

Explicit homogenized equation of a boundary-value problem in two-dimensional domains separated by an interface highly oscillating between two concentric ellipses

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THE MAIN PURPOSE OF THIS PAPER is to find the homogenized equation and the associate continuity condition in the explicit form of a boundary-value problem in two-dimensional domains separated by an interface oscillating rapidly between two concentric ellipses. This boundary-value problem originates from various mechanical problems. By the homogenization method and following the techniques presented recently by these authors the homogenized equation and the associate continuity condition in the explicit form are derived. Since the obtained homogenized equation is totally explicit it is convenient to use.

Key words: interfaces oscillating highly between two concentric ellipses, homogenization method, homogenized equation.

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1. Introduction

BOUNDARY-VALUE PROBLEMS IN DOMAINS with rough boundaries or interfaces are closely related to various practical problems such as scattering of elastic waves at rough boundaries and interfaces [1], transmission and reflection of waves on rough interfaces [2, 3, 4], mechanical problems concerning the plates with densely spaced stiffeners [5], flows over rough walls [6] and so on. When the amplitude (height) of the roughness is much smaller in comparison to its period, the problems are usually analyzed by perturbation methods. When the amplitude is much large than its period, i.e., the boundaries and interfaces are very rough, the homogenization method [7, 8] is required.

In [9] NEVARD and KELLER investigated a boundary-value problem in two-dimensional domains separated by a curve highly oscillating between two straight lines, namely

$$(1.1) \quad (\sigma U_x)_x + (\sigma U_z)_z - \lambda U = f(x, z), \quad (x, z) \notin L,$$

$$(1.2) \quad [U]_L = 0, \quad [\sigma U_n]_L = 0,$$

where U and f are scalar functions,

$$\phi_x := \partial\phi/\partial x, \quad \phi_z := \partial\phi/\partial z, \quad \phi_n = \phi_x n_x + \phi_z n_z, \quad n_x, \quad n_z$$

are the components of the unit normal to the curve L which rapidly oscillates between two straight lines $z = -A$ ($A > 0$) and $z = 0$ (Fig. 1), $U(x, z)$ is unknown, $f(x, z)$, σ , λ are given and

$$(1.3) \quad \sigma, \lambda = \begin{cases} \sigma_+, \lambda_+ & \text{for } (x, z) \in D_+, \\ \sigma_-, \lambda_- & \text{for } (x, z) \in D_-, \end{cases}$$

$\sigma_+, \sigma_-, \lambda_+, \lambda_-$ are constant and $[w]_L = w_+ - w_-$ on L , D_+ and D_- are separated by L . By using the homogenization method the authors derived the homogenized equation and the associate continuity condition in the explicit form. This problem was then considered in two-dimensional domains with an interface highly oscillating between two concentric circles [9], and the corresponding homogenized equation and associate continuity condition in the explicit form were also obtained.

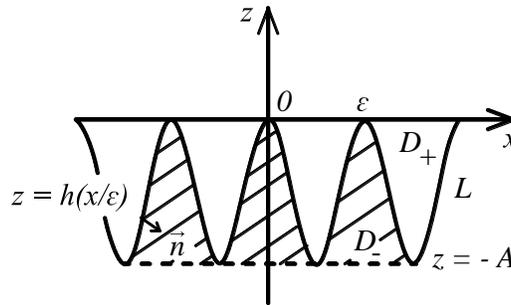


FIG. 1. Two-dimensional domains D_+ and D_- have a very rough interface L expressed by equation $z = h(x/\varepsilon) = h(y)$, where $h(y)$ is a periodic function with period 1. The curve L highly oscillates between the straight lines $z = 0$ and $z = -A$ ($A > 0$).

The main purpose of this paper is to extend the results of Nevard and Keller to the general problem of Eqs. (1.1)–(1.2), in particular, to find the homogenized equation and the associate continuity condition in the explicit form of the following problem:

$$(1.4) \quad (\sigma_{11}U_x + \sigma_{12}U_z)_x + (\sigma_{12}U_x + \sigma_{22}U_z)_z - \lambda U = f(x, z), \quad (x, z) \notin L,$$

$$(1.5) \quad [U]_L = 0, \quad [q_n]_L = 0,$$

where

$$(1.6) \quad q_n = (\sigma_{11}U_x + \sigma_{12}U_z)n_x + (\sigma_{12}U_x + \sigma_{22}U_z)n_z$$

and

$$(1.7) \quad \sigma_{ij}, \lambda = \begin{cases} \sigma_{ij}^+, \lambda^+, & \text{for } (x, z) \in D_+, \\ \sigma_{ij}^-, \lambda^-, & \text{for } (x, z) \in D_-, \end{cases}$$

$\sigma_{ij}^+, \sigma_{ij}^-, \lambda^+, \lambda^-$ are given constants, $U(x, z)$ is unknown, $f(x, z)$ is given, the interface L rapidly oscillates between two straight lines or two concentric ellipses. The matrix $(\sigma_{ij})_{2 \times 2}$ is assumed to be positive definite, i.e., there exists a positive constant α so that

$$(1.8) \quad \sigma_{ij}\eta_i\eta_j > \alpha\eta_k\eta_k \quad \text{for any real vector } \eta = (\eta_1, \eta_2).$$

When $\sigma_{12} = 0$ and $\sigma_{11} = \sigma_{22} = \sigma$, the problem (1.4)–(1.5) coincides with the problem (1.1)–(1.2). The boundary-value problem (1.4)–(1.5) originates from various mechanical problems, such as the steady thermal conductivity problem, the problem of harmonic wave propagation in anisotropic elastic media, and so on.

By the homogenization method and following the techniques presented recently in [10, 11, 12], the homogenized equation and the associate continuity condition in the explicit form are derived for both cases when L highly oscillates between two straight lines and L highly oscillates between two concentric ellipses. The obtained results recover the ones derived by NEVARD and KELLER [9] as special cases. Since the obtained homogenized equation is totally explicit it is convenient to use. Note that the technique used in this paper is different from the one employed by NEVARD and KELLER in [9] by which we can not derive the explicit homogenized equation for the general problem.

2. Homogenization of interfaces highly oscillating between two concentric ellipses

2.1. Interfaces highly oscillating between two concentric ellipses

Now we consider the boundary-value problem (1.4)–(1.5) in $D = D_+ \cup L \cup D_-$, where D_+, D_- are separated by the interface L expressed by

$$(2.1) \quad \begin{cases} x = a_1 h(\theta/\varepsilon) \cos \theta, \\ z = b_1 h(\theta/\varepsilon) \sin \theta, \end{cases} \quad 0 \leq \theta < 2\pi,$$

where a_1, b_1 are given positive numbers, $0 < \varepsilon = 2\pi/N \ll 1$, N is a sufficiently large positive integer number, $h(\varphi)$, $\varphi = \theta/\varepsilon$ is a periodic function with period

1 and its minimum value is 1 and its maximum value is $k = a_2 : a_1 > 1$, a_2 is a given positive number. One can see that the (closed) curve L oscillates highly between two concentric ellipses E_1 and E_2 (see Fig. 2) defined respectively by

$$(2.2) \quad \frac{x^2}{a_1^2} + \frac{z^2}{b_1^2} = 1 \quad \text{and} \quad \frac{x^2}{a_2^2} + \frac{z^2}{b_2^2} = 1, \quad a_2 : a_1 = b_2 : b_1 = k > 1.$$

The domain D_- (D_+) lies inside (outside) the closed curve L . We also assume that any ellipse $x^2/a^2 + z^2/b^2 = 1$, $a_1 < a < a_2$, $b_1 < b < b_2$, $a : b = a_1 : b_1 = a_2 : b_2$, has exactly two intersections with the curve L . From (2.1) we have

$$(2.3) \quad n_x : n_z = -z'(\theta) : x'(\theta).$$

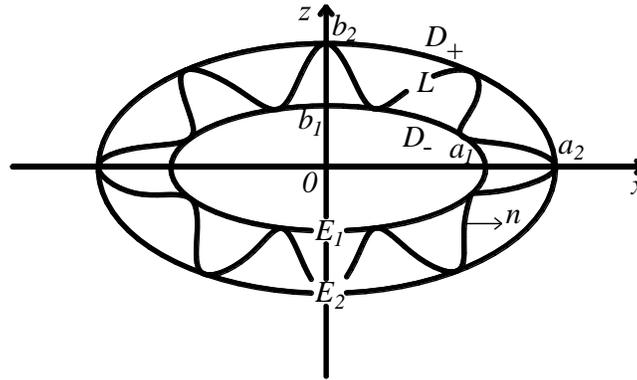


FIG. 2. The interface L , expressed by (2.1), oscillates highly between two concentric ellipses E_1 and E_2 defined by (2.2).

REMARK 1. Through the mapping:

$$(2.4) \quad X = x/a_1, \quad Z = z/b_1,$$

the curve L belonging to the plane (x, z) is mapped to the curve L^* belonging to the plane (X, Z) defined by

$$(2.5) \quad X = h(\theta/\varepsilon) \cos \theta, \quad Z = h(\theta/\varepsilon) \sin \theta, \quad 0 \leq \theta < 2\pi$$

that oscillates highly between two concentric circles $X^2 + Z^2 = 1$ and $X^2 + Z^2 = k^2$, denoted by E_1^* and E_2^* (see Fig. 3), respectively. These circles are images of the ellipses E_1 and E_2 through the mapping (2.4). In terms of the polar coordinates r, θ of the plane (X, Z) (i.e., $X = r \cos \theta$, $Z = r \sin \theta$), the curve L^* is expressed by $r = h(\theta/\varepsilon)$. Since $x = a_1 r \cos \theta$, $z = b_1 r \sin \theta$, the generalized polar coordinates of the plane (x, z) are r, θ .

Our purpose is to study asymptotic behavior of the boundary-value problems (1.4) and (1.5) when $\varepsilon \rightarrow 0$. In particular, we want to find the explicit homogenized equation of the problems (1.4) and (1.5), and the associate boundary conditions in terms of the generalized polar coordinates r, θ .

On view of Remark 1, it is convenient to study the problems (1.4) and (1.5) in the plane (X, Z) . By D_+^* and D_-^* we denote the images of D_+ and D_- , respectively, through the mapping (2.4) (see Fig. 3). The domains D_+^* and D_-^* are separated by L^* which highly oscillates between two concentric circles: E_1^* with radius 1 and E_2^* with radius k , and it is expressed by equation $r = h(\theta/\varepsilon)$.

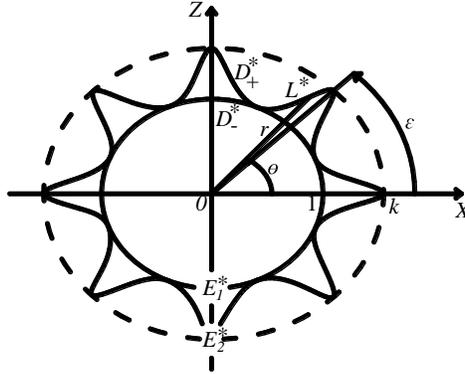


FIG. 3. The curve L^* , expressed by $r = h(\theta/\varepsilon)$, oscillates rapidly between two concentric circles $E_1 : X^2 + Z^2 = 1$ and $E_2 : X^2 + Z^2 = k^2$.

In terms of the variables X, Z , Eqs. (1.4) and (1.5) take the form:

$$(2.6) \quad \frac{1}{a_1^2}(\sigma_{11}U_X)_X + \frac{1}{a_1b_1}[(\sigma_{12}U_Z)_X + (\sigma_{12}U_X)_Z] + \frac{1}{b_1^2}(\sigma_{22}U_Z)_Z - \lambda U = f, \quad (X, Z) \notin L^*,$$

$$(2.7) \quad [U]_{L^*} = 0, \quad \left[\left(\frac{\sigma_{11}}{a_1}U_X + \frac{\sigma_{12}}{b_1}U_Z \right) z'(\theta) - \left(\frac{\sigma_{12}}{a_1}U_X + \frac{\sigma_{22}}{b_1}U_Z \right) x'(\theta) \right]_{L^*} = 0,$$

here $\phi_X := \partial\phi/\partial X$, $\phi_Z := \partial\phi/\partial Z$. Since $X = r \cos \theta$, $Z = r \sin \theta$, we have:

$$(2.8) \quad \begin{cases} \frac{\partial}{\partial X} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial Z} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \end{cases}$$

Introducing (2.8) into (2.6) yields:

$$(2.9) \quad (\sigma_M U_r)_r + \frac{1}{r^2} \left[k_3 (\sigma_{11} U_\theta)_\theta + k_4 (\sigma_{22} U_\theta)_\theta - k_9 (\tilde{\sigma} U_\theta)_\theta \right] \\ + \frac{1}{r} \sigma_T U_r + \frac{2}{r^2} \sigma_K U_\theta \\ - \frac{1}{r} \left[k_5 [(\sigma_{11} U_\theta)_r + (\sigma_{11} U_r)_\theta] - k_6 [(\sigma_{22} U_\theta)_r + (\sigma_{22} U_r)_\theta] \right. \\ \left. + k_7 [(\sigma_{12} U_\theta)_r + (\sigma_{12} U_r)_\theta] - k_8 [(\sigma_{12} U_\theta)_r + (\sigma_{12} U_r)_\theta] \right] - \lambda U = f,$$

where $\phi_r := \partial\phi/\partial r$, $\phi_\theta := \partial\phi/\partial\theta$ and

$$(2.10) \quad k_1 = \frac{\cos^2 \theta}{a_1^2}, \quad k_2 = \frac{\sin^2 \theta}{b_1^2}, \quad k_3 = \frac{\sin^2 \theta}{a_1^2}, \\ k_4 = \frac{\cos^2 \theta}{b_1^2}, \quad k_5 = \frac{\sin \theta \cos \theta}{a_1^2}, \quad k_6 = \frac{\sin \theta \cos \theta}{b_1^2}, \\ k_7 = \frac{\sin^2 \theta}{a_1 b_1}, \quad k_8 = \frac{\cos^2 \theta}{a_1 b_1}, \quad k_9 = \frac{\sin \theta \cos \theta}{a_1 b_1}, \\ \sigma_M = k_1 \sigma_{11} + k_2 \sigma_{22} + k_9 \tilde{\sigma}, \quad \sigma_T = k_3 \sigma_{11} + k_4 \sigma_{22} - k_9 \tilde{\sigma}, \\ \sigma_K = k_5 \sigma_{11} - k_6 \sigma_{22} + (k_7 - k_8) \sigma_{12}, \quad \tilde{\sigma} = 2 \sigma_{12}.$$

Note that $k_3 = 1 - k_1$, $k_4 = 1 - k_2$, $k_8 = 1 - k_7$. Similarly, in terms of the polar coordinates (r, θ) of the plane (X, Z) , the continuity condition (2.7) is of the form:

$$(2.11) \quad [U]_{L^*} = 0, \\ \left[\sigma_{11} U_r \left(-k_1 - \frac{k_5 h'}{\varepsilon r} \right) + \sigma_{12} U_r \left(-k_9 - \frac{k_7 h'}{\varepsilon r} \right) + \sigma_{12} U_r \left(-k_9 + \frac{k_8 h'}{\varepsilon r} \right) \right. \\ + \sigma_{22} U_r \left(-k_2 + \frac{k_6 h'}{\varepsilon r} \right) + \sigma_{11} U_\theta \left(\frac{k_5}{r} + \frac{k_3 h'}{\varepsilon r^2} \right) + \sigma_{12} U_\theta \left(-\frac{k_8}{r} - \frac{k_9 h'}{\varepsilon r^2} \right) \\ \left. + \sigma_{12} U_\theta \left(\frac{k_7}{r} - \frac{k_9 h'}{\varepsilon r^2} \right) + \sigma_{22} U_\theta \left(-\frac{k_6}{r} + \frac{k_4 h'}{\varepsilon r^2} \right) \right]_{L^*} = 0.$$

Equation (2.11)₂ is derived by using (2.1) and (2.8) in (2.7)₂, then dividing the resulting equation by $a_1 b_1$ and noting that $r = h(\theta/\varepsilon)$ on L^* .

2.2. Explicit homogenized equation

Following BENSOUSSAN *et al.* [7], SANCHEZ-PALENCIA [8] we can suppose that $U(r, \theta, \varepsilon) = u(r, \theta, \varphi, \varepsilon)$. Then we have

$$(2.12) \quad U_\theta = u_\theta + \frac{1}{\varepsilon} u_\varphi,$$

here $\phi_\varphi := \partial\phi/\partial\varphi$. Using (2.12) in Eq. (2.9) and (2.11)₂ leads to

$$(2.13) \quad \begin{aligned} & \frac{1}{\varepsilon^2 r^2} \left[k_3 (\sigma_{11} u_\varphi)_\varphi + k_4 (\sigma_{22} u_\varphi)_\varphi - k_9 (\tilde{\sigma} u_\varphi)_\varphi \right] \\ & + \frac{1}{\varepsilon r^2} \left[k_3 [(\sigma_{11} u_\varphi)_\theta + (\sigma_{11} u_\theta)_\varphi] + k_4 [(\sigma_{22} u_\varphi)_\theta + (\sigma_{22} u_\theta)_\varphi] - k_9 [(\tilde{\sigma} u_\varphi)_\theta + (\tilde{\sigma} u_\theta)_\varphi] \right] \\ & + \frac{1}{r^2} \left[k_3 (\sigma_{11} u_\theta)_\theta + k_4 (\sigma_{22} u_\theta)_\theta - k_9 (\tilde{\sigma} u_\theta)_\theta \right] \\ & - \frac{1}{\varepsilon r} \left[k_5 [(\sigma_{11} u_r)_\varphi + (\sigma_{11} u_\varphi)_r] - k_6 [(\sigma_{22} u_r)_\varphi + (\sigma_{22} u_\varphi)_r] + k_7 [(\sigma_{12} u_r)_\varphi + (\sigma_{12} u_\varphi)_r] \right. \\ & \qquad \left. - k_8 [(\sigma_{12} u_r)_\varphi + (\sigma_{12} u_\varphi)_r] \right] \\ & - \frac{1}{r} \left[k_5 [(\sigma_{11} u_r)_\theta + (\sigma_{11} u_\theta)_r] - k_6 [(\sigma_{22} u_r)_\theta + (\sigma_{22} u_\theta)_r] + k_7 [(\sigma_{12} u_r)_\theta + (\sigma_{12} u_\theta)_r] \right. \\ & \left. - k_8 [(\sigma_{12} u_r)_\theta + (\sigma_{12} u_\theta)_r] \right] + \frac{2}{r^2} \sigma_K u_\theta + \frac{2}{\varepsilon r^2} \sigma_K u_\varphi + \frac{1}{r} \sigma_T u_r + (\sigma_M u_r)_r - \lambda u = f, \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} & [u]_{L^*} = 0, \\ & \left[\sigma_{11} u_r \left(-k_1 - \frac{k_5 h'}{\varepsilon r} \right) + \sigma_{12} u_r \left(-k_9 - \frac{k_7 h'}{\varepsilon r} \right) + \sigma_{12} u_r \left(-k_9 + \frac{k_8 h'}{\varepsilon r} \right) \right. \\ & + \sigma_{22} u_r \left(-k_2 + \frac{k_6 h'}{\varepsilon r} \right) + \sigma_{11} u_\theta \left(\frac{k_5}{r} + \frac{k_3 h'}{\varepsilon r^2} \right) + \sigma_{12} u_\theta \left(-\frac{k_8}{r} - \frac{k_9 h'}{\varepsilon r^2} \right) \\ & + \sigma_{12} u_\theta \left(\frac{k_7}{r} - \frac{k_9 h'}{\varepsilon r^2} \right) + \sigma_{22} u_\theta \left(-\frac{k_6}{r} + \frac{k_4 h'}{\varepsilon r^2} \right) + \sigma_{11} u_\varphi \left(\frac{k_5}{\varepsilon r} + \frac{k_3 h'}{\varepsilon^2 r^2} \right) \\ & \left. + \sigma_{12} u_\varphi \left(-\frac{k_8}{\varepsilon r} - \frac{k_9 h'}{\varepsilon^2 r^2} \right) + \sigma_{12} u_\varphi \left(\frac{k_7}{\varepsilon r} - \frac{k_9 h'}{\varepsilon^2 r^2} \right) + \sigma_{22} u_\varphi \left(-\frac{k_6}{\varepsilon r} + \frac{k_4 h'}{\varepsilon^2 r^2} \right) \right]_{L^*} = 0. \end{aligned}$$

Following Vinh and Tung [10, 11, 12],

$$(2.15) \quad u = V + \varepsilon(N^1V + N^{11}V_r + N^{12}V_\theta) \\ + \varepsilon^2(N^2V + N^{21}V_r + N^{22}V_\theta + N^{211}V_{rr} + N^{212}V_{r\theta} + N^{222}V_{\theta\theta}) + O(\varepsilon^3),$$

where $V = V(\theta, r)$ (being independent of φ), $N^i, N^{ij}, N^{ijk}, \dots$ are functions of φ and r (not depending on θ) and they are φ -periodic with period 1. The functions $N^i, N^{ij}, N^{ijk}, \dots$ are chosen so that the equation (2.13) and the continuity conditions (2.14) are satisfied.

From (2.15), it is clear that the continuity (2.14)₁ is satisfied if

$$(2.16) \quad [N^i]_{L^*} = 0, \quad [N^{ij}]_{L^*} = 0, \quad [N^{ijk}]_{L^*} = 0, \quad \dots$$

Substituting (2.15) into (2.13) and (2.14)₂ yields equations which we call (e_1) and (e_2) , respectively. In order to make the coefficients of ε^{-1} and ε^0 of (e_1) zero we take

$$(2.17) \quad \frac{1}{r^2}(\sigma_T N_\varphi^1)_\varphi V + \left(\frac{1}{r^2} \sigma_T N_\varphi^{11} - \frac{1}{r} \sigma_K \right)_\varphi V_r + \frac{1}{r^2} \left(\sigma_T (N_\varphi^{12} + 1) \right)_\varphi V_\theta = 0$$

and

$$(2.18) \quad \left\{ \frac{1}{r^2} (\sigma_T N_\varphi^2)_\varphi - \frac{1}{r} [(\sigma_K N_\varphi^1)_r + (\sigma_K N_r^1)_\varphi] + \frac{2}{r^2} \sigma_K N_\varphi^1 \right\} V \\ + \left\{ \frac{1}{r^2} (\sigma_T N_\varphi^{21})_\varphi - \frac{1}{r} [\sigma_K N_\varphi^1 + (\sigma_K N^1)_\varphi + (\sigma_K N_\varphi^{11})_r + (\sigma_K N_r^{11})_\varphi] \right. \\ \left. + \frac{2}{r^2} \sigma_K N_\varphi^{11} + \frac{1}{r} \sigma_T \right\} V_r \\ + \left\{ \frac{1}{r^2} [(\sigma_T N^1)_\varphi + \sigma_T N_\varphi^1] - \frac{1}{r} [(\sigma_K N_\varphi^{12})_r + (\sigma_K N_r^{12})_\varphi] \right. \\ \left. + \frac{2}{r^2} \sigma_K (N_\varphi^{12} + 1) + \frac{1}{r^2} (\sigma_T N_\varphi^{22})_\varphi \right\} V_\theta \\ + \left\{ \frac{1}{r^2} [(\sigma_T N^{11})_\varphi + \sigma_T N_\varphi^{11} + (\sigma_T N_\varphi^{212})_\varphi] \right. \\ \left. - \frac{1}{r} [\sigma_K (N_\varphi^{12} + 1) + \sigma_K + (\sigma_K N^{12})_\varphi] \right\} V_{r\theta} \\ + \left\{ \frac{1}{r^2} (\sigma_T N_\varphi^{211})_\varphi - \frac{1}{r} [\sigma_K N_\varphi^{11} + (\sigma_K N^{11})_\varphi] \right\} V_{rr} \\ + \frac{1}{r^2} \left\{ (\sigma_T N_\varphi^{222})_\varphi + (\sigma_T N^{12})_\varphi + \sigma_T (N_\varphi^{12} + 1) \right\} V_{\theta\theta} + (\sigma_M V_r)_r - \lambda V = f.$$

Note that $\sigma_T, \sigma_K, \sigma_M$ are independent of r and φ in each domain D_+ and D_- . Vanishing of the coefficients of ε^{-1} and ε^0 of (e_2) gives

$$(2.19) \quad \frac{h'}{r^2}[\sigma_T N_\varphi^1]_{L^*} V + \frac{h'}{r} \left[\frac{1}{r} \sigma_T N_\varphi^{11} - \sigma_K \right]_{L^*} V_r + \frac{h'}{r^2} [\sigma_T (N_\varphi^{12} + 1)]_{L^*} V_\theta = 0,$$

and

$$(2.20) \quad \begin{aligned} & \left[\frac{h'}{r} \sigma_K N_r^1 - \frac{1}{r} \sigma_K N_\varphi^1 - \frac{h'}{r^2} \sigma_T N_\varphi^2 \right]_{L^*} V \\ & + \left[\frac{h'}{r} \sigma_K (N_r^{11} + N^1) + \sigma_M - \frac{1}{r} \sigma_K N_\varphi^{11} - \frac{h'}{r^2} \sigma_T N_\varphi^{21} \right]_{L^*} V_r \\ & + \left[\frac{h'}{r} \sigma_K N_r^{12} - \frac{1}{r} \sigma_K (N_\varphi^{12} + 1) - \frac{h'}{r^2} \sigma_T (N_\varphi^{22} + N^1) \right]_{L^*} V_\theta \\ & + \left[\frac{h'}{r} \sigma_K N^{12} - \frac{h'}{r^2} \sigma_T (N_\varphi^{212} + N^{11}) \right]_{L^*} V_{r\theta} + \left[\frac{h'}{r} \sigma_K N^{11} - \frac{h'}{r^2} \sigma_T N_\varphi^{211} \right]_{L^*} V_{rr} \\ & \quad - \frac{h'}{r^2} [\sigma_T (N^{12} + N_\varphi^{222})]_{L^*} V_{\theta\theta} = 0. \end{aligned}$$

To make (2.17) and (2.19) satisfied, the functions N^1, N^{11}, N^{12} are chosen as follows (taking into account (2.16)):

$$(2.21) \quad \begin{aligned} & (\sigma_T N_\varphi^1)_\varphi = 0, \quad 0 < \varphi < 1, \quad \varphi \neq \varphi_1, \varphi_2, \\ & [\sigma_T N_\varphi^1]_{L^*} = 0, \quad [N^1]_{L^*} = 0, \quad N^1(r, 0) = N^1(r, 1) = 0, \end{aligned}$$

$$(2.22) \quad \begin{aligned} & (\sigma_T [N_\varphi^{12} + 1])_\varphi = 0, \quad 0 < \varphi < 1, \quad \varphi \neq \varphi_1, \varphi_2, \\ & [\sigma_T (N_\varphi^{12} + 1)]_{L^*} = 0, \quad [N^{12}]_{L^*} = 0, \quad N^{12}(r, 0) = N^{12}(r, 1) = 0, \end{aligned}$$

$$(2.23) \quad \begin{aligned} & \left(\frac{1}{r} \sigma_T N_\varphi^{11} - \sigma_K \right)_\varphi = 0, \quad 0 < \varphi < 1, \quad \varphi \neq \varphi_1, \varphi_2, \\ & \left[\left(\frac{1}{r} \sigma_T N_\varphi^{11} - \sigma_K \right) \right]_{L^*} = 0, \quad [N^{11}]_{L^*} = 0, \quad N^{11}(r, 0) = N^{11}(r, 1) = 0, \end{aligned}$$

where φ_1 and φ_2 ($0 < \varphi_1 < \varphi_2 < 1$) are two roots of the equation $h(\varphi) = r$ for φ in the interval $(0, 1)$ in which r , as a parameter, belongs to the domain $(1, k)$. The functions $\varphi_1(r)$ and $\varphi_2(r)$ are two inverse branches of the function $r = h(\varphi)$. Note that, from a mentioned above assumption, any circle $X^2 + Z^2 = c^2$ ($1 < c < k$) has exactly two intersections with the curve L^* .

From (2.21)–(2.23), it is not difficult to show that

$$(2.24) \quad \begin{aligned} N^1 \equiv 0, \quad \langle \sigma_T N_\varphi^{11} \rangle &= r \left(\langle \sigma_K \rangle - \langle \sigma_T^{-1} \rangle^{-1} \left\langle \frac{\sigma_K}{\sigma_T} \right\rangle \right), \quad \langle \sigma_T (N_\varphi^{12} + 1) \rangle = \langle \sigma_T^{-1} \rangle^{-1}, \\ \sigma_T N_\varphi^{11} &= r \left(\sigma_K - \langle \sigma_T^{-1} \rangle^{-1} \left\langle \frac{\sigma_K}{\sigma_T} \right\rangle \right), \quad \sigma_T N_\varphi^{12} = \langle \sigma_T^{-1} \rangle^{-1} - \sigma_T, \end{aligned}$$

where:

$$(2.25) \quad \langle g \rangle = \int_0^1 g d\varphi = (\varphi_2 - \varphi_1)g_+ + (1 + \varphi_1 - \varphi_2)g_-,$$

g_+ and g_- are the values of g in D_+^* and D_-^* , respectively. Note that from (1.8) it follows $\sigma_T > 0$. Taking into account the fact $N^1 \equiv 0$, Eq. (2.18) now is simplified to

$$(2.26) \quad \begin{aligned} & \frac{1}{r^2} (\sigma_T N_\varphi^2)_\varphi V + \left\{ \frac{1}{r^2} (\sigma_T N_\varphi^{21})_\varphi - \frac{1}{r} [(\sigma_K N_\varphi^{11})_r + (\sigma_K N_r^{11})_\varphi] \right. \\ & \qquad \qquad \qquad \left. + \frac{2}{r^2} \sigma_K N_\varphi^{11} + \frac{1}{r} \sigma_T \right\} V_r \\ & + \left\{ \frac{2}{r^2} \sigma_K (N_\varphi^{12} + 1) + \frac{1}{r^2} (\sigma_T N_\varphi^{22})_\varphi - \frac{1}{r} [(\sigma_K N_\varphi^{12})_r + (\sigma_K N_r^{12})_\varphi] \right\} V_\theta \\ & + \left\{ \frac{1}{r^2} [(\sigma_T N^{11})_\varphi + \sigma_T N_\varphi^{11} + (\sigma_T N_\varphi^{212})_\varphi] - \frac{1}{r} [\sigma_K (N_\varphi^{12} + 1) + \sigma_K + (\sigma_K N^{12})_\varphi] \right\} V_{r\theta} \\ & + \left\{ \frac{1}{r^2} (\sigma_T N_\varphi^{211})_\varphi - \frac{1}{r} [\sigma_K N_\varphi^{11} + (\sigma_K N^{11})_\varphi] \right\} V_{rr} \\ & + \frac{1}{r^2} \left\{ (\sigma_T N_\varphi^{222})_\varphi + (\sigma_T N^{12})_\varphi + \sigma_T (N_\varphi^{12} + 1) \right\} V_{\theta\theta} + (\sigma_M V_r)_r - \lambda V = f. \end{aligned}$$

In order to satisfy (2.20) we take (noting that $N^1 \equiv 0$ by (2.24)₁):

$$(2.27) \quad \begin{aligned} & [\sigma_T N_\varphi^2]_{L^*} = 0, \\ & \left[\left\{ \sigma_M + \frac{h'}{r} \sigma_K N_r^{11} - \frac{1}{r} \sigma_K N_\varphi^{11} - \frac{h'}{r^2} \sigma_T N_\varphi^{21} \right\} \right]_{L^*} = 0, \\ & \left[\left\{ \frac{h'}{r} \sigma_K N_r^{12} - \frac{1}{r} \sigma_K (N_\varphi^{12} + 1) - \frac{h'}{r^2} \sigma_T N_\varphi^{22} \right\} \right]_{L^*} = 0, \\ & \left[\frac{1}{r} \sigma_K N^{12} - \frac{1}{r^2} \sigma_T (N^{11} + N_\varphi^{212}) \right]_{L^*} = 0, \\ & \left[\frac{1}{r} \sigma_K N^{11} - \frac{1}{r^2} \sigma_T N_\varphi^{211} \right]_{L^*} = 0, \quad [\sigma_T (N^{12} + N_\varphi^{222})]_{L^*} = 0. \end{aligned}$$

By integrating equation (2.26) along the circle $r = \text{const}$, $1 < r < k$ from $\varphi = 0$ to $\varphi = 1$ and taking into account the fact $\langle (\sigma_M V_r)_r \rangle = \langle \sigma_M \rangle V_{rr}$ we have

$$(2.28) \quad A_0 V + A_1 V_r + A_2 V_\theta + A_{11} V_{rr} + A_{12} V_{r\theta} + A_{22} V_{\theta\theta} - \langle \lambda \rangle V = f,$$

where:

$$(2.29) \quad \begin{aligned} A_0 &= \frac{1}{r^2} \langle (\sigma_T N_\varphi^2)_\varphi \rangle, \\ A_1 &= \left\langle \frac{1}{r^2} (\sigma_T N_\varphi^{21})_\varphi - \frac{1}{r} (\sigma_K N_r^{11})_\varphi \right\rangle - \frac{1}{r} \langle (\sigma_K N_\varphi^{11})_r \rangle \\ &\quad + \frac{2}{r^2} \langle \sigma_K N_\varphi^{11} \rangle + \frac{1}{r} \langle \sigma_T \rangle, \\ A_2 &= \left\langle \frac{1}{r^2} (\sigma_T N_\varphi^{22})_\varphi - \frac{1}{r} (\sigma_K N_r^{12})_\varphi \right\rangle + \frac{2}{r^2} \langle \sigma_K (N_\varphi^{12} + 1) \rangle \\ &\quad - \frac{1}{r} \langle (\sigma_K N_\varphi^{12})_r \rangle, \\ A_{11} &= \left\langle \frac{1}{r^2} (\sigma_T N_\varphi^{211})_\varphi - \frac{1}{r} (\sigma_K N^{11})_\varphi \right\rangle - \frac{1}{r} \langle \sigma_K N_\varphi^{11} \rangle + \langle \sigma_M \rangle, \\ A_{12} &= \left\langle \frac{1}{r^2} [(\sigma_T N^{11})_\varphi + (\sigma_T N_\varphi^{212})_\varphi] - \frac{1}{r} (\sigma_K N^{12})_\varphi \right\rangle + \frac{1}{r^2} \langle \sigma_T N_\varphi^{11} \rangle, \\ &\quad - \frac{1}{r} [\langle \sigma_K (N_\varphi^{12} + 1) \rangle + \langle \sigma_K \rangle], \\ A_{22} &= \frac{1}{r^2} [\langle (\sigma_T N_\varphi^{222})_\varphi + (\sigma_T N^{12})_\varphi \rangle + \langle \sigma_T (N_\varphi^{12} + 1) \rangle]. \end{aligned}$$

From (2.27)₁, (2.27)₄, (2.27)₅ and (2.27)₆ it is clear that

$$(2.30) \quad \begin{aligned} \langle (\sigma_T N_\varphi^2)_\varphi \rangle &= 0, \\ \left\langle \frac{1}{r^2} (\sigma_T N_\varphi^{211})_\varphi - \frac{1}{r} (\sigma_K N^{11})_\varphi \right\rangle &= 0, \\ \left\langle \frac{1}{r^2} [(\sigma_T N^{11})_\varphi + (\sigma_T N_\varphi^{212})_\varphi] - \frac{1}{r} (\sigma_K N^{12})_\varphi \right\rangle &= 0, \\ \frac{1}{r^2} \langle (\sigma_T N_\varphi^{222})_\varphi + (\sigma_T N^{12})_\varphi \rangle &= 0, \end{aligned}$$

therefore from (2.29)₁, (2.29)₄–(2.29)₆:

$$(2.31) \quad \begin{aligned} A_0 &= 0, \quad A_{11} = -\frac{1}{r} \langle \sigma_K N_\varphi^{11} \rangle + \langle \sigma_M \rangle, \\ A_{12} &= \frac{1}{r^2} \langle \sigma_T N_\varphi^{11} \rangle - \frac{1}{r} [\langle \sigma_K (N_\varphi^{12} + 1) \rangle + \langle \sigma_K \rangle], \\ A_{22} &= \frac{1}{r^2} \langle \sigma_T (N_\varphi^{12} + 1) \rangle. \end{aligned}$$

In order to evaluate A_1, A_2 we use the following equality (see Eq. (26) in [12]):

$$(2.32) \quad \langle F_r \rangle + d\delta F = \langle F \rangle_r,$$

where

$$(2.33) \quad d = [1/h'(\varphi_2) - 1/h'(\varphi_1)], \quad \delta F = (F_+ - F_-),$$

and F_+, F_- are the (constant-in φ) values of F in D_+, D_- , respectively.

Taking into account (2.27)₂ we have

$$(2.34) \quad \left\langle \frac{1}{r^2}(\sigma_T N_\varphi^{21})_\varphi - \frac{1}{r}(\sigma_K N_r^{11})_\varphi \right\rangle = d \left[\delta\sigma_M - \frac{1}{r}\delta(\sigma_K N_\varphi^{11}) \right].$$

Applying (2.32) for $F = \sigma_K N_\varphi^{11}$ we have

$$(2.35) \quad -\frac{1}{r}\langle (\sigma_K N_\varphi^{11})_r \rangle = -\frac{1}{r}\langle \sigma_K N_\varphi^{11} \rangle_r + \frac{1}{r}d\delta(\sigma_K N_\varphi^{11}).$$

Substituting (2.34) and (2.35) into (2.29)₂ yields:

$$(2.36) \quad A_1 = d\delta\sigma_M - \frac{1}{r}\langle \sigma_K N_\varphi^{11} \rangle_r + \frac{2}{r^2}\langle \sigma_K N_\varphi^{11} \rangle + \frac{1}{r}\langle \sigma_T \rangle.$$

Similarly, from (2.27)₃ it follows:

$$(2.37) \quad \left\langle \frac{1}{r^2}(\sigma_T N_\varphi^{22})_\varphi - \frac{1}{r}(\sigma_K N_r^{12})_\varphi \right\rangle = -\frac{1}{r}d\delta[\sigma_K(N_\varphi^{12} + 1)],$$

and applying (2.32) for $F = \sigma_K N_\varphi^{12}$ we have

$$(2.38) \quad -\frac{1}{r}\langle (\sigma_K N_\varphi^{12})_r \rangle = -\frac{1}{r}\langle \sigma_K N_\varphi^{12} \rangle_r + \frac{1}{r}d\delta(\sigma_K N_\varphi^{12}).$$

Therefore, on view of (2.29)₃, (2.37) and (2.38), A_2 is expressed by

$$(2.39) \quad A_2 = -\frac{1}{r}d\delta\sigma_K - \frac{1}{r}\langle \sigma_K N_\varphi^{12} \rangle_r + \frac{2}{r^2}\langle \sigma_K(N_\varphi^{12} + 1) \rangle.$$

From (2.25) one can see that

$$(2.40) \quad d\delta\sigma_M = \langle \sigma_M \rangle_r, \quad d\delta\sigma_K = \langle \sigma_K \rangle_r.$$

From (2.24) it follows:

$$(2.41) \quad \begin{aligned} \langle \sigma_K N_\varphi^{11} \rangle &= r[\langle \sigma_K^2 \sigma_T^{-1} \rangle - \langle \sigma_K \sigma_T^{-1} \rangle^2 \langle \sigma_T^{-1} \rangle^{-1}], \\ \langle \sigma_K N_\varphi^{12} \rangle &= \langle \sigma_K \sigma_T^{-1} \rangle \langle \sigma_T^{-1} \rangle^{-1} - \langle \sigma_K \rangle. \end{aligned}$$

Introducing (2.40) and (2.41) into (2.31), (2.36) and (2.39) and taking into account (2.24) lead to

$$\begin{aligned}
 A_1 &= \langle \sigma_M \rangle_r + \frac{1}{r} \langle \sigma_T \rangle - r \left[\frac{1}{r} \{ \langle \sigma_K^2 \sigma_T^{-1} \rangle - \langle \sigma_K \sigma_T^{-1} \rangle^2 \langle \sigma_T^{-1} \rangle^{-1} \} \right]_r, \\
 A_2 &= -r \left[\frac{1}{r^2} \{ \langle \sigma_K \sigma_T^{-1} \rangle \langle \sigma_T^{-1} \rangle^{-1} \} \right]_r, \\
 (2.42) \quad A_{11} &= \langle \sigma_M \rangle - \langle \sigma_K^2 \sigma_T^{-1} \rangle + \langle \sigma_K \sigma_T^{-1} \rangle^2 \langle \sigma_T^{-1} \rangle^{-1}, \\
 A_{12} &= -\frac{2}{r} \langle \sigma_K \sigma_T^{-1} \rangle \langle \sigma_T^{-1} \rangle^{-1}, \\
 A_{22} &= \frac{1}{r^2} \langle \sigma_T^{-1} \rangle^{-1}.
 \end{aligned}$$

The desired explicit homogenized equation finally takes the form:

$$\begin{aligned}
 (2.43) \quad A_1 V_r + A_2 V_\theta + A_{11} V_{rr} + A_{12} V_{r\theta} + A_{22} V_{\theta\theta} - \langle \lambda \rangle V &= f, \\
 1 < r < k, \quad 0 \leq \theta < 2\pi,
 \end{aligned}$$

in which the coefficients $A_1, A_2, A_{11}, A_{12}, A_{22}$ are given by (2.42). In the domains $0 < r < 1, 0 \leq \theta < 2\pi$ and $r > k, 0 \leq \theta < 2\pi$ (of the plane (x, z)) are satisfied the following equations:

$$\begin{aligned}
 (2.44) \quad \sigma_M^+ V_{rr} - \frac{2}{r} \sigma_K^+ V_{r\theta} + \frac{1}{r^2} \sigma_T^+ V_{\theta\theta} + \frac{1}{r} \sigma_T^+ V_r \\
 + \frac{2}{r^2} \sigma_K^+ V_\theta - \lambda^+ V = f, \quad r > k, \quad 0 \leq \theta < 2\pi, \\
 \sigma_M^- V_{rr} - \frac{2}{r} \sigma_K^- V_{r\theta} + \frac{1}{r^2} \sigma_T^- V_{\theta\theta} + \frac{1}{r} \sigma_T^- V_r \\
 + \frac{2}{r^2} \sigma_K^- V_\theta - \lambda^- V = f, \quad 0 < r < 1, \quad 0 \leq \theta < 2\pi,
 \end{aligned}$$

which come from (2.9). Note that Eq. (2.44)₁ [(2.44)₂] can be obtained from Eq. (2.43) by replacing σ_K, σ_M and σ_T by σ_K^+, σ_M^+ and σ_T^+ [by σ_K^-, σ_M^- and σ_T^-], respectively. In addition to Eq. (2.43) and Eq. (2.44) the continuity conditions are required on the ellipses E_1 and E_2 , namely:

$$\begin{aligned}
 (2.45) \quad [V]_{E_i} &= 0, \\
 \left[\left(\langle \sigma_M \rangle - \langle \sigma_K^2 \sigma_T^{-1} \rangle + \langle \sigma_T^{-1} \sigma_K \rangle^2 \langle \sigma_T^{-1} \rangle^{-1} \right) V_r \right. \\
 \left. - \frac{1}{r} \langle \sigma_T^{-1} \sigma_K \rangle \langle \sigma_T^{-1} \rangle^{-1} V_\theta \right]_{E_i} &= 0, \quad i = 1, 2.
 \end{aligned}$$

The ellipses E_1 and E_2 correspond to $r = 1$ and $r = k$, respectively. Note that the continuity condition $(2.45)_2$ originates from the condition: $[q_n]_{E_i} = 0$, $i = 1, 2$, n is the normal unit for E_i , or equivalently:

$$(2.46) \quad \left[\sigma_M u_r - \frac{\sigma_K}{r} (u_\theta + \varepsilon^{-1} u_\varphi) \right]_{E_i} = 0, \quad i = 1, 2.$$

Introducing (2.15) into (2.46) yields an equation denoted by (e_3) . Equating to zero the coefficient of ε^0 of (e_3) provides

$$(2.47) \quad \left[\left(\sigma_M - \frac{1}{r} \sigma_K N_\varphi^{11} \right) V_r - \frac{1}{r} \sigma_K (N_\varphi^{12} + 1) V_\theta \right]_{E_i} = 0, \quad i = 1, 2.$$

Integrating (2.47) along the lines $r = 1$ and $r = k$ from $\varphi = 0$ to $\varphi = 1$ and using (2.41) we arrive at $(2.45)_2$.

When the ellipses E_1 and E_2 become the circles with radii $a_1 (= b_1)$ and $a_2 (= b_2)$ and $\sigma_{12} = 0$, $\sigma_{11} = \sigma_{22} = \sigma$, we have: $\sigma_M = \sigma_T = \sigma/a_1^2$, $\sigma_K = 0$; therefore, by using (2.42):

$$(2.48) \quad A_1 = \frac{1}{a_1^2 r} [r \langle \sigma \rangle]_r, \quad A_2 = 0, \quad A_{11} = \frac{1}{a_1^2} \langle \sigma \rangle, \quad A_{12} = 0, \quad A_{22} = \frac{1}{a_1^2 r^2} \langle \sigma^{-1} \rangle^{-1}.$$

With (2.48) the homogenized equation (2.43) and the continuity condition (2.45) are simplified respectively to

$$(2.49) \quad \frac{1}{r} (r \langle \sigma \rangle V_r)_r + \frac{1}{r^2} \langle \sigma^{-1} \rangle^{-1} V_{\theta\theta} - \langle \lambda \rangle V = f,$$

and

$$(2.50) \quad [V]_{E_i} = 0, \quad [\langle \sigma \rangle V_r]_{E_i} = 0, \quad i = 1, 2,$$

where (r, θ) are the polar coordinates in the (x, z) -plane. The homogenized equation (2.49) and the continuity condition (2.50) coincide with Eq. (4.6) and its associate continuity condition in [9] obtained by NEVARD and KELLER in a different way.

3. Homogenization of interfaces highly oscillating between to straight lines

Applying Theorem 1 of the paper [10] for the matrices $\mathbf{A}_{ij} = (\sigma_{ij})_{1 \times 1}$, $\mathbf{u} = U \exp(i\omega t)$, $\mathbf{F} = -f \exp(i\omega t)$, $\lambda = -\rho\omega^2$ (ρ and ω being some fixed positive numbers, $i^2 = -1$) we arrive immediately at the explicit homogenized equation and the associate continuity condition of the problems (1.4)–(1.5) for the case

when L highly oscillates between two straight lines $z = -A$ and $z = 0$ (Fig. 1), namely:

$$(3.1) \quad \langle \sigma_{11}^{-1} \rangle^{-1} V_{xx} + \langle \sigma_{11}^{-1} \rangle^{-1} \langle \sigma_{11}^{-1} \sigma_{12} \rangle V_{xz} + [\langle \sigma_{11}^{-1} \rangle^{-1} \langle \sigma_{11}^{-1} \sigma_{12} \rangle V_x]_z + [\{ \langle \sigma_{22} \rangle + \langle \sigma_{11}^{-1} \rangle^{-1} \langle \sigma_{11}^{-1} \sigma_{12} \rangle^2 - \langle \sigma_{11}^{-1} \sigma_{12}^2 \rangle \} V_z]_z - \langle \lambda \rangle V = f, \quad -A < z < 0,$$

and

$$(3.2) \quad [\langle \sigma_{11}^{-1} \rangle^{-1} \langle \sigma_{11}^{-1} \sigma_{12} \rangle V_x + \{ \langle \sigma_{22} \rangle + \langle \sigma_{11}^{-1} \rangle^{-1} \langle \sigma_{11}^{-1} \sigma_{12} \rangle^2 - \langle \sigma_{11}^{-1} \sigma_{12}^2 \rangle \} V_z]_{\bar{L}} = 0, \\ [V]_{\bar{L}} = 0, \quad \bar{L} \text{ is lines } z = -A, z = 0.$$

Note that, since the matrix $(\sigma_{ij})_{2 \times 2}$ is positive definite, $\sigma_{11} > 0$. When $\sigma_{12} = 0$, $\sigma_{11} = \sigma_{22} = \sigma$, Eqs. (3.1) and (3.2) are simplified to

$$(3.3) \quad \langle \sigma^{-1} \rangle^{-1} V_{xx} + [\langle \sigma \rangle V_z]_z - \langle \lambda \rangle V = f, \quad -A < z < 0, \\ [V]_{\bar{L}} = 0, \quad [\langle \sigma \rangle V_z]_{\bar{L}} = 0, \quad \bar{L} \text{ is lines } z = -A, z = 0.$$

that coincide with the ones obtained by NEVARD and KELLER [9].

4. Conclusions

By using the homogenization method and the techniques introduced recently in [10, 11, 12], the authors derive the homogenized equation and the associate continuity condition in the explicit form of a boundary-value problem, originating from various problems in practical applications, in two-dimensional domains separated by an interface which highly oscillates between two concentric ellipses and between two straight lines. The obtained results recover the ones derived by NEVARD and KELLER [9] as special cases. Since the obtained homogenized equation and the associate continuity conditions are explicit, they are convenient to use.

Finally, one can see that the technique presented in this paper is still applicable for the case when the scalars σ_{ij} and λ are square matrices of order $n \geq 1$, U is a unknown n -dimensional vector and f is a given n -dimensional vector. Therefore, by this technique we can extend the results obtained in [12] to the case when the interface L highly oscillating between two concentric ellipses instead of two concentric circles.

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