

Faxén's law for arbitrary oscillatory Stokes flow past a porous sphere

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THE PRESENT ARTICLE DEALS WITH the study of the hydrodynamics of a porous sphere in an oscillatory viscous flow of an incompressible Newtonian fluid. Unsteady Stokes equations are used for the flow outside the porous sphere and Darcy's equation is used for the flow inside the porous sphere. Corresponding Faxén's law for drag and torque acting on the surface of the porous sphere is derived. Also the results are compared with few existing special cases. Examples like uniform flow, oscillating Stokeslet, oscillatory shear flow and quadratic shear flow are discussed.

Key words: Stokes flow, Darcy's law, Saffman's condition, Faxén's law.

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1. Introduction

THE STUDY OF PERIODIC AND OSCILLATORY FLOWS in porous media has recently received considerable attention because of its application to biology, environmental sciences and industry. Stokes flow past porous spherical particles has been studied by many researchers [1]– [5]. They have used Stokes equation outside the porous region and Brinkman equation/Darcy law, inside the porous region. The above studies considered linearized Navier–Stokes equations without the inertial term. The problem of a porous sphere in a viscous fluid has been studied by FENG and MICHAELIDES [6], considering steady Navier–Stokes equations including the inertial non-linear term. They have used Darcy's law inside the porous sphere and calculated the solution using matched asymptotic expansions. In case of porous sphere experiencing unsteady motion, less attention has been given. The unsteady Stokes equations for the microscopic flow in porous media subject to an oscillatory pressure gradient has been studied by

CHAPMAN and HIGDON [7], where the media consist of periodic array of spheres ranging from dilute systems with isolated spheres to highly concentrated consolidated media with overlapping spheres. The effect of oscillatory forcing on flow through porous media has been studied by GRAHAM and HIGDON [8] and the impact of inertial effects on the flow rate produced by a given pressure gradient is analyzed. Study on the hydrodynamics of a rigid, weakly permeable sphere undergoing translational oscillations in an incompressible Newtonian fluid has been done by LOOKER and CARNIE [9], where Darcy's law is used inside the porous sphere. It is shown in [10, 11] that homogenization of the full Navier–Stokes equations in a periodic porous medium yields Darcy's law. Also using asymptotic expansion for the velocity and pressure via homogenization, LOOKER and CARNIE [9] have shown that the macroscopic equations for unsteady Stokes flow in a periodic porous medium have the same form as for steady flow, i.e., Darcy's law.

Another important area where oscillatory forcing plays a significant role is during the analysis of convective mass transfer in porous catalysts [12, 13]. In case of small, highly porous catalyst, the particles diffusion alone may not account for the nutrient transport, and convective flow has a major role. It is evident that under steady state, convective flow within a porous catalyst is not so important, whereas oscillatory forcing at higher amplitude and/or lower frequencies enhances the mass transfer. NI *et al.* [14] observed that oscillatory flow improves the performance of a bed packed with spherical particles. CRITTENDEN *et al.* [15] studied the influence of oscillatory flow on axial dispersion in packed beds of spheres. They observed that the best reduction (up to 50%) in the axial dispersion coefficient from the non-oscillation base value is at the highest frequency considered and when the column to particle size is the smallest. Hence, the present study aims at understanding of the hydrodynamics of oscillatory Stokes flow past a porous sphere. Such an investigation not only gives an idea of the hydrodynamic forces acting on the surface of a porous sphere, but also the corresponding calculations can be used in order to understand the mass transfer inside porous pellets under oscillatory forcing. Another important application is to analyze acoustic properties of granular materials. UMNova *et al.* [16] have considered oscillatory flow of viscous incompressible flow around a spherical particle and used the cell model, in order to estimate the hydrodynamic drag due to oscillating flow in a stack of fixed identical rigid spheres. The present investigation is also useful in understanding acoustic properties of porous materials.

Recently, VAINSHTEIN and SHAPIRO [17] gave a theoretical investigation on forces acting on a porous sphere oscillating in a viscous fluid. The flow outside the porous sphere is governed by inhomogeneous Stokes equation and the Darcy–Brinkman equation that include an unsteady term for the flow inside the porous

sphere. They have used continuity of velocity components together with continuity of pressure in case of Darcy law and obtained a force acting on the surface of a porous sphere by considering an uniform oscillating flow. The corresponding expressions for the limiting values of low and high frequencies are obtained, when the flow inside the porous sphere is governed by Brinkman equation. It may be noted that the continuity of velocity components, shear stress together with the continuity of pressure are justified when Brinkman equation is used inside the porous region, while these boundary conditions need some attention when Darcy's law is used because in the latter case, BEAVERS and JOSEPH [18], and SAFFMAN [19] type slip boundary condition is more appropriate at a porous-liquid interface. In case of steady Newtonian flows, while employing Darcy's law inside a porous region, the Saffman's condition is used, along with continuity of normal velocity and continuity of pressure (see [5, 20, 21]). LOOKER and CARNIE [9] have shown that Saffman's boundary condition can be applied for oscillatory Stokes flows at low frequency.

The flow coupling between an external Stokes flow with the internal Darcy flow is generally via the continuity of normal velocity, which requires another supplement of pressure continuity. LOOKER and CARNIE [9] argued that due to the decoupling between external and internal flows, one can avoid using pressure continuity at the porous-liquid interface. It is shown that continuity of normal velocity reduces to no penetration condition in case of weakly permeable sphere. However, in general, while dealing with viscous flows past spherical porous bodies that are not weakly permeable, one has to consider the internal flow and in such cases, pressure continuity is a valid boundary condition. Therefore, the present study is to understand the hydrodynamics of arbitrary oscillatory flow past a porous sphere employing Darcy's law inside the porous region. It may be noted that the present study differs from that of VAINSHTEIN and SHAPIRO [17] in several issues. They have obtained forces acting on the surface of a permeable particle in oscillating flow by considering uniform flow, whereas the present study is focused on deriving more general expressions for the forces acting on the surface of porous sphere, in terms of Faxén's law, by considering an arbitrary oscillatory Stokes flow past a porous sphere. They have obtained expression for the drag corresponding to the Darcy case (as a low permeability of Brinkman's case) using continuity of velocity, continuity of pressure and continuity of shear stress. However, it is well-known that in case of Darcy flow, either Saffman-Beavers-Joseph slip condition is more appropriate (see [5, 20, 21]) and the present investigation is based on the Saffman's slip condition. The potential of the method is evident with the fact that arbitrary flow can be handled. We discuss examples like the uniform flow, oscillatory shear flow and flow due to an oscillating Stokeslet and several limiting cases are also discussed.

2. Mathematical formulation and method of solution

Let us consider a stationary porous sphere of radius a and permeability k in an arbitrary oscillatory flow of a viscous incompressible fluid. Let us assume that the flow inside the porous sphere ($r < a$) is governed by the Darcy's law and continuity equation:

$$(2.1) \quad \mathbf{V}^i = -\frac{k}{\mu} \nabla P^i,$$

$$(2.2) \quad \nabla \cdot \mathbf{V}^i = 0,$$

where k is the permeability of the porous medium, μ is the coefficient of viscosity of the fluid. The flow outside the porous sphere ($r > a$) is described by the unsteady Stokes and continuity equations:

$$(2.3) \quad \rho \frac{\partial \mathbf{v}^e}{\partial t} = -\nabla p^e + \mu \nabla^2 \mathbf{v}^e,$$

$$(2.4) \quad \nabla \cdot \mathbf{v}^e = 0,$$

where ρ is the density of the fluid. Since we are interested in the study of oscillatory flow with frequency ω , we set the velocity and pressure fields \mathbf{v}^e and p^e as $\mathbf{v}^e = \mathbf{V}^e e^{-i\omega t}$ and $p^e = P^e e^{-i\omega t}$. The complex treatment is possible since linear equations are considered. Thus, the governing equations transform to

$$(2.5) \quad -i\rho\omega \mathbf{V}^e = -\nabla P^e + \mu \nabla^2 \mathbf{V}^e,$$

$$(2.6) \quad \nabla \cdot \mathbf{V}^e = 0.$$

Here \mathbf{V}^e and P^e represent the velocity and pressure fields outside the porous sphere, and \mathbf{V}^i and P^i are those of the flow inside the porous sphere. The physical quantities are non-dimensionalized by using the transformation

$$\tilde{\mathbf{X}} = \frac{\mathbf{X}}{a}, \quad \tilde{\mathbf{V}} = \frac{\mathbf{V}}{U}, \quad \tilde{P} = \frac{P}{\mu \frac{Ua}{k}},$$

where U is a characteristic velocity.

Therefore, the non-dimensional equations for the flow inside the porous region ($r < 1$) take the form

$$(2.7) \quad \mathbf{V}^i = -\nabla P^i,$$

$$(2.8) \quad \nabla \cdot \mathbf{V}^i = 0,$$

and the corresponding equations for the fluid region ($r > 1$) reduce to

$$(2.9) \quad \left(\nabla^2 \frac{k}{a^2} + \frac{i\omega a^2}{\nu} \frac{k}{a^2} \right) \mathbf{V}^e = \nabla P^e,$$

$$(2.10) \quad \nabla \cdot \mathbf{V}^e = 0.$$

Note that we have omitted the symbol \sim in Eqs. (2.7)–(2.10).

Now (2.9) together with (2.10) can be written as

$$(2.11) \quad (\nabla^2 - \lambda^2)\mathbf{W}^e = \nabla P^e,$$

$$(2.12) \quad \nabla \cdot \mathbf{W}^e = 0,$$

where $\lambda^2 = -i\omega a^2/\nu$, $\mathbf{W}^e = Da\mathbf{V}^e$ and $Da = k/a^2$ is the Darcy number.

In addition, since the unsteady oscillatory Stokes equation has a similar form (mathematically) to the Brinkman equation, we use the following representation of the velocity and pressure fields, \mathbf{W}^e and P^e :

$$(2.13) \quad \mathbf{W}^e = \nabla \times \nabla \times (A^e \mathbf{X}) + \nabla \times (B^e \mathbf{X}),$$

$$(2.14) \quad P^e = p_0 + \frac{\partial}{\partial r}[r(\nabla^2 - \lambda^2)A^e],$$

which, in fact, yield the complete general solution of the Brinkman and continuity equations (see [4, 22]). Here \mathbf{X} is the position vector of the current point, p_0 is a constant, and A^e and B^e are unknown scalar functions satisfying the equations

$$(2.15) \quad \nabla^2(\nabla^2 - \lambda^2)A^e = 0, \quad (\nabla^2 - \lambda^2)B^e = 0.$$

Let us now assume that the velocity field \mathbf{W}_0 of the basic flow, i.e. of the unperturbed flow, is given in absence of any boundaries by

$$(2.16) \quad \mathbf{W}_0 = \nabla \times \nabla \times (A_0 \mathbf{X}) + \nabla \times (B_0 \mathbf{X}),$$

$$(2.17) \quad A_0 = \sum_{n=1}^{\infty} [\alpha_n r^n + \beta_n f_n(\lambda r)] S_n(\theta, \varphi),$$

$$B_0 = \sum_{n=1}^{\infty} \gamma_n f_n(\lambda r) T_n(\theta, \varphi),$$

where $f_n(\lambda r)$ are modified spherical Bessel functions of the first kind, which are finite at the center of porous sphere and $S_n(\theta, \varphi)$ and $T_n(\theta, \varphi)$ are spherical harmonics of the form

$$(2.18) \quad S_n(\theta, \varphi) = \sum_{m=0}^n P_n^m(\xi)(A_{nm} \cos m\varphi + B_{nm} \sin m\varphi), \quad \xi = \cos \theta,$$

$$(2.19) \quad T_n(\theta, \varphi) = \sum_{m=0}^n P_n^m(\cos \theta)(C_{nm} \cos m\varphi + D_{nm} \sin m\varphi),$$

where P_n^m are associated Legendre polynomials and $A_{nm}, B_{nm}, C_{nm}, D_{nm}$ are the known coefficients. The coefficients $\alpha_n, \beta_n, \gamma_n$ are arbitrary constants and corresponding to a given basic flow; in absence of any boundaries, $\alpha_n, \beta_n, \gamma_n$ take a suitable form. For example, in case of uniform flow along the z -axis, we have $\alpha_1 = 1/2, \beta_1 = 0, \gamma_1 = 0$. In addition, the scalar functions A_0 and B_0 satisfy (2.15). It may be noted that the scalars A, B represent the flow field and the vector equations are now reduced to equivalent scalar equations.

On the other hand, if the basic flow with the velocity field \mathbf{W}_0 is perturbed by the presence of a stationary porous sphere with the radius $r = 1$, then the velocity field \mathbf{W}^e of the resulting flow outside the porous sphere is given by the relation $\mathbf{W}^e = \mathbf{W}_0 + \mathbf{W}^*$, where \mathbf{W}^* is the velocity due to the disturbed flow such that $\mathbf{W}^* \rightarrow 0$ as $r \rightarrow \infty$. Hence, the resulting flow in the exterior region ($r > 1$) is given by

$$(2.20) \quad A^e = \sum_{n=1}^{\infty} \left[\alpha_n r^n + \frac{\alpha'_n}{r^{n+1}} + \beta_n f_n(\lambda r) + \beta'_n g_n(\lambda r) \right] S_n(\theta, \varphi),$$

$$(2.21) \quad B^e = \sum_{n=1}^{\infty} [\gamma_n f_n(\lambda r) + \gamma'_n g_n(\lambda r)] T_n(\theta, \varphi),$$

where $g_n(\lambda r)$ are modified spherical Bessel functions of the second kind. Since in the porous region ($r < 1$) the pressure field is harmonic and finite at the origin, it can be expressed as

$$(2.22) \quad P^i = p_0 + \sum_{n=1}^{\infty} \delta_n r^n S_n(\theta, \varphi),$$

where (r, θ, φ) are spherical coordinates with respect to the origin, chosen at the center of the sphere $r = 1$. In the above expressions $\alpha'_n, \beta'_n, \gamma'_n$ and δ_n are unknown constants. The unknowns are to be determined from the boundary conditions.

2.1. Boundary conditions

The obvious choice of boundary condition at a permeable interface is the continuity of normal component of velocity, which is the consequence of incompressibility. In order to have a completely determined flow of the free fluid, a certain condition on the tangential component of the free fluid velocity needs to be specified at the interface. Based on experiments, BEAVERS [18] (BJ) proposed a condition involving porous-liquid interface. Further, TAYLOR [23] and RICHARDSON [24] provided support for the BJ condition. A mathematical justification of this interface condition was obtained by SAFFMAN [19]. A common choice for

boundary conditions while matching Darcy's law with the Stokes equation, is continuity of pressure and continuity of normal velocity components, along with Saffman's slip condition for tangential velocity components [5, 20, 21, 25]. But, when the flow is of oscillatory nature, applicability of Saffman's condition at the porous–liquid interface is validated by LOOKER and CARNIE [9], however under low frequency. In analogy to the restriction on the Darcy number proposed by LOOKER and CARNIE [9], the present system is assumed to satisfy the following conditions:

$$(2.23) \quad \frac{\sqrt{k}}{a} \ll 1, \quad \varpi = \frac{\omega a^2}{\nu} = O(1),$$

and a is assumed to be much smaller than the wavelength of sound in the fluid. This ensures that Saffman's slip condition can be applied at the surface of the porous sphere in case of oscillatory flow [9]. It can be seen from (2.9) that all the terms resulting from the oscillatory flow are $O((a\sqrt{Da}\sqrt{\omega/\nu})^2)$. Hence, provided that

$$(2.24) \quad a\sqrt{Da}\sqrt{\frac{\omega}{\nu}} \ll 1,$$

the flow near the boundary may be treated as steady with new velocity $\mathbf{W}^e = Da\mathbf{V}^e$. Hence, one can use Saffman's slip condition. The relation (2.24) is equivalent to

$$(2.25) \quad \omega \ll \frac{\nu}{a^2 Da}.$$

Therefore, we consider the following conditions at the boundary between the porous and fluid regions, i.e., for $r = 1$.

(i) Continuity of the pressure field:

$$(2.26) \quad P^e = P^i.$$

(ii) Continuity of the normal velocity component:

$$(2.27) \quad V_r^e = V_r^i \Rightarrow W_r^e = DaV_r^i.$$

(iii) Saffman's boundary condition for the tangential components of the velocity field:

$$(2.28) \quad W_\theta^e = \frac{\sqrt{Da}}{\alpha} \frac{\partial W_\theta^e}{\partial r}, \quad W_\varphi^e = \frac{\sqrt{Da}}{\alpha} \frac{\partial W_\varphi^e}{\partial r},$$

where α is the dimensionless slip coefficient.

Now, using these boundary conditions, the unknown coefficients $\alpha'_n, \beta'_n, \gamma'_n$ and δ_n are determined in terms of the known coefficients α_n, β_n and γ_n , and are given by:

$$\begin{aligned}
(2.29) \quad \alpha'_n &= (n+1) \frac{[\{X_n g_n(\lambda) + \lambda(\alpha+l)(1-l^2\lambda^2)g_{n+1}(\lambda)\}\alpha_n + \lambda(\alpha+l)Y_n\beta_n]}{Z_n}, \\
\beta'_n &= -\frac{(n+1)\{(1-l^2\lambda^2)\alpha_n + f_n(\lambda)\beta_n\} + (n+1+n l^2\lambda^2)\alpha'_n}{(n+1)g_n(\lambda)}, \\
\gamma'_n &= \frac{\gamma_n\{(\alpha-nl)f_n(\lambda) - l\lambda f_{n+1}(\lambda)\}}{(nl-\alpha)g_n(\lambda) - l\lambda g_{n+1}(\lambda)}, \\
\delta_n &= \lambda^2\{n\alpha'_n - (n+1)\alpha_n\},
\end{aligned}$$

where

$$\begin{aligned}
(2.30) \quad X_n &= \lambda^2\{l^2(n+1)\alpha - l^3(n^2 + \lambda^2 - 1) - l\}, \\
Y_n &= f_n(\lambda)g_{n+1}(\lambda) + f_{n+1}(\lambda)g_n(\lambda), \\
Z_n &= [l\{n(n+1)(n+2) - (n^2 + \lambda^2 - 1)(n\lambda^2 l^2 + n + 1)\} \\
&\quad + \alpha(n+1)(n\lambda^2 l^2 + 2n + 1)]g_n(\lambda) \\
&\quad - \lambda(\alpha+l)(n\lambda^2 l^2 + n + 1)g_{n+1}(\lambda),
\end{aligned}$$

and $l = \sqrt{Da}$.

The use of Saffman's condition brings limitations on the permeability range. LOOKER and CARNIE [9] concluded that Saffman's condition is applicable at low frequency. VAINSHTEIN and SHAPIRO [17] calculated the force acting on a permeable particle in oscillatory flow using the Brinkman and the Darcy equation.

It may be noted that in case of the Brinkman equation, it is customary to use continuity of velocity components together with the continuity of stress components and these boundary conditions are accepted by a large community. VAINSHTEIN and SHAPIRO [17] have also used the same boundary conditions in case of Brinkman's equation, whereas in case of the Darcy equation, the continuity of tangential velocity needs to be replaced by Beavers-Joseph/Saffman-type slip condition. VAINSHTEIN and SHAPIRO [17] reported a critical value of the Brinkman parameter, a/\sqrt{k} , which is expected to control the applicability of the Darcy equation. They also observed that this critical value diminished with decreasing frequency of oscillations and reaches that of a non-oscillating particle ≈ 10 . It appears that critical value can be as large as 200 for high frequency of oscillations. For low and moderate values of frequency, this critical value can be readily identified. Hence, the hydrodynamic problem of oscillatory flow past a porous body, considering Darcy equation inside together with Saffman's condition on the porous-liquid interface, brings such trade-off between frequency and the range of permeability.

3. Faxén's law for porous sphere in oscillatory flow

Faxén derived expressions for the drag and torque exerted by an exterior steady Stokes flow on a rigid sphere (see [26]). This enables us to express the drag force and torque in terms of the basic flow. Faxén's law in terms of singularity solutions for fluid-fluid, fluid-solid and solid-solid dispersions has been given by KIM and LU [27]. By using the singularity method, similar results for unsteady Stokes flow were obtained [26, 28, 29]. Next, we try to obtain Faxén's law for arbitrary oscillatory Stokes flow past a porous sphere.

It is well-known that the drag \mathbf{D} exerted by an exterior flow on a spherical surface $r = 1$, as well as the torque \mathbf{T} , are given by

$$(3.1) \quad \mathbf{D} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[T_{rr}^e \hat{\mathbf{e}}_r + T_{r\theta}^e \hat{\mathbf{e}}_\theta + T_{r\varphi}^e \hat{\mathbf{e}}_\varphi \right] r^2 \sin \theta \, d\theta \, d\varphi \Big|_{r=1},$$

$$(3.2) \quad \mathbf{T} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[r T_{r\theta}^e \hat{\mathbf{e}}_\varphi - r T_{r\varphi}^e \hat{\mathbf{e}}_\theta \right] r^2 \sin \theta \, d\theta \, d\varphi \Big|_{r=1},$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\varphi$ are the unit vectors corresponding to the spherical coordinates (r, θ, φ) , and T_{rr}^e , $T_{r\theta}^e$ and $T_{r\varphi}^e$ are the components of the stress tensor.

We now derive the corresponding Faxén's law which provide expressions for the drag and torque acting on a porous sphere in an unbounded, arbitrary oscillatory Stokes flow. Computing the stress components and using the relations (3.1) and (3.2), we obtain the following expressions for drag and torque:

$$(3.3) \quad \mathbf{D} = \frac{8\pi}{3} \lambda^2 (A_{11} \hat{\mathbf{i}} + B_{11} \hat{\mathbf{j}} + A_{10} \hat{\mathbf{k}}) N_1,$$

where

$$(3.4) \quad N_1 = \frac{[\{3l\lambda^2(2l^2 - 1)g_1(\lambda) - 3\lambda(\alpha + l)g_2(\lambda)\}\alpha_1 - N_2\beta_1]}{(6l + 6\alpha + 2\alpha l^2\lambda^2 - 2l\lambda^2 - l^3\lambda^4)g_1(\lambda) - \lambda(\alpha + l)(2 + l^2\lambda^2)g_2(\lambda)},$$

$$N_2 = \lambda(\alpha + l)(3 + l^2\lambda^2)\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\},$$

and

$$(3.5) \quad \mathbf{T} = \frac{8\pi}{3} \lambda(l - \alpha) \frac{\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(l - \alpha)g_1(\lambda) - l\lambda g_2(\lambda)} (C_{11} \hat{\mathbf{i}} + D_{11} \hat{\mathbf{j}} + C_{10} \hat{\mathbf{k}}) \gamma_1.$$

In the above expressions, $f_1(\lambda)$, $f_2(\lambda)$ and $g_1(\lambda)$, $g_2(\lambda)$ are given by

$$(3.6) \quad f_1(\lambda) = \frac{\lambda \cosh \lambda - \sinh \lambda}{\lambda^2}, \quad f_2(\lambda) = \frac{(\lambda^2 + 3) \sinh \lambda - 3\lambda \cosh \lambda}{\lambda^3},$$

$$g_1(\lambda) = \frac{e^{-\lambda}(\lambda + 1)}{\lambda^2}, \quad g_2(\lambda) = \frac{e^{-\lambda}(\lambda^2 + 3\lambda + 3)}{\lambda^3}.$$

Note that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors corresponding to an orthogonal frame of Cartesian coordinates with the origin at the center of the sphere, and the coefficients A_{11} , A_{10} , B_{11} , C_{11} , C_{10} , D_{11} are due to the spherical harmonics $S_n(\theta, \varphi)$ and $T_n(\theta, \varphi)$ as given in (2.18)–(2.19). The expression for drag and torque given in (3.3) and (3.5) can be reduced to a compact form which is the corresponding Faxén's law. This can be done as follows.

Let us consider the unperturbed velocity \mathbf{W}_0 given by

$$(3.7) \quad \begin{aligned} \mathbf{W}_0 &= \nabla \times \nabla \times (A_0 \mathbf{X}) + \nabla \times (B_0 \mathbf{X}) \\ &= 2\nabla A_0 + (\mathbf{X} \cdot \nabla) \nabla A_0 - \mathbf{X} \nabla^2 A_0 - (\mathbf{X} \times \nabla) B_0. \end{aligned}$$

Then assuming the form of A_0 , B_0 as in (2.17), we get

$$(3.8) \quad [\mathbf{W}_0]_0 = 2[\nabla A_0]_0 = 2(\alpha_1 + \frac{\lambda}{3}\beta_1)(A_{11}\hat{\mathbf{i}} + B_{11}\hat{\mathbf{j}} + A_{10}\hat{\mathbf{k}}),$$

$$(3.9) \quad [\nabla^2 \mathbf{W}_0]_0 = \frac{2\lambda^3}{3}\beta_1(A_{11}\hat{\mathbf{i}} + B_{11}\hat{\mathbf{j}} + A_{10}\hat{\mathbf{k}}),$$

$$(3.10) \quad [\nabla \times \mathbf{W}_0]_0 = \frac{2\lambda}{3}\gamma_1(C_{11}\hat{\mathbf{i}} + D_{11}\hat{\mathbf{j}} + C_{10}\hat{\mathbf{k}}).$$

Using (3.8)–(3.10), the expressions given in (3.3) and (3.5) can be expressed in terms of the basic flow as follows:

$$(3.11) \quad \mathbf{D} = \frac{4\pi}{3}\lambda^2 E[\mathbf{W}_0]_0 - 4\pi \left[\frac{E}{3} + F \right] [\nabla^2 \mathbf{W}_0]_0,$$

$$(3.12) \quad \mathbf{T} = 4\pi(l - \alpha) \frac{\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(l - \alpha)g_1(\lambda) - l\lambda g_2(\lambda)} [\nabla \times \mathbf{W}_0]_0,$$

where \mathbf{W}_0 is the velocity field corresponding to the basic flow; the notation $[]_0$ means evaluation at the origin $r = 0$, and

$$(3.13) \quad \begin{aligned} E &= \frac{3l\lambda^2(2l^2 - 1)g_1(\lambda) - 3\lambda(\alpha + l)g_2(\lambda)}{(6l + 6\alpha + 2\alpha l^2\lambda^2 - 2l\lambda^2 - l^3\lambda^4)g_1(\lambda) - \lambda(\alpha + l)(2 + l^2\lambda^2)g_2(\lambda)}, \\ F &= \frac{(\alpha + l)(3 + l^2\lambda^2)\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(6l + 6\alpha + 2\alpha l^2\lambda^2 - 2l\lambda^2 - l^3\lambda^4)g_1(\lambda) - \lambda(\alpha + l)(2 + l^2\lambda^2)g_2(\lambda)}. \end{aligned}$$

Therefore, the decomposition given in (2.16) helps us to express the drag and torque, exerted on a porous sphere by an arbitrary, oscillatory Stokes flow in terms of Faxén's law. Since there are restrictions on the validity of Saffman's condition in case of oscillatory flow, the above expressions are of mathematical interest only to make the study complete. In order to obtain physically meaningful results, one has to take into account the restrictions involved as described

in Subsection 2.1. Hence, we consider the case of low permeability of the porous sphere and expanding the formulae (3.11) and (3.12) up to $O(l^2)$, we get the following results:

$$(3.14) \quad \mathbf{D} = 2\pi \left[\lambda^2 + 3\lambda + 3 - \frac{3l}{\alpha}(\lambda + 1)^2 + O(l^2) \right] [\mathbf{W}_0]_0 \\ - 2\pi \left[1 + \frac{3}{\lambda} + \frac{3}{\lambda^2} - \frac{3l}{\alpha\lambda^2}(\lambda + 1)^2 \right. \\ \left. - \frac{3e^\lambda}{\lambda^2} \left\{ 1 - \frac{l}{\alpha}(\lambda + 1) \right\} + O(l^2) \right] [\nabla^2 \mathbf{W}_0]_0,$$

$$(3.15) \quad \mathbf{T} = 4\pi e^\lambda \left[1 - \lambda + \lambda^2 + \frac{l}{\alpha}(2\lambda^3 + 2\lambda^2 + 3\lambda - 2) + O(l^2) \right] [\nabla \times \mathbf{W}_0]_0.$$

It may be noted that the expressions given in (3.11) and (3.12) that represent Faxén's law for arbitrary oscillatory Stokes flow past a porous sphere, and the expressions given in (3.14) and (3.15) that represent the drag and torque acting on the surface of a weakly permeable porous sphere corresponding to arbitrary oscillatory Stokes flow, are new in the literature.

In the limiting case of $l \rightarrow 0$, the above expressions reduce to

$$(3.16) \quad \mathbf{D} = 2\pi(\lambda^2 + 3\lambda + 3)[\mathbf{W}_0]_0 - 2\pi \left(1 + \frac{3}{\lambda} + \frac{3}{\lambda^2} - \frac{3e^\lambda}{\lambda^2} \right) [\nabla^2 \mathbf{W}_0]_0,$$

$$(3.17) \quad \mathbf{T} = 4\pi \left[\frac{e^\lambda}{\lambda + 1} \right] [\nabla \times \mathbf{W}_0]_0.$$

These formulae, which give the drag and torque in the case of an arbitrary oscillatory Stokes flow past an impermeable sphere, have been obtained by POZRIKIDIS [29].

3.1. Uniform oscillatory flow past a porous sphere

If the basic flow is uniform along the z -axis, then the corresponding expressions for A_0 and B_0 in nondimensional form are $A_0 = \frac{1}{2}r \cos \theta$, $B_0 = 0$. Comparing it with the general expressions given in (2.17), we have $\alpha_1 = \frac{1}{2}$ and $\beta_1 = 0$. Hence, the expressions for drag and torque given in (3.11) and (3.12) reduce to

$$(3.18) \quad \mathbf{D} = 4\pi\lambda^2 \frac{(2l^2 - 1)l\lambda^2 g_1(\lambda) - \lambda(\alpha + l)g_2(\lambda)}{(6l + 6\alpha + 2\alpha l^2 \lambda^2 - 2l\lambda^2 - l^3 \lambda^4)g_1(\lambda) - \lambda(\alpha + l)(2 + l^2 \lambda^2)g_2(\lambda)} \hat{\mathbf{k}},$$

$$(3.19) \quad \mathbf{T} = \mathbf{0}.$$

The general expression for drag acting on the surface of a porous sphere in case of uniform flow is given by Eq. (3.18). Expanding the expression (3.18) up to $O(l^2)$, we get the following formula:

$$(3.20) \quad \mathbf{D} = 2\pi \left[\lambda^2 + 3\lambda + 3 - \frac{3l}{\alpha}(\lambda + 1)^2 + O(l^2) \right] \hat{\mathbf{k}},$$

which is the drag acting on the surface of weakly permeable sphere in uniform flow. In the low frequency limit ($\lambda \ll 1$), the modified spherical Bessel functions behave like

$$(3.21) \quad f_n(\lambda) \sim \frac{\lambda^n}{(2n+1)!}, \quad g_n(\lambda) \sim \frac{(2n-1)!}{\lambda^{(n+1)}},$$

and hence the corresponding expression for drag given in (3.18) reduces to

$$(3.22) \quad \mathbf{D} = 4\pi \frac{\lambda^2(\lambda^2 l + 6\alpha + 6l - 2\lambda^2 l^3)}{(6l + 6\alpha + 4\alpha\lambda^2 l^2 + 2\lambda^2 l + \lambda^4 l^3 + 6\lambda^2 l^3)} \hat{\mathbf{k}}.$$

In case of weakly permeable sphere, the expression (3.22) reduces to

$$(3.23) \quad \mathbf{D} = 4\pi\lambda^2 \left(1 - \frac{\lambda^2 l}{6\alpha} + O(l^2) \right) \hat{\mathbf{k}}.$$

In the limit $l \rightarrow 0$, the formula (3.18) reduces to

$$(3.24) \quad \mathbf{D} = 6\pi \left(1 + \lambda + \frac{\lambda^2}{3} \right) \hat{\mathbf{k}}.$$

This formula was obtained independently by BOUSSINESQ and BASSET (see [30, 31]). It may be noted that the expression (3.24) can be obtained from (3.14) in the limit of $l \rightarrow 0$ and $\mathbf{V}_0 \rightarrow U\hat{\mathbf{k}}$. In the case of steady uniform flow past a stationary porous sphere, i.e., for $\lambda \rightarrow 0$, expanding the expression (3.18) in terms of l and considering terms up to $O(l^3)$, we get

$$(3.25) \quad \mathbf{D} = 6\pi \left[1 - \frac{l}{\alpha} + l^2 \left(\alpha^{-2} - \frac{1}{2} \right) + O(l^3) \right] \hat{\mathbf{k}},$$

which is also due to DAVIS and STONE [21]. If we consider the expression (3.18) in the limit of $\lambda \rightarrow 0$ (steady case) and take $\alpha = 1$, then we get

$$(3.26) \quad \mathbf{D} = 6\pi \frac{2(1+l)}{3l^3 + l^2 + 4l + 2} \hat{\mathbf{k}},$$

which agrees with the result in [20]. It may be noted that when $\alpha = 1$ in (3.25), which is due to DAVIS and STONE [21], it does not reduce to the result by NEALE *et al.* [20]. This is because DAVIS and STONE [21] considered terms up to $O(l^3)$ only, whereas the present analysis considers the general case and hence, full agreement with NEALE *et al.* [20].

3.2. Flow due to an oscillating Stokeslet

The flow due to an oscillatory point force located at the point \mathbf{y} in free space, whose strength is given by the real or imaginary part of $\mathbf{b} \exp^{-i\omega t}$, where \mathbf{b} is a constant vector, is called the oscillating Stokeslet. The velocity and pressure of such an oscillatory Stokeslet, in terms of the corresponding tensor notation in \mathbb{R}^3 , is given by (see [28])

$$(3.27) \quad \begin{aligned} v_j(\mathbf{x}) &= \frac{1}{8\pi\mu} \mathcal{G}_{jk}^{\lambda^2}(\mathbf{x} - \mathbf{y}) b_k, \\ p(\mathbf{x}) &= \frac{1}{8\pi} \Pi_k^{\lambda^2}(\mathbf{x} - \mathbf{y}) b_k, \quad j, k = 1, 2, 3. \end{aligned}$$

The components of the fundamental oscillatory Stokes tensor \mathcal{G}^{λ^2} and those of its associated pressure vector Π^{λ^2} , which determine the fundamental solution $(\mathcal{G}^{\lambda^2}, \Pi^{\lambda^2})$ of the oscillatory Stokes system, are given by

$$(3.28) \quad \begin{aligned} \mathcal{G}_{jk}^{\lambda^2}(\mathbf{x} - \mathbf{y}) &= \frac{\delta_{jk}}{|\mathbf{x} - \mathbf{y}|} A_1(\lambda|\mathbf{x} - \mathbf{y}|) + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x} - \mathbf{y}|^3} A_2(\lambda|\mathbf{x} - \mathbf{y}|), \\ \Pi_j^{\lambda^2}(\mathbf{x} - \mathbf{y}) &= 2 \frac{x_j - y_j}{|\mathbf{x} - \mathbf{y}|^3}, \end{aligned}$$

where

$$(3.29) \quad \begin{aligned} A_1(\mathcal{R}) &= 2e^{-\mathcal{R}}(1 + \mathcal{R}^{-1} + \mathcal{R}^{-2}) - 2\mathcal{R}^{-2}, \\ A_2(\mathcal{R}) &= -2e^{-\mathcal{R}}(1 + 3\mathcal{R}^{-1} + 3\mathcal{R}^{-2}) + 6\mathcal{R}^{-2}, \\ \mathcal{R} &= \lambda r. \end{aligned}$$

It can be seen from the expressions (3.11)–(3.12) that in order to compute Faxén's law corresponding to a given basic velocity \mathbf{W}_0 , it is enough to compute $[\mathbf{W}_0]_0$, $[\nabla^2 \mathbf{W}_0]_0$ and $[\nabla \times \mathbf{W}_0]_0$. Hence, even if the basic flow is not expressed in terms of the decomposition, one can compute the corresponding Faxén's law. However, the use of the decomposition in arriving at the compact form of Faxén's law, is implicit as shown in (3.11)–(3.12). Hence, to compute Faxén's law corresponding to an oscillatory Stokeslet, we consider the location of the oscillatory Stokeslet to be at $(0, 0, c)$ where $c > a$ and the strength being $b_1/8\pi\mu$ with axis along the positive x -axis. The velocity components of such an oscillatory Stokeslet in Cartesian form are given by

$$(3.30) \quad \begin{aligned} u &= \frac{b_1}{8\pi\mu} \left(\frac{A_1(\lambda r)}{r} + \frac{x^2}{r^3} A_2(\lambda r) \right), \\ v &= \frac{b_1}{8\pi\mu} \left(\frac{xy}{r^3} A_2(\lambda r) \right), \\ w &= \frac{b_1}{8\pi\mu} \left(\frac{x(z - c)}{r^3} A_2(\lambda r) \right), \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + (z - c)^2}$. Hence, we get

$$\begin{aligned}
 [\mathbf{W}_0]_0 &= \frac{b_1}{4\pi\mu} \left(\left(\frac{1}{c} + \frac{1}{\lambda c^2} + \frac{1}{\lambda^2 c^3} \right) \exp^{-\lambda c} - \frac{1}{\lambda^2 c^3} \right) \hat{\mathbf{i}}, \\
 (3.31) \quad [\nabla^2 \mathbf{W}_0]_0 &= \frac{b_1}{4\pi\mu} \left(\frac{\lambda^2}{c} + \frac{\lambda}{c^2} + \frac{1}{c^3} \right) \exp^{-\lambda c} \hat{\mathbf{i}}, \\
 [\nabla \times \mathbf{W}_0]_0 &= \frac{b_1}{4\pi\mu} \left(\frac{1}{c^2} + \frac{\lambda}{c} \right) \exp^{-\lambda c} \hat{\mathbf{j}}.
 \end{aligned}$$

The corresponding Faxén's law, when the basic flow is due to an oscillatory Stokeslet, is obtained as

$$\begin{aligned}
 (3.32) \quad \mathbf{D} &= b_1 d_{10} \left(\left(\frac{1}{c} + \frac{1}{\lambda c^2} + \frac{1}{\lambda^2 c^3} \right) \exp^{-\lambda c} - \frac{1}{\lambda^2 c^3} \right) \hat{\mathbf{i}} \\
 &\quad + b_1 d_{20} \left(\frac{\lambda^2}{c} + \frac{\lambda}{c^2} + \frac{1}{c^3} \right) \exp^{-\lambda c} \hat{\mathbf{i}},
 \end{aligned}$$

$$(3.33) \quad \mathbf{T} = b_1 t_{10} \left(\frac{1}{c^2} + \frac{\lambda}{c} \right) \exp^{-\lambda c} \hat{\mathbf{j}},$$

where

$$\begin{aligned}
 (3.34) \quad d_{10} &= \lambda^2 \frac{E}{3}, \\
 d_{20} &= - \left(\frac{E}{3} + F \right), \\
 t_{10} &= (l - \alpha) \frac{\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(l - \alpha)g_1(\lambda) - l\lambda g_2(\lambda)},
 \end{aligned}$$

where E, F are as those given in (3.13). The interesting limiting values are corresponding to low and high frequency. We give here expressions for the low frequency limit and the corresponding limiting values for high frequency can also be obtained.

Low frequency limit

In order to consider the case of low frequency oscillations (small values of the frequency parameter λ), we expand \mathcal{G}^{λ^2} in a Taylor series with respect to λ as follows (see [28]):

$$(3.35) \quad \mathcal{G}^{\lambda^2} = \mathbf{G}^{(0)} + \lambda \mathbf{G}^{(1)} + \lambda^2 \mathbf{G}^{(2)} + \dots$$

where

$$\begin{aligned}
 G_{jk}^{(0)}(\mathbf{x} - \mathbf{y}) &= \frac{\delta_{jk}}{r} + \frac{(x_j - y_j)(x_k - y_k)}{r^3}, \\
 (3.36) \quad G_{jk}^{(1)}(\mathbf{x} - \mathbf{y}) &= -\frac{4}{3}\delta_{jk}, \\
 G_{jk}^{(2)}(\mathbf{x} - \mathbf{y}) &= \frac{r^2}{4} \left(3\frac{\delta_{jk}}{r} - \frac{(x_j - y_j)(x_k - y_k)}{r^3} \right).
 \end{aligned}$$

It may be noted that $G_{jk}^{(0)}$ is nothing else but the steady Stokeslet and $G_{jk}^{(1)}$ represents a uniform flow.

Also, in this low frequency limit, the coefficients given in (3.34), behave like

$$\begin{aligned}
 d_{10} &\sim \frac{\lambda^2(\lambda^2 l + 6\alpha + 6l - 2\lambda^2 l^3)}{(6l + 6\alpha + 4\alpha\lambda^2 l^2 + 2\lambda^2 l + \lambda^4 l^3 + 6\lambda^2 l^3)}, \\
 (3.37) \quad d_{20} &\sim \frac{1}{120} \frac{360\lambda^2 l^3 - 117\lambda^2 l - 360\alpha - 360l + 120\alpha\lambda^2 l^2 + \alpha\lambda^4 l^2 + 3\alpha\lambda^2 + \lambda^4 l^3}{6l + 6\alpha + 4\alpha\lambda^2 l^2 + 2\lambda^2 l + \lambda^4 l^3 + 6\lambda^2 l^3}, \\
 t_{10} &\sim -\frac{1}{120} \frac{(l - \alpha)(120 + \lambda^2)}{5l + \alpha}.
 \end{aligned}$$

Hence, the corresponding Faxén's law reduces to

$$\begin{aligned}
 (3.38) \quad \mathbf{D} &= \frac{b_1}{2} \frac{1}{(6l + 6\alpha + 4\alpha\lambda^2 l^2 + 2\lambda^2 l + \lambda^4 l^3 + 6\lambda^2 l^3)} \\
 &\quad \times \left[(\lambda^2 l + 6\alpha + 6l - 2\lambda^2 l^3) \left(\frac{\lambda^2}{c} - \frac{4\lambda^3}{3} + \frac{3\lambda^4 c}{4} \right) \right. \\
 &\quad \left. + \frac{1}{60} (360\lambda^2 l^3 - 117\lambda^2 l - 360\alpha - 360l + 120\alpha\lambda^2 l^2 \right. \\
 &\quad \left. + \alpha\lambda^4 l^2 + 3\alpha\lambda^2 + \lambda^4 l^3) \left(\frac{2}{c^3} + \frac{\lambda^2}{c} \right) \right] \hat{\mathbf{i}}, \\
 (3.39) \quad \mathbf{T} &= -\frac{b_1}{2} \frac{1}{120} \frac{(l - \alpha)(120 + \lambda^2)}{5l + \alpha} \left(\frac{2}{c^2} - \lambda^2 \right) \hat{\mathbf{j}}.
 \end{aligned}$$

The total drag in this case is the superposition of the drag due to steady Stokeslet, together with that of a uniform flow and an additional perturbation term. In case of weakly permeable sphere expanding the expressions (3.38)–(3.39) up to $O(l^2)$, we get the following results:

$$\begin{aligned}
 (3.40) \quad \mathbf{D} &= \frac{b_1}{2} \left[\left\{ 1 - \frac{\lambda^2 l}{6\alpha} + O(l^2) \right\} \left(\frac{\lambda^2}{c} - \frac{4\lambda^3}{3} + \frac{3\lambda^4 c}{4} \right) \right. \\
 &\quad \left. - \left\{ 1 - \frac{\lambda^4 l}{360\alpha} + O(l^2) \right\} \left(\frac{2}{c^3} + \frac{\lambda^2}{c} \right) \right] \hat{\mathbf{i}},
 \end{aligned}$$

$$(3.41) \quad \mathbf{T} = \frac{b_1}{2} \frac{1}{120} (\lambda^2 + 120) \left\{ 1 - \frac{6l}{\alpha} + O(l^2) \right\} \hat{\mathbf{j}}.$$

The expressions (3.40)–(3.41) are the Faxén law for oscillating Stokeslet in case of a weakly permeable sphere.

3.3. A porous sphere in a linear oscillatory shear flow

We consider the porous sphere in a linear oscillatory shear flow along the z -axis. Therefore, we have the far-field basic velocity in dimensionless form as $\mathbf{W}_0 = \tau\varpi x \hat{\mathbf{k}}$, where the coordinates x , y , and z have been non-dimensionalized by the radius of porous sphere, the shear rate has been non-dimensionalized by U/a , and the frequency is non-dimensionalized by $\varpi = \omega a^2/\nu$. Here τ is the shear rate coefficient. We can see that $[\mathbf{W}_0]_0 = 0$, $[\nabla^2 \mathbf{W}_0]_0 = 0$ and $[\nabla \times \mathbf{W}_0]_0 = \tau\varpi \hat{\mathbf{j}}$. Hence, we have

$$(3.42) \quad \mathbf{D} = \mathbf{0},$$

$$(3.43) \quad \mathbf{T} = 4\pi\tau\varpi(l - \alpha) \frac{\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(l - \alpha)g_1(\lambda) - l\lambda g_2(\lambda)} \hat{\mathbf{j}}.$$

This is the general expression for drag and torque acting on the surface of a porous sphere. In case of weakly permeable sphere expanding the expression (3.42) up to $O(l^2)$, we get

$$(3.44) \quad \mathbf{D} = \mathbf{0},$$

$$(3.45) \quad \mathbf{T} = 4\pi\tau\varpi e^\lambda \left[1 - \lambda + \lambda^2 + \frac{l}{\alpha} (2\lambda^3 + 2\lambda^2 + 3\lambda - 2) + O(l^2) \right] \hat{\mathbf{j}}.$$

3.4. A porous sphere in a quadratic oscillatory shear flow

We consider the porous sphere in oscillatory shear flow along the z -axis. Hence, we have the far-field basic velocity in dimensionless form as $\mathbf{W}_0 = \tau\varpi(x - x^2)\hat{\mathbf{k}}$. We can see that

$$[\mathbf{W}_0]_0 = 0, \quad [\nabla^2 \mathbf{W}_0]_0 = 2\tau\varpi \hat{\mathbf{k}}, \quad [\nabla \times \mathbf{W}_0]_0 = \tau\varpi \hat{\mathbf{j}}.$$

Hence

$$(3.46) \quad \mathbf{D} = -8\pi\tau\varpi \left[\frac{E}{3} + F \right] \hat{\mathbf{k}},$$

$$(3.47) \quad \mathbf{T} = 4\pi\tau\varpi(l - \alpha) \frac{\{f_1(\lambda)g_2(\lambda) + f_2(\lambda)g_1(\lambda)\}}{(l - \alpha)g_1(\lambda) - l\lambda g_2(\lambda)} \hat{\mathbf{j}}.$$

In case of weakly permeable sphere expanding the expression (3.46) up to $O(l^2)$, we get

$$(3.48) \quad \mathbf{D} = -4\pi\tau\varpi \left[1 + \frac{3}{\lambda} + \frac{3}{\lambda^2} - \frac{3l}{\alpha\lambda^2}(\lambda + 1)^2 - \frac{3e^\lambda}{\lambda^2} \left\{ 1 - \frac{l}{\alpha}(\lambda + 1) \right\} + O(l^2) \right] \hat{\mathbf{k}},$$

$$(3.49) \quad \mathbf{T} = 4\pi\tau\varpi e^\lambda \left[1 - \lambda + \lambda^2 + \frac{l}{\alpha}(2\lambda^3 + 2\lambda^2 + 3\lambda - 2) + O(l^2) \right] \hat{\mathbf{j}}.$$

It may be noted that linear shear gives zero drag, while quadratic shear produces a non-zero drag. The torque in both the cases remain the same. Also, the corresponding expressions due to low frequency limit can be obtained using the coefficients in (3.37).

4. Results and discussion

Expressions for drag and torque acting on a porous sphere due to arbitrary oscillatory Stokes flow are derived and expressed in the form of Faxén's law. These expressions are verified by some of the existing results in the literature, due to various limiting cases. The limit as $l \rightarrow 0$ corresponds to the case of impermeable sphere. In this limit, (3.11) and (3.12), reduce to the results given by Pozrikidis for arbitrary oscillatory flow past an impermeable sphere. We have shown that for the steady porous sphere case, i.e., $\lambda \rightarrow 0$, the results are in full agreement with the results of (NEALE *et al.* [20]), whereas similar results of Looker, differ from $O(l^2)$ onwards, because the present investigations considered the internal flow to the porous sphere as well.

Let us now assess the effect of parameters involved. We consider $l = 0.01, 0.025, 0.05$, i.e., for various Darcy numbers, the frequency of oscillation between 1 and 10 MHz, as considered by Looker, and $a^2/\nu = 10^{-6}$ s. A representative value for dimensionless slip coefficient $\alpha = 0.7$ is considered in the analysis.

Uniform flow

The variation of the slip velocity with respect to permeability and frequency is considered (Fig. 1). The slip velocity is defined as the difference between the tangential components of the fluid velocity outside the particle and the filter velocity inside the porous medium, both evaluated at the surface. Hence, Fig. 1 shows that the slip velocity increases almost linearly with both the permeability and frequency.

Figure 2 shows the magnitude of drag (3.18) with respect to permeability and frequency in case of uniform flow through a porous sphere. Figure 2a shows

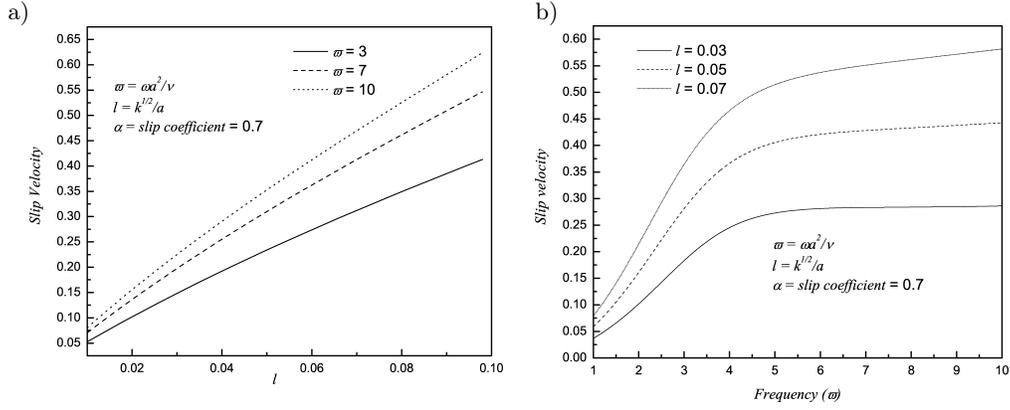


FIG. 1. Variation of slip velocity with a) $l = \sqrt{k}/a$ for different frequencies, b) frequency for different l .

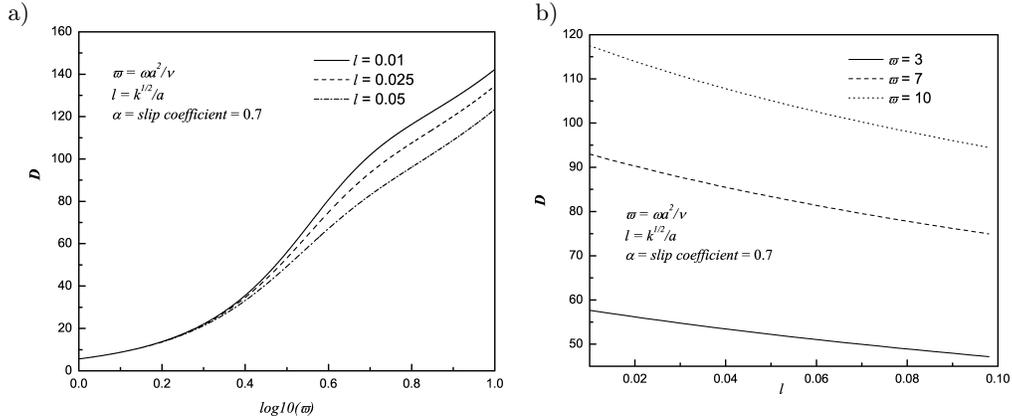


FIG. 2. Variation of drag in case of uniform flow with a) frequency for different l , b) l for different frequency.

that the magnitude of drag decreases with increasing permeability. This can be accounted by the following two factors. The first factor is the slip velocity increasing with permeability, and the other factor is more fluid penetrating the surface of the sphere with increase in permeability. Increasing of the slip velocity (as shown in Fig. 1) results in more fluid slipping over the sphere, hence reducing the force. Drag also decreases with increasing permeability due to the fluid penetrating the surface of the sphere. This is not true in [21], as they have considered terms up to $O(l^2)$ only. Figure 2b shows that the magnitude of drag increases with increasing frequency. Increasing frequency enhances the magnitude of drag due to larger resistance.

Oscillating Stokeslet

The variation in magnitude of drag with position of Stokeslet is shown (Fig. 3) for different frequencies. It is observed that the magnitude of drag decreases with position of Stokeslet for different frequencies because while Stokeslet moves far from the body, the body experiences less resistance, hence reduction in drag is seen. However, it may be noted that when the Stokeslet is located close to the porous sphere, larger frequency induces marginal increase in magnitude compared to that of smaller frequency. But, beyond a certain critical location, oscillating Stokeslet with smaller frequency induces larger magnitude of force. The variation in magnitude of drag and torque with frequency at different permeabilities is shown (Fig. 4) in case of oscillating Stokeslet with ($b_1 = 1$, $c = 3$), where

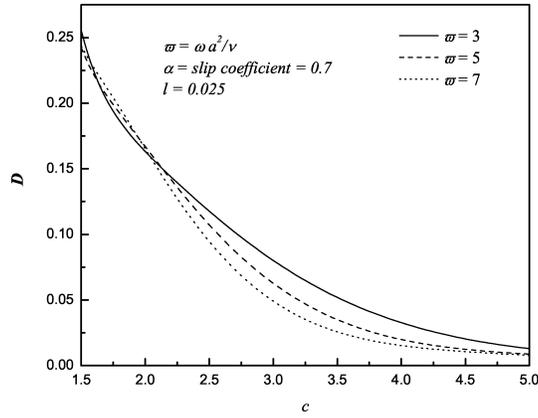


FIG. 3. Variation of drag with position of oscillating Stokeslet for different ϖ , $b_1 = 1$, $l = 0.025$.

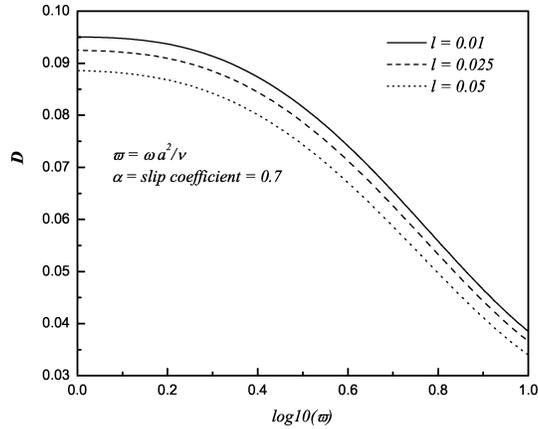


FIG. 4. Variation of drag with frequency for different l for oscillating Stokeslet, $b_1 = 1$, $c = 3$.

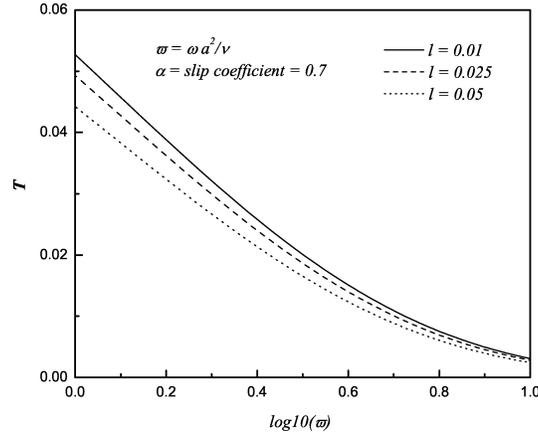


FIG. 5. Variation of torque with frequency for different l for oscillating Stokeslet, $b_1 = 1$, $c = 3$.

b_1 is the strength and c is the position of Stokeslet. The direction of drag is along the x -axis, It can be seen that the drag decreases with increase in permeability ($l = \sqrt{k}/a$). The direction of torque is along the y -axis, and it can be seen (Fig. 5) that the magnitude of torque decreases with increase in permeability. This may be due to the fact that increasing permeability reduces shear stress that contributes to the torque.

Oscillatory shear flow

The magnitude of drag in case of oscillatory quadratic shear flow is plotted on frequency (ω) for fixed shear rate, $\tau = 2.2$ (Fig. 6). The qualitative behavior with permeability is the same as in the case of uniform flow. The magnitude of

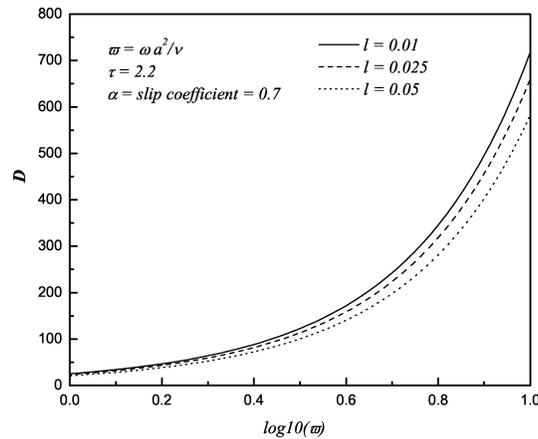


FIG. 6. Variation of drag with frequency in case of oscillatory quadratic shear flow.

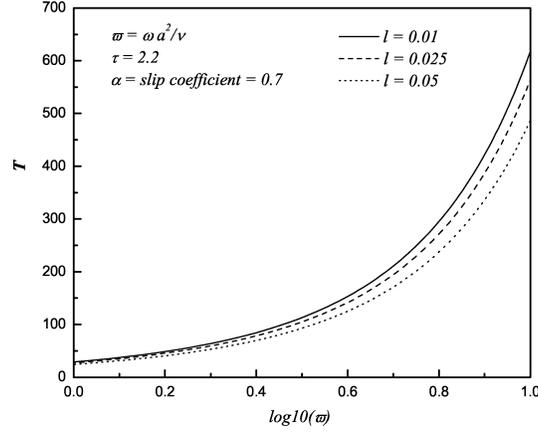


FIG. 7. Variation of torque with frequency in case of oscillatory linear/quadratic shear flow.

torque in case of oscillatory linear/quadratic shear flow is plotted with frequency (Fig. 7) for fixed shear rate, $\tau = 2.2$. The magnitude of torque decreases with increase of the permeability.

5. Conclusion

The objective of the present article is to obtain Faxén's law for an arbitrary oscillatory Stokes flow past a porous sphere. The internal flow is assumed to be governed by the usual Darcy's law, together with the equation of continuity, and that the external flow by the continuity and unsteady Stokes equations. Saffman's interfacial boundary condition for tangential velocity is used together with continuity of pressure and continuity of normal velocity to match the internal and external flows. The power of the solution procedure lies in the solenoidal decomposition of the velocity field, in order to discuss hydrodynamics of permeable sphere in arbitrary oscillatory Stokes flow. Because this reduces the problem in terms of solving scalar equations of the type given in (2.15) and also gives a scope to handle arbitrary flow. The expressions for drag matches with (see [9]) up to $O(l^2)$. Also, the results obtained are verified by some existing cases. Examples like uniform flow, oscillating Stokeslet, oscillatory shear flow and quadratic shear flow are discussed.

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