

Compliance minimization of thin plates made of material with predefined Kelvin moduli.

Part I. Solving the local optimization problem

G. DZIERŻANOWSKI, T. LEWIŃSKI

*Warsaw University of Technology
Faculty of Civil Engineering,
al. Armii Ludowej 16
00-637 Warszawa, Poland
e-mails: g.dzierzanowski@il.pw.edu.pl; t.lewinski@il.pw.edu.pl*

THE PAPER DEALS WITH COMPLIANCE MINIMIZATION of a transversely homogeneous plate, subjected to the in-plane and transverse loadings acting simultaneously. The set of design variables includes the eigenstates of Hooke's tensor whose eigenvalues, i.e. Kelvin moduli fields, are assumed to be fixed on the middle plane of the plate, but no isoperimetric condition is imposed. The optimization task reduces to an equilibrium problem of an effective hyperelastic plate. The effective potential is explicitly expressed in terms of the invariants of both the strain fields involved.

Key words: free material optimization, compliance minimization, anisotropic plates

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1. Introduction

IN THE THREE-DIMENSIONAL ELASTICITY, the elastic properties of a material are determined by six independent moduli of stress measure (Pa) and by fifteen nondimensional geometric independent parameters. This characterization follows from the spectral decomposition of Hooke's tensor, cf. [31], where the Author proposed to call these six elastic moduli by the name of Lord Kelvin. The theory of spectral decomposition of Hooke's tensors was also developed in papers by RYCHLEWSKI [32–34], BLINOWSKI *et al.* [5–7], MEHRABADI and COWIN [25], THEOCARIS and SOKOLIS [40], MOAKHER and NORRIS [26], and NORRIS [27]. The meaning of moduli and geometric parameters involved in the spectral representation is best seen if one takes a closer look at the geometric properties of the RVE cells, while considering the theory of non-homogeneous media from the point of view of the mechanics of composites. The mathematical structure of Hooke's tensor of crystals with various internal symmetries has been discussed in [38].

In two-dimensional problems, the constitutive properties are determined by three Kelvin moduli ($\lambda_1, \lambda_2, \lambda_3$) and by three geometric parameters, see [6, 9, 10].

In the present paper this spectral representation will be applied to describe the stiffness distribution of a thin elastic transversely homogeneous plate, whose model is assumed to satisfy the so-called generalized plane stress. It is worth pointing out that both the membrane and bending stiffnesses can be determined by one tensor \mathbf{A} of Hooke's symmetry.

In the present paper we attempt to construct the optimal layout of geometric parameters of tensor \mathbf{A} in the problem of simultaneous in-plane and transverse deformation of a plate. Kelvin moduli of this tensor are kept fixed, i.e. they do not undergo optimization. In general these moduli constitute the fields $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$, x being an arbitrary point at the middle plane Ω . The aim of the optimization is to minimize the compliance of a plate made from the material of such class, which is equivalent to the stiffening of a plate with respect to the given loading. Thus three types of loadings are discussed: (M), or membrane-type, bending-type: (B) and a composition of both: (M-B). We shall prove that the optimal design corresponding to the (M-B) loading comprises the cases of (M) and (B) types and that it tends to these specific solutions if one of the loading is absent.

To make the formulation of the minimum compliance problem well posed, it is in many cases necessary to augment it with an isoperimetric condition usually imposing a restriction on the volume of the material used. In the discussion below, the isoperimetric condition is absent, which makes the problem more universal.

However, upon solving the task considered in this paper we arrive at the starting point for a new optimization problem in which the Kelvin moduli can be viewed as design variables. Then it is natural to introduce an isoperimetric condition, constraining the distribution of these moduli within the domain Ω , in the form

$$(1.1) \quad \int_{\Omega} g(\lambda_1, \lambda_2, \lambda_3) dx \leq \text{const},$$

and one can choose the integrand as either the Euclidean norm

$$(1.2) \quad g(\lambda_1, \lambda_2, \lambda_3) = [(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2]^{1/2},$$

or assume that

$$(1.3) \quad g(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3.$$

Note that by (1.2) we assume the boundedness of the integral of the Frobenius norm of tensor \mathbf{A} , while (1.3) means that the integral of the trace of tensor \mathbf{A} is bounded. Such optimization problems, called free material optimization (FMO)

or free material design (FMD), have been originated by BENDSØE *et al.* [2, 3] and then developed in papers by KOČVARA *et al.*, see [22, 23] and STINGL *et al.* [35–37]. Its applications can be found in GAILE *et al.* [17], HÖRNLEIN *et al.* [21]. An overview of the FMO state-of-art has been recently delivered in HASLINGER *et al.* [20].

The FMO setting is closely related to the optimal material modeling addressed in RINGERTZ [28], BANICHUK [1], RODRIGUES *et al.* [30], TAYLOR [39], DU and TAYLOR [14] and GUEDES *et al.* [18, 19].

The approach proposed in the present paper makes it possible to formulate the FMO problem using different methodology enabling a deep insight into the mathematical structure of the stiffness moduli tensor. Consequently, it paves the way for the new version of FMO, with alternative isoperimetric conditions concerning the Kelvin moduli. Partial results of the research were announced in [15, 16, 24].

The plan of the paper is as follows: The optimization task is formulated in terms of displacements, which means that the minimum compliance problem admits a saddle point formulation, see (2.19). This, in turn, results in the local problem (2.23) which can be solved explicitly by using vector interpretation of tensors from the set \mathbb{E}_s^2 (symmetric tensors of second order referred to a plane). Compliance optimization is thus reduced to finding of an equilibrium of an effective plate with hyperelastic potential W_λ of the form (3.13). The strain fields solving the posed problem determine the spectral representation of tensor \mathbf{A} in a point-wise manner. It turns out that the problem derived is well posed, the proof being delivered in the second part of the present paper. The usual summation convention for repeated indices is adopted. The small Greek indices assume the values 1, 2, while the Latin indices assume values 1, 2, 3.

2. Problem statement

2.1. Equilibrium equations of a transversely homogeneous plate

Consider a plate with constantly varying thickness h and the middle plane Ω parameterized by a Cartesian system (x_1, x_2) with the basis $(\mathbf{i}_1, \mathbf{i}_2)$ and write $x = (x_1, x_2)$, $x \in \Omega$. Take x_3 as an axis orthogonal to Ω and let the coordinate system (x_1, x_2, x_3) be counterclockwise. Assume that $h(x) > 0$, $x \in \Omega$ and small if compared to the diameter of Ω . Moreover, assume that the material of a plate is homogeneous with respect to its thickness and set $x_3 = \text{const.}$ as planes of material symmetry. Suppose that the deformation of such defined structure is described by the theory of thin plates, therefore the transverse deformations are neglected in the analysis. Let the fields $\mathbf{u}(x) = (u_1(x), u_2(x))$ and $w(x)$ represent the displacement fields along the axes x_1, x_2, x_3 respectively and adopt

the membrane and bending strain fields definitions within the linear theory of plates: $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\mathbf{u})$, $\varkappa_{\alpha\beta} = \varkappa_{\alpha\beta}(w)$, where

$$(2.1) \quad \varepsilon_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \varkappa_{\alpha\beta}(w) = -w_{,\alpha\beta}$$

and $(\cdot)_{,\alpha} = \partial/\partial x_\alpha$.

Let the tensor $\mathbf{C} = [C^{\alpha\beta\lambda\mu}]$ comprise all moduli of the generalized plane stress state and set

$$(2.2) \quad A^{\alpha\beta\lambda\mu} = hC^{\alpha\beta\lambda\mu}, \quad D^{\alpha\beta\lambda\mu} = \frac{h^3}{12}C^{\alpha\beta\lambda\mu}.$$

Stress resultants $\mathbf{N} = [N^{\alpha\beta}]$ and couple resultants $\mathbf{M} = [M^{\alpha\beta}]$ are linked with the strains by linear equations

$$(2.3) \quad N^{\alpha\beta} = A^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \quad M^{\alpha\beta} = D^{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu}.$$

If the plate is subjected to the in-plane and transversal loadings of intensities $\mathbf{p}(x) = (p^1(x), p^2(x))$ and $q(x)$ respectively, then the virtual work of these loadings on the test in-plane displacements $\mathbf{v}(x) = (v_1(x), v_2(x))$ and transverse displacements $v(x)$ is expressed by

$$(2.4) \quad f(\mathbf{v}, v) = \int_{\Omega} (p^\alpha v_\alpha + qv) dx.$$

Assume that the part Γ_1 of the boundary $\partial\Omega$ is clamped, i.e.

$$(2.5) \quad \mathbf{u} = \mathbf{0}, \quad w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_1,$$

where $\mathbf{n} = (n_1, n_2)$ stands for the unit vector outward normal to Γ_1 . The other part of the boundary, denoted by Γ_2 , is free. Let V stand for a linear space of appropriately regular fields (\mathbf{u}, w) satisfying (2.5).

Stress resultants \mathbf{N} and \mathbf{M} satisfy the variational equilibrium equation within Ω and the natural boundary conditions along Γ_2

$$(2.6) \quad \int_{\Omega} (N^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\varkappa_{\alpha\beta}(v))dx = f(\mathbf{v}, v), \quad \forall(\mathbf{v}, v) \in V.$$

Substituting (2.1) and (2.3) in (2.6) gives two governing equations of Lax-Milgram type with separated membrane and bending mode of deformation, however the optimization procedure presented in the sequel results in re-coupling $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ in one formula.

2.2. Compliance minimization of a plate with predefined Kelvin moduli

Second-order plane symmetric tensors, e.g. $\mathbf{a} = a_{\alpha\beta} \mathbf{i}_\alpha \otimes \mathbf{i}_\beta$, and fourth-order tensors of Hooke's symmetry, e.g. $\mathbf{A} = A_{\alpha\beta\lambda\mu} \mathbf{i}_\alpha \otimes \mathbf{i}_\beta \otimes \mathbf{i}_\lambda \otimes \mathbf{i}_\mu$, constitute the spaces denoted by \mathbb{E}_s^2 and \mathbb{E}_s^4 respectively. A certain geometrical analogy, see [33], allows to treat the objects belonging to these spaces as vectors and second-order tensors in \mathbb{R}^3 . Indeed, if we adopt a basis

$$(2.7) \quad \mathbf{B}_1 = \mathbf{i}_1 \otimes \mathbf{i}_1, \quad \mathbf{B}_2 = \mathbf{i}_2 \otimes \mathbf{i}_2, \quad \mathbf{B}_3 = \frac{1}{\sqrt{2}}(\mathbf{i}_1 \otimes \mathbf{i}_2 + \mathbf{i}_2 \otimes \mathbf{i}_1),$$

then we can represent $\mathbf{a} \in \mathbb{E}_s^2$ and $\mathbf{A} \in \mathbb{E}_s^4$ as $\mathbf{a} = a_i \mathbf{B}_i$ and $\mathbf{A} = A_{ij} \mathbf{B}_i \otimes \mathbf{B}_j$, where

$$(2.8) \quad [a_i] = \begin{bmatrix} a_{11} \\ a_{22} \\ \sqrt{2}a_{12} \end{bmatrix}, \quad [A_{ij}] = \begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{1122} & A_{2222} & \sqrt{2}A_{1222} \\ \sqrt{2}A_{1112} & \sqrt{2}A_{1222} & 2A_{1212} \end{bmatrix}.$$

Obviously, components of such defined representations depend on the choice of basis in Ω . For brevity of further derivation, define the following operations on objects from \mathbb{E}_s^2 and \mathbb{E}_s^4 :

$$(2.9) \quad \begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \sum_{i=1}^3 a_i b_i, & \mathbf{a} \in \mathbb{E}_s^2, \mathbf{b} \in \mathbb{E}_s^2, \\ \mathbf{A} : \mathbf{B} &= \sum_{i=1}^3 \sum_{j=1}^3 A_{ij} B_{ij}, & \mathbf{A} \in \mathbb{E}_s^4, \mathbf{B} \in \mathbb{E}_s^4, \\ (\mathbf{A} \mathbf{b})_i &= \sum_{j=1}^3 A_{ij} b_j, & \mathbf{A} \in \mathbb{E}_s^4, \mathbf{b} \in \mathbb{E}_s^2, \\ (\mathbf{A} \mathbf{B})_{ik} &= \sum_{j=1}^3 A_{ij} B_{jk}, & \mathbf{A} \in \mathbb{E}_s^4, \mathbf{B} \in \mathbb{E}_s^4. \end{aligned}$$

The first and second equation in Eqs. (2.9) denote scalar products in the corresponding spaces. Respective norms are thus defined as $\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}$ and $\|\mathbf{A}\| = (\mathbf{A} : \mathbf{A})^{1/2}$.

Due to the transversal symmetry of a plate, the components of tensor \mathbf{A} are constant in this direction, thus they are treated as fields referred to Ω . Spectral decomposition of \mathbf{A} admits the following form, see [31, 33, 34] and [6],

$$(2.10) \quad \mathbf{A} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3,$$

where $\lambda_1 > \lambda_2 > \lambda_3$ stand for the Kelvin moduli and

$$(2.11) \quad \mathbf{P}_1 = \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1, \quad \mathbf{P}_2 = \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2, \quad \mathbf{P}_3 = \boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3$$

denote the eigentensors of \mathbf{A} having the properties of projection operators.

Tensors $\boldsymbol{\omega}_j$, $j = 1, 2, 3$ stand for the eigenstates, or proper states, of \mathbf{A} satisfying the orthogonality conditions

$$(2.12) \quad \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij}.$$

The unit tensor \mathbf{I}_4 in the space \mathbb{E}_s^4 of tensors obeying the Hooke's symmetry can be decomposed as

$$(2.13) \quad \mathbf{I}_4 = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3.$$

Eigenstates $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ are determined by three angular parameters, see [9]. In the present paper no use is made of these parameters, hence there is no need to recall their representations here.

Denote the set of Hooke's tensors of the given Kelvin moduli λ_1 , λ_2 , λ_3 and arbitrary eigenstates $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ by \mathcal{T}_λ . Next, let $\mathcal{T}_\lambda(\Omega)$ stand for the set of Hooke's tensor fields determined at each point $x \in \Omega$ by the Kelvin moduli $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$ and arbitrary eigenstates $\boldsymbol{\omega}_1(x)$, $\boldsymbol{\omega}_2(x)$, $\boldsymbol{\omega}_3(x)$.

Assuming that the values of moduli $\lambda_1(x) > \lambda_2(x) > \lambda_3(x) > 0$ are fixed within Ω and the orientation of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ is unknown, consider the following optimum design problem: at each point $x \in \Omega$ find such orientation of tensors $\boldsymbol{\omega}_i$ that minimizes the compliance $C = f(\mathbf{u}, w)$.

Making use of the theorem on the minimum of the total elastic potential of the plate, define the functional

$$(2.14) \quad J(\mathbf{A}, \mathbf{v}, v) = \frac{1}{2} \int_{\Omega} \left[\boldsymbol{\varepsilon}(\mathbf{v}) \cdot (\mathbf{A} \boldsymbol{\varepsilon}(\mathbf{v})) + \frac{h^2}{12} \boldsymbol{\varkappa}(v) \cdot (\mathbf{A} \boldsymbol{\varkappa}(v)) \right] dx - f(\mathbf{v}, v).$$

In order to get the optimal orientation of the eigenstates thus minimizing the total plate compliance, it is necessary to find the solution to the following equation:

$$(2.15) \quad C_0 = \min_{\mathbf{A} \in \mathcal{T}_\lambda(\Omega)} C(\mathbf{A}),$$

where $C(\mathbf{A}) = f(\mathbf{u}(\mathbf{A}), w(\mathbf{A}))$ and $(\mathbf{u}(\mathbf{A}), w(\mathbf{A}))$ solves the equilibrium problem governed by Eqs. (2.6) and (2.3). By combining (2.6) with (2.14) one obtains

$$(2.16) \quad f(\mathbf{u}(\mathbf{A}), w(\mathbf{A})) = -2J(\mathbf{A}, \mathbf{u}(\mathbf{A}), w(\mathbf{A}))$$

where

$$(2.17) \quad J(\mathbf{A}, \mathbf{u}(\mathbf{A}), w(\mathbf{A})) = \min_{(\mathbf{v}, v) \in V} J(\mathbf{A}, \mathbf{v}, v).$$

Rewriting (2.15) as a maximization problem leads to

$$(2.18) \quad C_0 = - \max_{\mathbf{A} \in \mathcal{T}_\lambda(\Omega)} \{-f(\mathbf{u}(\mathbf{A}), w(\mathbf{A}))\}$$

or

$$(2.19) \quad C_0 = -2 \max_{\mathbf{A} \in \mathcal{T}_\lambda(\Omega)} \min_{(\mathbf{v}, v) \in V} J(\mathbf{A}, \mathbf{v}, v).$$

The next step of the analysis requires switching the “min” and “max” in (2.19). The justification of this interchange is given in Sec. 3.4 of Part II of the present paper. Assuming that this is possible, set

$$(2.20) \quad \hat{C}_0 = -2 \min_{(\mathbf{v}, v) \in V} \max_{\mathbf{A} \in \mathcal{T}_\lambda(\Omega)} J(\mathbf{A}, \mathbf{v}, v),$$

introduce the functional

$$(2.21) \quad J_\lambda(\mathbf{v}, v) = \max_{\mathbf{A} \in \mathcal{T}_\lambda(\Omega)} J(\mathbf{A}, \mathbf{v}, v)$$

and pass with the “max” operation under the integral sign in (2.14), thus obtaining

$$(2.22) \quad J_\lambda(\mathbf{v}, v) = \int_{\Omega} W_{\lambda(x)}(\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varkappa}(v)) dx - f(\mathbf{v}, v),$$

where

$$(2.23) \quad W_{\lambda(x)}(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = \frac{1}{2} \max_{\mathbf{A} \in \mathcal{T}_\lambda(x)} \left\{ \boldsymbol{\varepsilon} \cdot (\mathbf{A}\boldsymbol{\varepsilon}) + \frac{h^2}{12} \boldsymbol{\varkappa} \cdot (\mathbf{A}\boldsymbol{\varkappa}) \right\}.$$

The equivalence of Eqs. (2.21) and (2.22) follows from the Rockafellar theorem, see [29].

The minimum compliance problem (2.20) assumes the form

$$(2.24) \quad \hat{C}_0 = -2 \min_{(\mathbf{v}, v) \in V} J_\lambda(\mathbf{v}, v) \quad (P^*)$$

equivalent to the equilibrium problem of an effective plate with hyperelastic constitutive properties. Indeed, the condition of stationarity imposed on the functional (2.22) leads to (2.6) with

$$(2.25) \quad \mathbf{N} = \frac{\partial W_{\lambda(x)}(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M} = \frac{\partial W_{\lambda(x)}(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})}{\partial \boldsymbol{\varkappa}}.$$

The stress resultants \mathbf{N} , \mathbf{M} in (2.25) are linked with the strain measures in (2.1) which in turn are dependent on the displacement fields according to the formulae known from the linear plate theory.

The solution to (P^*) will be denoted by $\hat{\mathbf{N}}, \hat{\mathbf{M}}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{\varkappa}}, \hat{\mathbf{u}}, \hat{\mathbf{w}}$. The equality $C_0 = \hat{C}_0$ will be proved in Part II.

Derivation of the potential (2.23) in its explicit form is the goal of this part of the paper. Thus the obtained solution of the problem in (2.24) gives the complete information on the distribution of the fields $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ within Ω , in this way determining the spatial distribution of all components of tensor \mathbf{C} .

3. Explicit formulation of the compliance minimization problem

3.1. Deformation energy potential of an optimal plate

The aim of this section is to find the potential (2.23) and the effective constitutive equations (2.25) in their explicit forms. To this end, by using representation (2.10), write

$$(3.1) \quad \boldsymbol{\varepsilon} \cdot (\mathbf{A}\boldsymbol{\varepsilon}) = \sum_{k=1}^3 \lambda_k (\boldsymbol{\omega}_k \cdot \boldsymbol{\varepsilon})^2, \quad \boldsymbol{\varkappa} \cdot (\mathbf{A}\boldsymbol{\varkappa}) = \sum_{k=1}^3 \lambda_k (\boldsymbol{\omega}_k \cdot \boldsymbol{\varkappa})^2$$

and to make the units of deformation tensors uniform, set

$$(3.2) \quad \boldsymbol{\kappa} = \frac{h}{\sqrt{12}} \boldsymbol{\varkappa}$$

which allows to rewrite (2.23) in the equivalent form

$$(3.3) \quad W_\lambda(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = W_\lambda\left(\boldsymbol{\varepsilon}, \frac{\sqrt{12}}{h} \boldsymbol{\kappa}\right) = 2U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}),$$

where

$$(3.4) \quad 2U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2} \max_{\mathbf{A} \in \mathcal{T}_\lambda} \{ \boldsymbol{\varepsilon} \cdot (\mathbf{A}\boldsymbol{\varepsilon}) + \boldsymbol{\kappa} \cdot (\mathbf{A}\boldsymbol{\kappa}) \}$$

with the argument x being omitted since further discussion is pointwise in Ω .

Formula (3.4) is equivalent to

$$(3.5) \quad 4U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \max \left\{ \sum_{k=1}^3 \lambda_k [(\boldsymbol{\omega}_k \cdot \boldsymbol{\varepsilon})^2 + (\boldsymbol{\omega}_k \cdot \boldsymbol{\kappa})^2] \mid \boldsymbol{\omega}_i \in \mathbb{E}_s^2, \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij} \right\}$$

and the following representations

$$(3.6) \quad \|\boldsymbol{\varepsilon}\|^2 = \sum_{k=1}^3 (\boldsymbol{\omega}_k \cdot \boldsymbol{\varepsilon})^2, \quad \|\boldsymbol{\kappa}\|^2 = \sum_{k=1}^3 (\boldsymbol{\omega}_k \cdot \boldsymbol{\kappa})^2$$

are inferred from (2.13). Consequently,

$$(3.7) \quad \sum_{k=1}^3 \lambda_k (\boldsymbol{\omega}_k \cdot \boldsymbol{\varepsilon})^2 = \lambda_3 \|\boldsymbol{\varepsilon}\|^2 + \mu_1 (\boldsymbol{\omega}_1 \cdot \boldsymbol{\varepsilon})^2 + \mu_2 (\boldsymbol{\omega}_2 \cdot \boldsymbol{\varepsilon})^2$$

where

$$(3.8) \quad \mu_1 = \lambda_1 - \lambda_3, \quad \mu_2 = \lambda_2 - \lambda_3$$

and a similar formula holds for $\boldsymbol{\kappa}$. Rearranging of (3.5) gives

$$(3.9) \quad 4U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \lambda_3 (\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2) + 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$$

with

$$(3.10) \quad 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \max_{\substack{\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\omega}_\beta = \delta_{\alpha\beta} \\ \boldsymbol{\omega}_\alpha \in \mathbb{B}_s^2}} \left\{ \sum_{\alpha=1}^2 \mu_\alpha [(\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\varepsilon})^2 + (\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\kappa})^2] \right\}$$

and $\alpha, \beta = 1, 2$. Calculations in (3.9) and (3.10) lead to

$$(3.11) \quad 2U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{4}(\lambda_1 + \lambda_2)(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2) \\ + \frac{1}{4}(\lambda_1 - \lambda_2)[(\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2]^{1/2},$$

$$(3.12) \quad 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2}(\mu_1 + \mu_2)(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2) \\ + \frac{1}{2}(\mu_1 - \mu_2)[(\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2]^{1/2},$$

thus determining the potential $W_\lambda(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})$ by (3.3), or explicitly

$$(3.13) \quad W_\lambda(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = \frac{1}{4}(\lambda_1 + \lambda_2) \left(\|\boldsymbol{\varepsilon}\|^2 + \frac{h^2}{12} \|\boldsymbol{\varkappa}\|^2 \right) \\ + \frac{1}{4}(\lambda_1 - \lambda_2) \left[\left(\|\boldsymbol{\varepsilon}\|^2 - \frac{h^2}{12} \|\boldsymbol{\varkappa}\|^2 \right)^2 + \frac{h^2}{3} (\boldsymbol{\varepsilon} \cdot \boldsymbol{\varkappa})^2 \right]^{1/2}.$$

The remainder of this section is dedicated to the derivation of (3.12) in full details.

Assume that $\|\boldsymbol{\varepsilon}\| \neq 0$, $\|\boldsymbol{\kappa}\| \neq 0$; introduce the tensors of unit norms

$$(3.14) \quad \hat{\boldsymbol{\varepsilon}} = \frac{\boldsymbol{\varepsilon}}{\|\boldsymbol{\varepsilon}\|}, \quad \hat{\boldsymbol{\kappa}} = \frac{\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|}$$

and an auxiliary tensor

$$(3.15) \quad \tilde{\mathbf{k}} = \begin{cases} \hat{\mathbf{k}} & \text{if } \hat{\boldsymbol{\varepsilon}} \cdot \hat{\mathbf{k}} > 0, \\ -\hat{\mathbf{k}} & \text{if } \hat{\boldsymbol{\varepsilon}} \cdot \hat{\mathbf{k}} < 0. \end{cases}$$

Obviously, $U_\lambda^*(\hat{\boldsymbol{\varepsilon}}, \hat{\mathbf{k}}) = U_\lambda^*(\hat{\boldsymbol{\varepsilon}}, \tilde{\mathbf{k}})$. Next, define a non-dimensional invariant

$$(3.16) \quad \xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2},$$

obtaining in this way

$$(3.17) \quad 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \mu_1 \|\boldsymbol{\varepsilon}\|^2 \max_{\substack{\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\omega}_\beta = \delta_{\alpha\beta} \\ \boldsymbol{\omega}_\alpha \in \mathbb{E}_s^2}} U^*(\hat{\boldsymbol{\varepsilon}}, \tilde{\mathbf{k}}).$$

where $\alpha, \beta = 1, 2$ and

$$(3.18) \quad U^*(\hat{\boldsymbol{\varepsilon}}, \tilde{\mathbf{k}}) = (\boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\varepsilon}})^2 + d(\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\varepsilon}})^2 + \xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})(\boldsymbol{\omega}_1 \cdot \tilde{\mathbf{k}})^2 + d\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})(\boldsymbol{\omega}_2 \cdot \tilde{\mathbf{k}})^2,$$

with

$$(3.19) \quad d = \frac{\mu_2}{\mu_1}, \quad d = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}.$$

Note that $d \in (0, 1)$.

In order to perform the maximization in (3.17), it is convenient to represent tensors from \mathbb{E}_s^2 as vectors belonging to \mathbb{R}^3 , see (2.8), as follows:

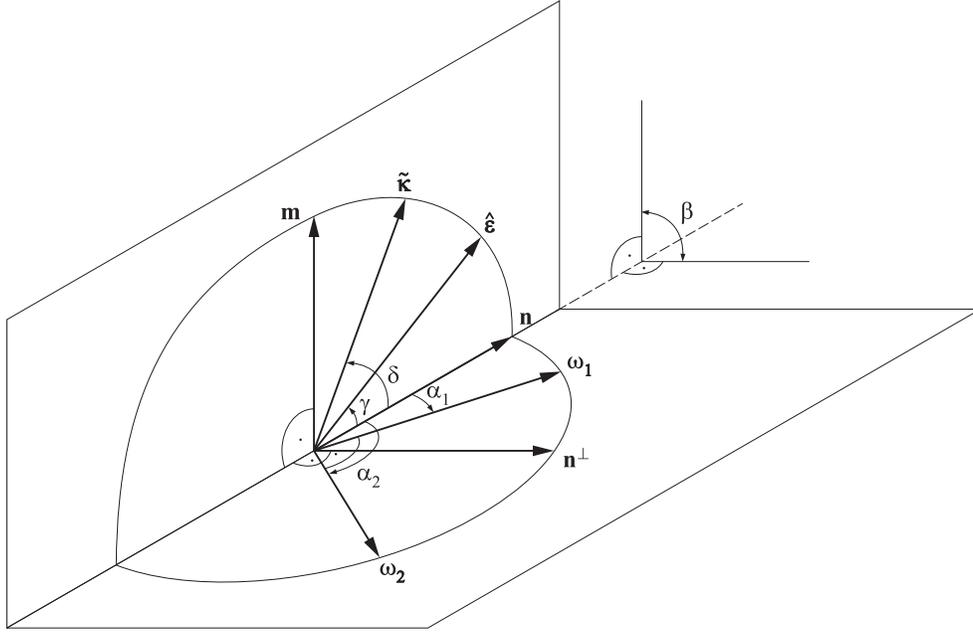
$$(3.20) \quad \boldsymbol{\omega}_\alpha = \begin{bmatrix} \omega_\alpha^{11} \\ \omega_\alpha^{22} \\ \sqrt{2}\omega_\alpha^{12} \end{bmatrix}, \quad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \sqrt{2}\hat{\varepsilon}_{12} \end{bmatrix}, \quad \tilde{\mathbf{k}} = \begin{bmatrix} \tilde{k}_{11} \\ \tilde{k}_{22} \\ \sqrt{2}\tilde{k}_{12} \end{bmatrix},$$

and to consider two planes: Π_{12} spanned by vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ and $\Pi_{\varepsilon\kappa}$ determined by vectors $\hat{\boldsymbol{\varepsilon}}, \tilde{\mathbf{k}}$ with the unit vector \mathbf{n} , denoting the vector at the edge of both planes, see Fig. 1.

Assume that vector \mathbf{n}^\perp belongs to Π_{12} , it is orthogonal to \mathbf{n} and such that $\boldsymbol{\omega}_2 \cdot \mathbf{n}^\perp > 0$. Similarly, suppose that vector \mathbf{m} belongs to $\Pi_{\varepsilon\kappa}$, it is orthogonal to \mathbf{n} and such that $\mathbf{m} \cdot \hat{\boldsymbol{\varepsilon}} > 0$. Moreover, set $\beta = \angle(\mathbf{n}^\perp, \mathbf{m})$, $\alpha_1 = \angle(\mathbf{n}, \boldsymbol{\omega}_1)$ and $\alpha_2 = \pi/2 - \alpha_1$. Next, decompose $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \hat{\boldsymbol{\varepsilon}}$ and $\tilde{\mathbf{k}}$ in the basis $(\mathbf{n}, \mathbf{n}^\perp)$

$$(3.21) \quad \begin{aligned} \boldsymbol{\omega}_1 &= \cos \alpha_1 \mathbf{n} + \sin \alpha_1 \mathbf{n}^\perp, & \hat{\boldsymbol{\varepsilon}} &= \cos \gamma \mathbf{n} + \sin \gamma \mathbf{m}, \\ \boldsymbol{\omega}_2 &= \cos \alpha_2 \mathbf{n} + \sin \alpha_2 \mathbf{n}^\perp, & \tilde{\mathbf{k}} &= \cos \delta \mathbf{n} + \sin \delta \mathbf{m}, \end{aligned}$$

and compute the scalar products

FIG. 1. Juxtaposition of vectors $\hat{\epsilon}$, $\tilde{\mathbf{k}}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$.

$$\begin{aligned}
 \boldsymbol{\omega}_1 \cdot \hat{\epsilon} &= \cos \alpha_1 \cos \gamma + \sin \alpha_1 \sin \gamma \cos \beta, \\
 \boldsymbol{\omega}_2 \cdot \hat{\epsilon} &= \cos \alpha_2 \cos \gamma + \sin \alpha_2 \sin \gamma \cos \beta, \\
 \boldsymbol{\omega}_1 \cdot \tilde{\mathbf{k}} &= \cos \alpha_1 \cos \delta + \sin \alpha_1 \sin \delta \cos \beta, \\
 \boldsymbol{\omega}_2 \cdot \tilde{\mathbf{k}} &= \cos \alpha_2 \cos \delta + \sin \alpha_2 \sin \delta \cos \beta,
 \end{aligned}
 \tag{3.22}$$

where $\alpha_2 = \alpha_1 + \pi/2$. Substitution of (3.22) into (3.18) gives

$$U^*(\hat{\epsilon}, \tilde{\mathbf{k}}) = g(\cos \beta)
 \tag{3.23}$$

with g being a second-order polynomial

$$g(z) = az^2 + bz + c
 \tag{3.24}$$

with $a > 0$. The explicit formulae for $a(\alpha_1, \gamma, \delta)$, $b(\alpha_1, \gamma, \delta)$, $c(\alpha_1, \gamma, \delta)$ are not necessary in the calculations, hence their derivation is omitted. Note that

$$\begin{aligned}
 (3.25) \quad & \max_{\substack{\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\omega}_\beta = \delta_{\alpha\beta} \\ \boldsymbol{\omega}_\alpha \in \mathbb{B}_s^2}} U^*(\hat{\epsilon}, \tilde{\mathbf{k}}) \\
 &= \max_{\alpha_1, \gamma, \delta, \beta \in \mathbb{R}} \{a(\alpha_1, \gamma, \delta) \cos^2 \beta + b(\alpha_1, \gamma, \delta) \cos \beta + c(\alpha_1, \gamma, \delta)\} \\
 &= \max_{\alpha_1, \gamma, \delta \in \mathbb{R}} \max \{a(\alpha_1, \gamma, \delta) \pm b(\alpha_1, \gamma, \delta) + c(\alpha_1, \gamma, \delta)\},
 \end{aligned}$$

since maximum is attained for $\cos \beta = \pm 1$.

The result above implies that Π_{12} and $\Pi_{\varepsilon\kappa}$ are coplanar, thus vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\hat{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\kappa}}$ belong to the same plane. Moreover, the arbitrariness of the directions of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ leads to the assumption that the juxtaposition of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\hat{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\kappa}}$ shown in Fig. 2 is possible. Let $\alpha = \angle(\hat{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\kappa}})$ and note that $\alpha \in (0, \pi/2)$.

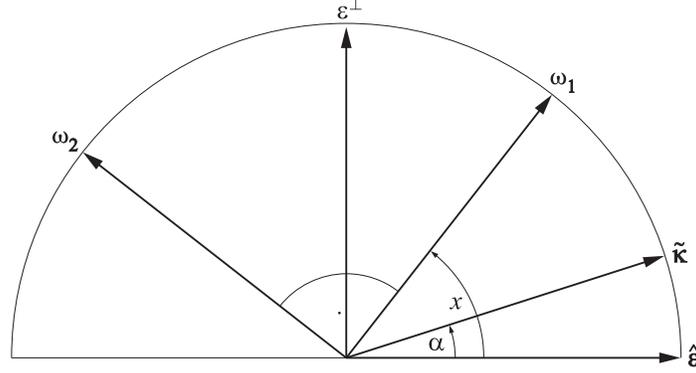


FIG. 2. Juxtaposition of vectors $\hat{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\kappa}}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ after maximization over β .

Let $x = \angle(\hat{\boldsymbol{\varepsilon}}, \boldsymbol{\omega}_1)$. Then

$$(3.26) \quad \begin{aligned} \boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\varepsilon}} &= \cos x, & \boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\varepsilon}} &= -\sin x, \\ \boldsymbol{\omega}_1 \cdot \tilde{\boldsymbol{\kappa}} &= \cos(x - \alpha), & \boldsymbol{\omega}_2 \cdot \tilde{\boldsymbol{\kappa}} &= -\sin(x - \alpha) \end{aligned}$$

which gives

$$(3.27) \quad U^*(\hat{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\kappa}}) = H(x),$$

see (3.18), where

$$(3.28) \quad H(x) = \cos^2 x + \xi \cos^2(x - \alpha) + d \sin^2 x + d\xi \sin^2(x - \alpha)$$

with ξ and α depending on $\hat{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\kappa}}$.

The next task is to maximize (3.28) with respect to x , since it is easily seen that

$$(3.29) \quad \begin{aligned} \max \{U^*(\hat{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\kappa}}) \mid \boldsymbol{\omega}_\alpha \cdot \boldsymbol{\omega}_\beta = \delta_{\alpha\beta}, \boldsymbol{\omega}_\alpha \in \mathbb{E}_s^2, \alpha, \beta = 1, 2\} \\ = \max \{H(x) \mid x \in \mathbb{R}\}. \end{aligned}$$

To perform this maximization set

$$(3.30) \quad H(x) = \tilde{H}(2x), \quad \tilde{H}(y) = \frac{1}{2} [(1 + \xi)(1 + d) + (1 - d)f(y)]$$

where

$$(3.31) \quad f(y) = \cos y + \xi \cos(y - y_0)$$

and $y_0 = 2\alpha$. By

$$(3.32) \quad \max\{H(x) \mid x \in \mathbb{R}\} = \max\{\tilde{H}(y) \mid y \in \mathbb{R}\}$$

one obtains

$$(3.33) \quad \max\{H(x) \mid x \in \mathbb{R}\} = \frac{1}{2}(1 + \xi)(1 + d) + \frac{1}{2}(1 - d) \max_{y \in \mathbb{R}} f(y).$$

Applying (A.2), see Appendix A, leads to

$$(3.34) \quad \max_{y \in \mathbb{R}} f(y) = [1 + 2\xi \cos(2\alpha) + \xi^2]^{1/2}$$

which determines the potential

$$(3.35) \quad 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2}(\mu_1 + \mu_2) (\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2) + \frac{1}{2}(\mu_1 - \mu_2) [\|\boldsymbol{\varepsilon}\|^4 + 2\|\boldsymbol{\varepsilon}\|^2\|\boldsymbol{\kappa}\|^2 \cos(2\alpha) + \|\boldsymbol{\kappa}\|^4]^{1/2},$$

where $\cos \alpha = \pm(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa} / \|\boldsymbol{\varepsilon}\| \|\boldsymbol{\kappa}\|)$, hence the sign of $\cos \alpha$ does not affect (3.35) thus proving (3.12).

REMARK. The discussion in this paper is based on the assumed equivalence between Eqs. (2.19) and (2.20), which will be proved in Part II of the present paper.

3.2. Representation of the optimal material and its constitutive properties

The following question arises: which eigenstates $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ realize their maxima in (3.5)? In this section the general case of $\|\boldsymbol{\varepsilon}\| \neq 0$, $\|\boldsymbol{\kappa}\| \neq 0$ is analyzed. Two extreme cases $\boldsymbol{\varepsilon} = \mathbf{0}$ and $\boldsymbol{\kappa} = \mathbf{0}$ are treated separately in the sequel.

Introduce $\boldsymbol{\varepsilon}^\perp$, see Fig. 2, co-planar with $\hat{\boldsymbol{\varepsilon}}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, orthogonal to $\hat{\boldsymbol{\varepsilon}}$ and such that the angle $\angle(\boldsymbol{\omega}_1, \boldsymbol{\varepsilon}^\perp)$ is acute. Next, assume that $\alpha \neq 0$. In this way $\boldsymbol{\varepsilon}^\perp$ can be expressed as

$$(3.36) \quad \boldsymbol{\varepsilon}^\perp = -\cot \alpha \hat{\boldsymbol{\varepsilon}} + \frac{1}{\sin \alpha} \tilde{\boldsymbol{\kappa}}.$$

Decomposition of $\tilde{\boldsymbol{\kappa}}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ in the basis $(\hat{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}^\perp)$ takes the form

$$(3.37) \quad \begin{aligned} \tilde{\boldsymbol{\kappa}} &= \cos \alpha \hat{\boldsymbol{\varepsilon}} + \sin \alpha \boldsymbol{\varepsilon}^\perp, \\ \boldsymbol{\omega}_1 &= \cos x_0 \hat{\boldsymbol{\varepsilon}} + \sin x_0 \boldsymbol{\varepsilon}^\perp, \\ \boldsymbol{\omega}_2 &= -\sin x_0 \hat{\boldsymbol{\varepsilon}} + \cos x_0 \boldsymbol{\varepsilon}^\perp, \end{aligned}$$

with x_0 being the maximizer in (3.29), hence one may express the angle x_0 by formula (A.3), see Appendix A, or

$$(3.38) \quad \tan(2x_0) = \frac{\xi \sin(2\alpha)}{1 + \xi \cos(2\alpha)}.$$

Substituting (3.37) in (3.36) gives

$$(3.39) \quad \boldsymbol{\omega}_1 = \gamma_1 \hat{\boldsymbol{\epsilon}} + \gamma_2 \tilde{\boldsymbol{\kappa}}, \quad \boldsymbol{\omega}_2 = \delta_1 \hat{\boldsymbol{\epsilon}} + \delta_2 \tilde{\boldsymbol{\kappa}},$$

where

$$(3.40) \quad \begin{aligned} \gamma_1 &= \cos x_0 - \cot \alpha \sin x_0, & \gamma_2 &= \frac{\sin x_0}{\sin \alpha}, \\ \delta_1 &= -\sin x_0 - \cot \alpha \cos x_0, & \delta_2 &= \frac{\cos x_0}{\sin \alpha}, \end{aligned}$$

with x_0 given by (3.38). Note that both x_0 and $x_0 + \pi/2$ solve (3.38) with the latter case resulting in switching $\boldsymbol{\omega}_1$ with $\boldsymbol{\omega}_2$, and the definition of $\boldsymbol{\omega}_3$ is unnecessary here since $\boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3$ was eliminated from the calculations by (2.13).

Assume now that $\alpha = 0$. Then $\hat{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\kappa}}$ and by (3.18) one obtains

$$(3.41) \quad U^*(\hat{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\kappa}}) = (1 + \xi) [(\boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\epsilon}})^2 + d(\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\epsilon}})^2].$$

Inserting (3.26) into (3.41) and taking into account that $\alpha = 0$ leads to

$$(3.42) \quad U^*(\hat{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\kappa}}) = (1 + \xi)[\cos^2 x + d \sin^2 x]$$

or

$$(3.43) \quad U^*(\hat{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\kappa}}) = \frac{1}{2}(1 + \xi)[(1 + d) + (1 - d) \cos 2x].$$

Since $1 - d > 0$, the maximum in Eq. (3.29) is attained for $\cos 2x = 1$. Then $x = 0$ or $x = \pi$, hence $\boldsymbol{\omega}_1 = \pm \hat{\boldsymbol{\epsilon}}$ and $\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\epsilon}} = 0$, therefore

$$(3.44) \quad \max_{x \in \mathbb{R}} U^*(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\kappa}}) = (1 + \xi),$$

which is compatible with (3.29)–(3.34) provided $\alpha = 0$.

It follows that the optimal choice of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ is given by

$$(3.45) \quad \boldsymbol{\omega}_1 = \begin{cases} \pm(\gamma_1 \hat{\boldsymbol{\epsilon}} + \gamma_2 \tilde{\boldsymbol{\kappa}}) & \text{if } \angle(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\kappa}}) \neq 0, \\ \pm \hat{\boldsymbol{\epsilon}} & \text{if } \angle(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\kappa}}) = 0, \end{cases}$$

$$(3.46) \quad \boldsymbol{\omega}_2 = \begin{cases} \pm(\delta_1 \hat{\boldsymbol{\epsilon}} + \delta_2 \tilde{\boldsymbol{\kappa}}) & \text{if } \angle(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\kappa}}) \neq 0, \\ \text{arbitrary vector} & \\ \text{perpendicular to } \hat{\boldsymbol{\epsilon}} & \text{if } \angle(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\kappa}}) = 0 \end{cases}$$

and $\boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$.

Consequently, by substituting the defined $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ in (2.10), one may construct tensor \mathbf{A} in two following cases:

(a) Case of $\angle(\hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{\kappa}}) \neq 0$,

$$(3.47) \quad \mathbf{A} = (\lambda_1 - \lambda_3)(\gamma_1 \hat{\boldsymbol{\varepsilon}} + \gamma_2 \tilde{\boldsymbol{\kappa}}) \otimes (\gamma_1 \hat{\boldsymbol{\varepsilon}} + \gamma_2 \tilde{\boldsymbol{\kappa}}) \\ + (\lambda_2 - \lambda_3)(\delta_1 \hat{\boldsymbol{\varepsilon}} + \delta_2 \tilde{\boldsymbol{\kappa}}) \otimes (\delta_1 \hat{\boldsymbol{\varepsilon}} + \delta_2 \tilde{\boldsymbol{\kappa}}) + \lambda_3 \mathbf{I}_4$$

with $\boldsymbol{\omega}_2$ being arbitrary and orthogonal to $\hat{\boldsymbol{\varepsilon}}$, and

(b) Case of $\angle(\hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{\kappa}}) = 0$,

$$(3.48) \quad \mathbf{A} = (\lambda_1 - \lambda_3) \hat{\boldsymbol{\varepsilon}} \otimes \hat{\boldsymbol{\varepsilon}} + (\lambda_2 - \lambda_3) \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2 + \lambda_3 \mathbf{I}_4.$$

Making use of (3.2) and (3.3) one may find the explicit expressions for the constitutive Eqs. (2.25)

$$(3.49) \quad \mathbf{N} = 2 \frac{\partial U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M} = 2 \frac{h}{\sqrt{12}} \frac{\partial U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\kappa}}.$$

For the brevity of notation, define the following scalar functions:

$$(3.50) \quad \phi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2}{G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}, \quad \psi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{2(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})}{G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}, \\ G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = ((\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2)^2 + 4(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2)^{1/2}.$$

Then write

$$(3.51) \quad 8U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = (\lambda_1 + \lambda_2) \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$$

where

$$(3.52) \quad \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2 + \nu G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}).$$

Obviously,

$$(3.53) \quad \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \tilde{U}^*(\boldsymbol{\kappa}, \boldsymbol{\varepsilon}), \quad \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \geq 0.$$

Note that the dependence on λ has been suppressed, while the ratio

$$(3.54) \quad \nu = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$$

satisfies: $0 < \nu < 1$, since $\lambda_1 > \lambda_2 > 0$.

The following rules of differentiation will be frequently used in the sequel

$$(3.55) \quad \frac{\partial \|\boldsymbol{\varepsilon}\|^2}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}, \quad \frac{\partial(\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\kappa}$$

leading to

$$(3.56) \quad \frac{\partial G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon}} = 2\mathbf{L}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$$

with

$$(3.57) \quad \mathbf{L}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \phi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})\boldsymbol{\varepsilon} + \psi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})\boldsymbol{\kappa}.$$

Consequently, one may write

$$(3.58) \quad \mathbf{N} = \frac{1}{4}(\lambda_1 + \lambda_2) \frac{\partial \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{K} = \frac{1}{4}(\lambda_1 + \lambda_2) \frac{\partial \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\kappa}}$$

where $\mathbf{K} = \frac{\sqrt{12}}{h}\mathbf{M}$ or, by (3.55)–(3.57),

$$(3.59) \quad \mathbf{N} = \frac{1}{2}(\lambda_1 + \lambda_2) [\boldsymbol{\varepsilon} + \nu \mathbf{L}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})], \quad \mathbf{K} = \frac{1}{2}(\lambda_1 + \lambda_2) [\boldsymbol{\kappa} + \nu \mathbf{L}(\boldsymbol{\kappa}, \boldsymbol{\varepsilon})].$$

It is worth mentioning that formulae (3.49) make sense if and only if $G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \neq 0$ in (3.50). For $\boldsymbol{\varepsilon}, \boldsymbol{\kappa}$ such that $\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2 = 0$ and $\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa} = 0$, it is easily seen from (3.52), (3.51) and (3.49) that (3.59) become

$$\mathbf{N} = \frac{1}{2}(\lambda_1 + \lambda_2)\boldsymbol{\varepsilon}, \quad \mathbf{K} = \frac{1}{2}(\lambda_1 + \lambda_2)\boldsymbol{\kappa}.$$

3.3. Extreme cases of $\boldsymbol{\varepsilon} = \mathbf{0}$ or $\boldsymbol{\kappa} = \mathbf{0}$

For analyzing whether (3.11) applies in the case of $\boldsymbol{\varepsilon} = \mathbf{0}, \boldsymbol{\kappa} \neq \mathbf{0}$, one has to turn back to (3.9), (3.10), thus obtaining

$$(3.60) \quad 4U_1^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \max_{\substack{\boldsymbol{\omega}_\alpha \cdot \boldsymbol{\omega}_\beta = \delta_{\alpha\beta}, \\ \boldsymbol{\omega}_\alpha \in \mathbb{E}_s^2}} \{ \mu_1(\boldsymbol{\omega}_1 \cdot \boldsymbol{\kappa})^2 + \mu_2(\boldsymbol{\omega}_2 \cdot \boldsymbol{\kappa})^2 \}.$$

Choosing $\boldsymbol{\omega}_1 = \hat{\boldsymbol{\kappa}}$, see Eq. (3.14), gives $\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\kappa}} = 0$ and

$$(3.61) \quad \mu_1(\boldsymbol{\omega}_1 \cdot \boldsymbol{\kappa})^2 + \mu_2(\boldsymbol{\omega}_2 \cdot \boldsymbol{\kappa})^2 = \mu_1\|\boldsymbol{\kappa}\|^2.$$

Indeed, let $x = \angle(\hat{\boldsymbol{\kappa}}, \boldsymbol{\omega}_1)$. Then

$$(3.62) \quad \boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\kappa}} = \cos x, \quad \boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\kappa}} = \cos\left(\frac{\pi}{2} + x\right) = -\sin x.$$

Next, by making use of $\mu_1 \geq \mu_2$, one can estimate

$$(3.63) \quad \mu_1(\boldsymbol{\omega}_1 \cdot \boldsymbol{\kappa})^2 + \mu_2(\boldsymbol{\omega}_2 \cdot \boldsymbol{\kappa})^2 \leq \mu_1 [(\boldsymbol{\omega}_1 \cdot \boldsymbol{\kappa})^2 + (\boldsymbol{\omega}_2 \cdot \boldsymbol{\kappa})^2] \\ = \mu_1\|\boldsymbol{\kappa}\|^2 [(\boldsymbol{\omega}_1 \cdot \hat{\boldsymbol{\kappa}})^2 + (\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{\kappa}})^2] = \mu_1\|\boldsymbol{\kappa}\|^2$$

and it is obvious by (3.61) that this bound is attainable for $\boldsymbol{\omega}_1 = \hat{\boldsymbol{\kappa}}$. This implies

$$(3.64) \quad 4U_1^*(\mathbf{0}, \boldsymbol{\kappa}) = \mu_1\|\boldsymbol{\kappa}\|^2, \quad 4U_\lambda^*(\mathbf{0}, \boldsymbol{\kappa}) = \lambda_1\|\boldsymbol{\kappa}\|^2$$

which is compatible with (3.35) and (3.11).

By analogy,

$$(3.65) \quad 4U_1^*(\boldsymbol{\varepsilon}, \mathbf{0}) = \mu_1 \|\boldsymbol{\varepsilon}\|^2, \quad 4U_\lambda^*(\boldsymbol{\varepsilon}, \mathbf{0}) = \lambda_1 \|\boldsymbol{\varepsilon}\|^2$$

which is also compatible with (3.35). We thus conclude that (3.35) and (3.11) hold for arbitrary $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$.

Final remarks

The free material optimization problem (2.15) has been reduced to the equilibrium problem (P^*) or (2.24) of a hyperelastic plate with an effective potential expressed by (3.13). The relevant constitutive equations (2.25) have been transformed to the form (3.59). The convexity of the potential (3.13) will be proved in the second part of the present paper. The mentioned convexity property implies strict monotonicity of the constitutive equations, which implies uniqueness of the solutions of the optimization problem discussed.

Acknowledgements

Authors express their thanks to Zdzisław Naniewicz and Jan Sokołowski for useful comments and bibliographical remarks.

The paper was prepared within the Research Grant N506 071338, financed by the Polish Ministry of Science and Higher Education, entitled: *Topology Optimization of Engineering Structures. Simultaneous shaping and local material properties determination.*

Appendix A

Let $\zeta > 0$ and define the function

$$(A.1) \quad f(y) = \cos y + \zeta \cos(y - y_0).$$

Then

$$(A.2) \quad \max_y f(y) = (1 + 2\zeta \cos y_0 + \zeta^2)^{1/2}$$

the maximizer being

$$(A.3) \quad y = \arctan \left(\frac{\zeta \sin y_0}{1 + \zeta \cos y_0} \right).$$

To prove (A.3) it is sufficient to solve the equation $f'(y) = 0$. The formula (A.2) has a geometric interpretation shown in Fig. 3.

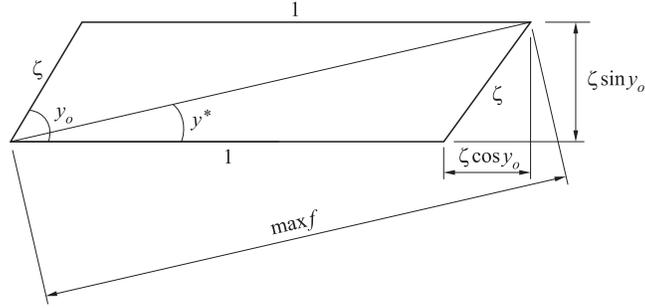


FIG. 3. Expression (A.2) represents the length of the longer diagonal of a parallelogram of sides 1 and ζ and angle y_0 .

The result (A.2) can be inferred from the rule

$$(A.4) \quad A_1 \cos(x + \varphi_1) + A_2 \cos(x + \varphi_2) = A \cos(x + \varphi)$$

where

$$(A.5) \quad A = (A_1^2 + A_2^2 + 2A_1 A_2 \cos(\varphi_1 - \varphi_2))^{1/2},$$

$$(A.6) \quad \tan \varphi = \frac{A_1 \sin \varphi_1 + A_2 \sin \varphi_2}{A_1 \cos \varphi_1 + A_2 \cos \varphi_2},$$

see [8]. Note that (A.4) attains its maximum for $x + \varphi = 0$. Problem (A.2) admits $A_1 = 1$, $A_2 = \zeta$, $\varphi_1 = 0$, $\varphi_2 = -y_0$ and (A.3) follows from (A.6).

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Received January 21, 2011; revised version June 9, 2011.
