]. Figure 4 illustrates the effect of K_W on natural frequency with different nonlocal parameter for different aspect ratios.

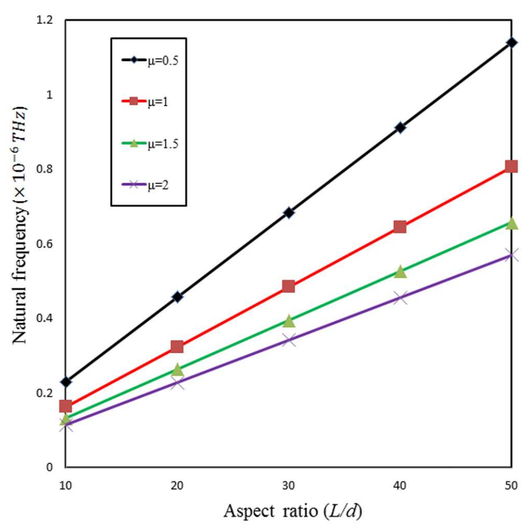

FIG. 4. Variation of natural frequency with aspect ratio L/d for different nonlocal parameters ($K_W = 10^3$ N/m^2, $m = 1$).

It is obvious that by increasing in nonlocal parameter (μ), the natural frequencies decrease, but with increasing in aspect ratio (L/d), the natural frequencies increase. In this case, by using Winkler's foundation, the natural frequencies are slightly reduced with respect to case 2. Figure 5 shows the effect of mode number m on natural frequency for constant aspect ratio and Winkler modulus parameter. It is noticeable that increasing the mode number and nonlocal parameter has the effect of decreasing the natural frequencies.

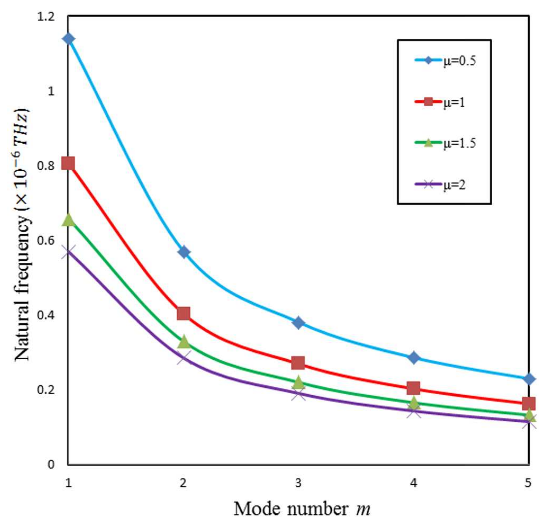

Fig. 5. Effect of mode number on natural frequency for different nonlocal parameters μ ($K_W = 10^3$ N/m^2 $L/d = 50$).

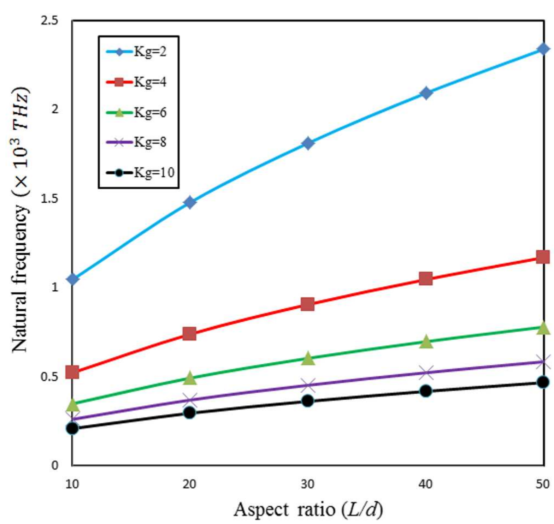

Fig. 6. The effect of shear modulus parameter K_G on natural frequency of a SWCNT for different aspect ratios L/d ($K_W = 10^2$ N/m^2, $m = 1$, $\mu = 0.5$).

Case 4. $\bar{N} = 0$. Figure 6 shows the effect of Pasternak's foundation on the natural frequency of SWCNT in the absence of thermal field. In this case, we assume K_W is constant and K_G is considered for five different values.

Moreover, by increasing in aspect ratio, the natural frequencies increase significantly as compared with the case 3. The variation of Pasternak's shear modulus on natural frequency with different mode number is shown in Fig. 7.

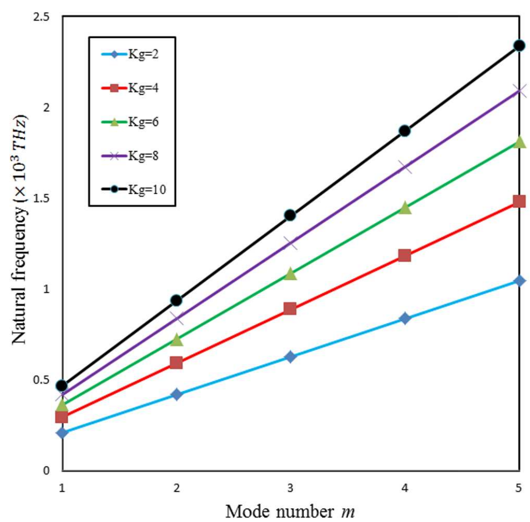

FIG. 7. The natural frequency of a SWCNT with different mode number $m$and shear modulus K_G ($K_W = 10^2$ N/m^2, $\mu = 0.5$, $L/d = 50$).

As illustrated in Fig. 7, by increasing the shear modulus parameter K_G in a constant aspect ratio, the natural frequencies of a SWCNT increase significantly and the frequencies are higher than the values in Fig. 4.

Case 5. The general case, including all different parameters, is considered here. The thermal expansion coefficient for CNTs is taken as -1.6×10^{-6} [40]. In Fig. 8, the effect of temperature on natural frequency is considered.

It is clear that with increasing in temperature, the natural frequency changes irregularly but the result is that with variation on temperature, the natural frequencies decrease. In this case, the temperature changes are assumed to be uniform. An important result is that unlike the previous cases, in a constant aspect ratio and temperature, difference in nonlocal parameter plays an important role on natural frequencies. Figure 9 depicts the effect of nonlocal parameter for different aspect ratios on natural frequency under temperature field.

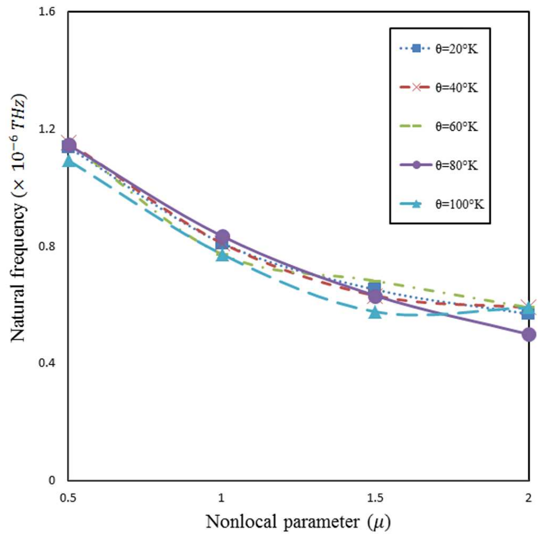

Fig. 8. Effect of temperature on frequency of a SWCNT with different nonlocal parameter μ ($L/d = 50$, $K_G = K_W = 0$, $m = 1$).

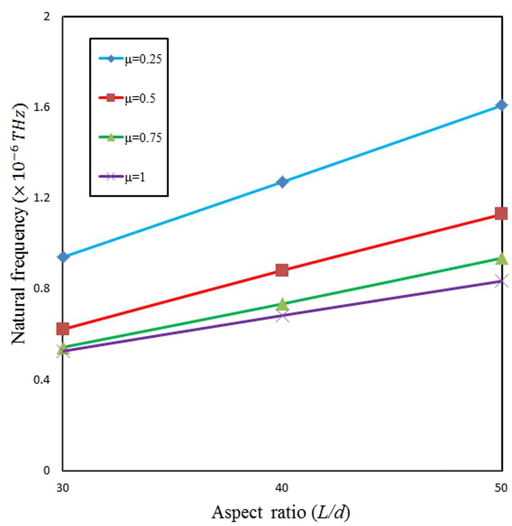

Fig. 9. Effect of nonlocal parameter μ on natural frequency with different aspect ratios L/d ($K_w = K_G = 0$, $m = 1$, $\theta = 50^\circ$ K).

It illustrates that with using temperature, the natural frequencies related to Fig. 6 increase slightly. Figure 10 shows the effect of mode number m and shear modulus parameter on frequency of a SWCNT.

Therefore, with increasing in mode number m and shear modulus parameter K_G, the frequencies increase strongly. Comparing Fig. 10 and Fig. 13, it is

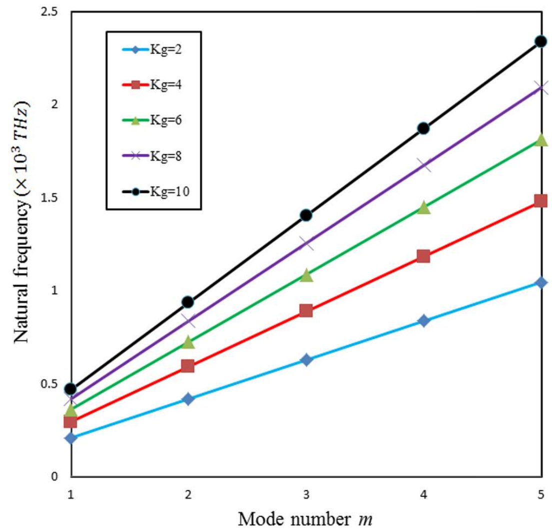

FIG. 10. Effect of mode numbers on natural frequency for different shear modulus parameter K_G ($\mu = 0.5$, $L/d = 50$, $K_W = 10^2$ N/m^2, $\theta = 50°$ K).

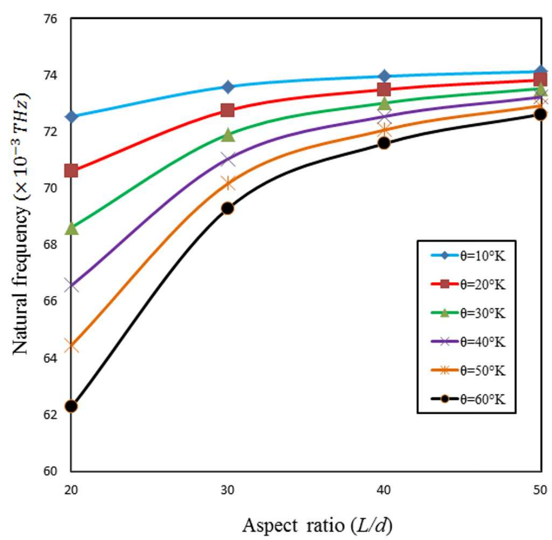

FIG. 11. Variation of natural frequency with temperature for different aspect ratio L/d and high Winkler's modulus K_W ($\mu = 1$, $K_G = 0$, $m = 1$, $K_W = 10^7$ N/m^2).

concluded that temperature have not significant effects on frequencies. In Fig. 11, as temperature changes, the effect of Winkler's foundation is considered on natural frequency of a SWCNT.

It is obvious that for $K_W = 10^7$ N/m^2, with increasing in temperature, the natural frequency decreases. Effects of aspect ratio with Pasternak's foundation and a fixed nonlocal parameter are shown in Fig. 12.

It is observed that with increasing in nonlocal parameter, natural frequencies decrease, but in this case, for different temperatures, natural frequencies change slightly.

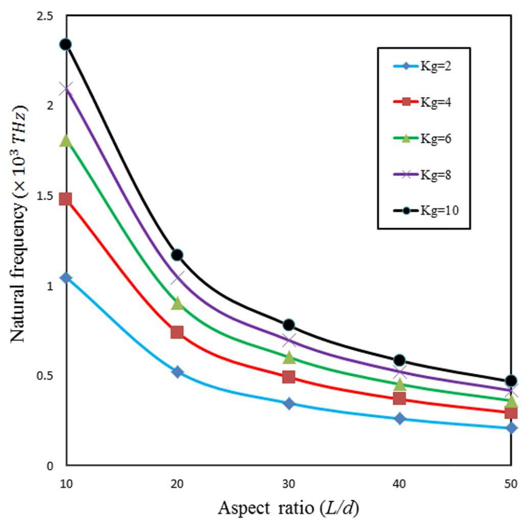

FIG. 12. Effect of aspect ratio L/d and Pasternak's foundation K_G on frequency of a SWCNT ($K_W = 10^2$ N/m^2, $\theta = 50°$ K, $m = 1$, $\mu = 0$).

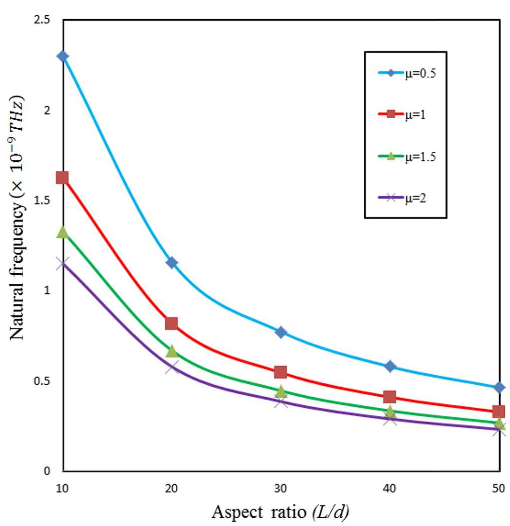

FIG. 13. The omission of rotary inertia m_2 and its effect on natural frequency for different nonlocal parameters.

Case 6. $m_2 = 0$, $\bar{N} = 0$, $K_G = 0$, $K_W = 0$. In this section, we examine the effect of rotary inertia on natural frequency. In previous cases, this parameter was considered in constitutive equation. Here, we set the rotary inertia to zero. Fig. 13 shows the variation of natural frequency with aspect ratio for different nonlocal parameter.

Therefore, it is concluded that by omitting the rotary inertia, related to case 2, the natural frequencies significantly reduced and its relationship with aspect ratio is a curve.

6. Conclusions

In this paper, based on nonlocal elasticity theory, the linear vibration characteristics of a single walled carbon nanotube embedded in two-parameter elastic medium called Pasternak's foundation were investigated in thermal environments. Analytical solution was used to solve the constitutive equations. Results for a SWCNT show that an increase in nonlocal parameter leads to decrease of the natural frequency. However, by increasing the Winkler and Pasternak's moduli in a constant aspect ratio, the natural frequencies are increased. Furthermore, in the absence of nonlocal parameter, with increasing in Pasternak's shear parameter, the natural frequencies increase slightly.

Moreover, for a constant nonlocal parameter, with increasing in temperature, the natural frequencies decrease and they are lower than the results of case 2.

Generally, it is concluded that the effect of Pasternak's foundation on natural frequency is more significant as compared to the effects of thermal loading, Winkler's modulus and nonlocal parameter and should be considered in vibration analysis of carbon nanotubes. Finally, it should be noted that the rotary inertia, nonlocal parameter and thermal loading have the effect of reducing natural frequency.

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