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Steady streaming effect on the flow of a couple stress fluid through a constricted annulus

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The oscillatory flow of an incompressible couple stress fluid through an annulus with mild constriction at the outer wall is considered. The mean pressure drops and the mean wall shear stress are calculated across the constricted region. The steady streaming effect on the flow is presented. The variations in the mean pressure drop and wall shear stress with the size of the catheter, the velocity of the catheter and couple stress fluid parameter, are studied through graphs.

Key words: catheterized artery, constriction, annulus, couple stress fluid, wall shear stress.

1. Introduction

The flows of the fluid in pipes of different shapes are important in many biological and biomedical systems, the human cardiovascular system and in several technological devices. The abnormal and unnatural growth in the lumen of an artery is called stenosis. Localized atherosclerotic constrictions in arteries (arterial stenosis) are found predominantly in the internal carotid artery, which supplies blood to the brain, the coronary artery, which supplies blood to the cardiac muscles, and the femoral artery, which supplies blood to the lower limbs. Catheterization refers to a procedure in which a long, thin, flexible plastic tube (catheter) is inserted into the artery. Catheter procedures can both diagnose and treat heart and blood-vessel conditions. Angiography, which is used for diagnosis, is the most common type of heart catheter procedure. The insertion of a catheter in an artery will form an annular region between the walls of the catheter and the artery. This will alter the flow field, modify the pressure distribution and increase the resistance. Hence, the pressure or pressure gradient recorded by a transducer attached to the catheter will differ from that of an uncatheterized artery, how small the size of the catheter may be. Therefore, it is important to study the effect due to presence of a catheter in the physiological artery flows.

A simple mathematical model for studying blood flow in a stenosed artery when a catheter is inserted into it, is the flow of the fluid through an annulus with mild constriction at the outer wall. In recent years, attention has been given to study blood flow characteristics due to presence of catheter in the lumen of the artery. Several researchers have studied the flow of blood in catheterized artery by modeling blood as a Newtonian or non-Newtonian fluid. ROOSE and LYKODIS [1] studied the fluid mechanics of the uretor with an inserted catheter by considering the peristaltic wave moving along the stationary cylinder. MACDONALD [2] considered the pulsatile blood flow in a catheterized artery and obtained theoretical estimates for pressure gradient corrections of the catheters, which are positioned eccentrically, as well as coaxially with the artery. KARAHALIOS [3] has studied the effect of catheterization on various flow characteristics in an artery with or without stenosis. DARIPA and DASH [4] have analyzed the numerical study of pulsatile blood flow in an eccentric catheterized artery, using a fast algorithm treating blood as a Newtonian fluid.

It is well known that, blood being a suspension of cells, behaves like a non-Newtonian fluid at low shear rates and during its flow through narrow blood vessels. SHUKLA *et al.* [5] investigated the effects of stenosis on non-Newtonian flow of the blood in an artery. PHILIP and PEEYUSH CHANDRA [6] have studied the flow of blood, which has been modeled by a simple micro-fluid in the core region with a Newtonian fluid peripheral layer, in a tube in the presence of very mild stenosis. DASH *et al.* [7] considered the steady and pulsatile flow in a narrow artery when a catheter is inserted into it and estimated the increase in frictional resistance in the artery due to catheterization, using Casson fluid model. SANKAR *et al.* [8] discussed the steady flow of Herschel–Bulkley fluid through a catheterized artery. SANKAR and LEE [9] analyzed a two-fluid Herschel–Bulkley model for blood flow in catheterized arteries. SRIVASTAVA and SRIVASTAVA [10] have considered the particulate suspension blood flow through a narrow catheterized artery. SRIVATSAVA and RASTOGI [11] have investigated the problem of blood flow through a narrow, catheterized artery with an axially non-symmetrical stenosis using a two-phase macroscopic model of blood (i.e., a suspension of red cells in plasma). Blood flow through a catheterized artery is analyzed by SANKAR and LEE [12], assuming the flow is steady and blood is treated as a two-fluid model with suspension of all the erythrocytes, in the core region as a Casson fluid and the plasma in the peripheral region as a Newtonian fluid.

The couple stress fluid theory represents the simplest generalization of the classical viscous fluid theory that allows for polar effects in the fluids. This fluid theory shows all the important features and effects of couple stresses and results in equations that are similar to Navier–Stokes equations. The main effect of couple stresses will be to introduce a size-dependent effect that is not present in the classical non-polar theories [13]. Blood is a suspension of blood cells [14] and blood cells influence flow characteristics significantly. The model using Newtonian/non-Newtonian concepts have no additive equations for accounting these size effects in the flow, however micro-continumm theories proposed by Stokes [13], Eringen [15] and Cowin [16] have additive constitutive equation for accounting the size effects. In view of the above, blood is considered to be represented by couple stress fluid in the present study. Chaturani has analyzed the problem of pulsatile flow of couple stress fluid with application to blood flow [17]. An analysis of the effects of couple stresses on the blood flow through thin artery with mild stenosis has been carried out by Sinha and Singh [18]. Srivastava [19] considered the flow of couple stress fluid through stenotic blood vessels. Srinivasacharya and Srikanth [20] studied the effect of couple stresses on the pulsatile flow through constricted annulus.

A class of problems of interest for researchers in fluid mechanics is the contribution of the steady streaming effect, due to the oscillatory nature of the flow. In an oscillatory flow, the steady motion of the fluid may occur when the real parts of the components of the velocity vector oscillate with a phase difference other than $\pi/2$. This is due to the generation of a non-zero average force on the fluid and a non-zero mean flux across the surface in the fluid. The consequent steady motion of the fluid leads to extensive migration of fluid elements in an apparently oscillatory system. This drift motion is called the steady streaming. Schlichting [21] studied the phenomenon of steady streaming about a solid circular cylinder in a vibrating fluid.

It is well-known that a fluid flow which is dominated by fluctuating components will realize in a non-zero time averaged component – a contribution due to the nonlinearity of the governing equations [22, 23]. This observation was missed in most of the studies on pulsatile flow of blood, since the models were based on the linearized Navier–Stokes equations. Although the analyzes were quite satisfactory in explaining many of the essential characteristics of blood flow in small arteries, they failed to be adequate in special situations. In fact, when an oscillatory viscous flow is set up over a wavy surface or wall, the Reynolds stresses within the fluid generate a steady streaming – a time-independent component of motion. Stuart [22] in his classical paper showed that there is an outer boundary layer within which the steady streaming velocity decays to zero. Riley [23] reviewed the earlier works on steady streaming analysis and discussed the formulation of the problem of calculating the steady flow driven by an oscillating body in fluid in terms of matched asymptotic expansions, under several different conditions influencing the dynamics of the flow. Hall [24] investigated the steady streaming in a pipe of slowly varying cross-section when an oscillatory pressure difference was maintained between the ends. The results showed regions of recirculation zones in the Stokes layer near the wall for large values of the steady streaming Reynolds number. Recently, SARKAR and JAYARAMAN [25] analyzed a theoretical model to study the combined effects of the catheter and elastic properties of the arterial wall, on the pulsatile nature of the blood flow.

In the present study, the oscillatory flow of an incompressible couple stress fluid through annulus with mild constriction at the outer wall is investigated. The mean pressure drop and the mean wall shear stress are calculated across the constricted region. The effects of steady streaming, couple stress fluid parameter and size of the catheter on mean pressure drop and mean wall shear stress are reported.

2. Formulation of the problem

Consider the flow of an incompressible couple stress fluid through the artery, an axisymmetric rigid tube of radius a with a catheter, a coaxial flexible tube of radius ka $(k < 1)$. The oscillatory nature of the flow will have an influence on the instantaneous position of the flexible catheter. The movement of the catheter is considered to be in phase with the rate of flow with small constant amplitude. The flow is assumed to be axi-symmetric and oscillatory in nature. The stenosis over a length of the artery is assumed to have developed in an axi-symmetric manner. The velocity of the catheter is taken as $\tilde{w}_c = w_c \cos(\tau - \tau_0)$, where $w_c < 1$ is the maximum amplitude of the moving catheter and τ_0 is the phase lead of the catheter oscillations.

The equations governing the flow of an incompressible couple stress fluid in the absence of body force and body couple are:

$$\text{(2.1)} \qquad \operatorname{div} \mathbf{q} = 0,$$

$$\text{(2.2)} \qquad \rho \frac{d\mathbf{q}}{dt} = -\operatorname{grad} P - \mu \operatorname{curl} \operatorname{curl} \mathbf{q} - \eta \operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} \mathbf{q},$$

where ρ is the density, $\mathbf{q}$ is the velocity vector, η is the couple stress fluid parameter, P is the fluid pressure, and μ is the fluid viscosity.

The force stress tensor $\boldsymbol{\tau}_{ij}$ and the couple stress tensor $\mathbf{m}_{ij}$ that appear in the theory, are given by

$$\text{(2.3)} \qquad \boldsymbol{\tau}_{ij} = (-P + \lambda \operatorname{div} \mathbf{q})\delta_{ij} + 2\mu d_{ij} - (1/2)\varepsilon_{ijk}[\mathbf{m}_{,k} + 4\eta \omega_{k,rr} + \rho C_k)],$$

$$\text{(2.4)} \qquad \mathbf{m}_{ij} = 4\eta \boldsymbol{\omega}_{j,i} + 4\eta' \boldsymbol{\omega}_{i,j},$$

where $2\boldsymbol{\omega} = \operatorname{curl} \mathbf{q}$ is the spin vector, $\boldsymbol{\omega}_{i,j}$ is the spin tensor, d_{ij} is the rate of deformation vector derived from the velocity vector, p is the fluid pressure and ρC_k is the body couple vector. The quantities λ and μ are the viscosity coefficients and η' and η are the constants associated with couple stresses. These material constants satisfy the following inequalities:

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \qquad \eta \geq 0, \qquad \eta' \leq \eta. \tag{2.5}$$

The problem has been studied in cylindrical coordinate system (r, θ, z). Since the flow is axisymmetric, all the variables are independent of θ. Hence, for this flow the velocity is given by $\mathbf{q} = (u(r, z), 0, w(r, z))$. An approximation to the stenosed wall is taken as

$$r_s(z) = \begin{cases} a(1 - f(z)) & 0 \leq z \leq L_0, \\ a & \text{otherwise}, \end{cases} \tag{2.6}$$

where a is the radius of the artery in non-stenosed portion, L_0 is the magnitude of the distance along the artery over which the stenosis is spread out, and h is the maximum height of the stenosis. In the absence of the catheter, the model corresponds to the oscillatory flow in annular region between two concentric tubes, one of constant radius and the other one of a cross-section varying along the Z-axis. If the catheter moves considerably away from the axis of the tube, then the eccentric annulus does not allow for any axisymmetric character of the flow.

The non-dimensional variables are defined as:

$$\begin{aligned} &r = a\tilde{r}, \quad t = \tilde{t}/\omega, \quad \delta = \tilde{\delta}a, \quad w = u_0\tilde{w}, \quad u = \delta u_0 \tilde{u}, \quad P = (\rho u_0^2 \tilde{p})/\delta, \\ &z = (a\tilde{z})/\delta, \quad w_c = \tilde{w}_c/u_0, \end{aligned} \tag{2.7}$$

where $\delta = a/L_0$ emerges as the geometric parameter, ω is the frequency of the prescribed flow rate, u_0 is a typical axial velocity, P is the pressure and q the flow rate. $\delta \ll 1$ corresponds to the slowly varying cross-section and enables the use of lubrication theory.

Introducing the non-dimensional variables into Eqs. (2.1) and (2.2) and dropping the tildes, we get

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.8}$$

$$\begin{aligned} &\delta^2 \,\mathrm{Rw}^2 \frac{\partial u}{\partial t} + R_e \delta^3 \left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} \right) \\ &\qquad = -\,\mathrm{Re}\,\frac{\partial P}{\partial r} - \frac{\delta^2}{\alpha^2}\left[F^2 - \frac{1}{r^2}\right]^2 u + \delta^2 \left[F^2 - \frac{1}{r^2}\right] u, \end{aligned} \tag{2.9}$$

$$\mathrm{Rw}^2 \frac{\partial w}{\partial t} + \mathrm{Re}\,\delta \left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} \right) = -\,\mathrm{Re}\,\frac{\partial P}{\partial z} + F^2 w - \frac{1}{\alpha^2} F^4 w, \tag{2.10}$$

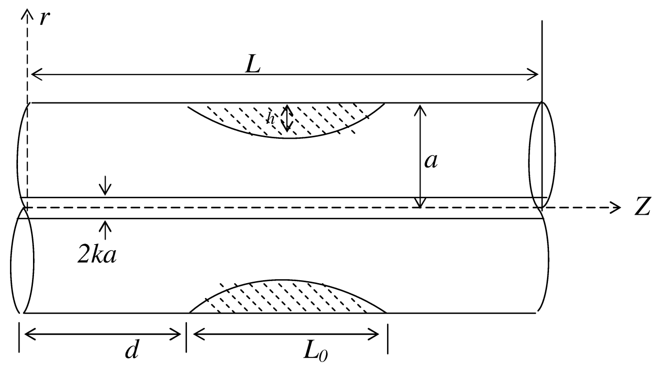

Fig. 1. Schematic diagram of catheterized stenosed artery.

where $F^2 = \partial^2/\partial r^2 + 1/r\partial/\partial r + \delta^2\partial^2/\partial z^2$, $\mu a^2/\eta = \alpha^2$, $\mathrm{Re} = \rho a u_0/\mu$ is the Reynolds number and $\mathrm{Rw}^2 = \rho a^2\omega/\mu$ is the Womersley number.

The boundary conditions are the usual no slip and hyper-stick conditions:

$$
\begin{aligned}
&w = u = 0 && \text{at } r = r_s(z),\\
&w = \tilde{w}_c(t), \quad u = 0 && \text{at } r = k,\\
&\left(\frac{\partial^2 w}{\partial r^2} - \frac{\sigma}{r}\frac{\partial w}{\partial r}\right) = 0 && \text{at } r = r_s(z) \text{ and } r = k,
\end{aligned}
\tag{2.11}
$$

where $r_s(z)$ represents the stenotic wall and $\sigma = \eta'/\eta$ is a couple stress fluid parameter. Boundary condition (2.11) shows that couple stresses (2.4) vanish at the tube wall and catheter wall.

In addition, we assume constant volume flux at every instant along the tube, i.e.,

$$Q = \cos(t) \qquad \forall z. \tag{2.12}$$

This helps us to find the alterations in the pressure gradient and hence the pressure due to the presence as well as the movement of the catheter.

3. Solution of the problem

The above equations obtained are non-linear and hence only a series solution for the velocity field is determined in powers of the geometric parameter δ. This amounts to solving the problem for a slowly varying cross-section of the annular region. This will also require that $\delta \ll 1$ while Re remains $O(1)$.

Let

$$u = u_0 + u_1\delta + u_2\delta^2 + \cdots, \tag{3.1}$$

where

$$u_0 = \frac{1}{2}\{u_{00}(r,z)e^{it} + \bar{u}_{00}(r,z)e^{-it}\}, \tag{3.2}$$

and where $\bar{u}_{00}$ is the complex conjugate of u_{00}. Similar expressions can be written for w and p. Substituting (3.1) in (2.8)–(2.10) and collecting the coefficients of various powers of δ on both the sides, we obtain the following set of coupled linear differential equations.

Zeroth order in δ

The equations corresponding to $O(1)$ terms, i.e. w_{00}, u_{00}, and p_{00} which are proportional to e^{it}, are

$$\frac{\partial u_{00}}{\partial r} + \frac{u_{00}}{r} + \frac{\partial w_{00}}{\partial z} = 0, \tag{3.3}$$

$$\frac{\partial p_{00}}{\partial r} = 0, \tag{3.4}$$

$$i\,\mathrm{Rw}^2\, w_{00} + \mathrm{Re}\,\frac{\partial p_{00}}{\partial z} = \left(\frac{\partial^2 w_{00}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{00}}{\partial r}\right) - \frac{1}{\alpha^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w_{00}. \tag{3.5}$$

The corresponding non-dimensional boundary conditions become

$$\begin{aligned}
&w_{00} = u_{00} = 0 && \text{at } r = r_s(z),\\
&w_{00} = w_c e^{-it} && \text{at } r = k,\\
&u_{00} = 0 && \text{at } r = k,
\end{aligned}$$

$$\left(\frac{\partial^2 w_{00}}{\partial r^2} - \frac{\sigma}{r}\frac{\partial w_{00}}{\partial r}\right) = 0 \quad \text{at } r = r_s(z) \text{ and } r = k, \tag{3.6}$$

$$Q_{00} = 1 \quad \forall z. \tag{3.7}$$

It can be noted from Eq. (3.4) that p_{00} is a function of z only. Eq. (3.5) can be simplified to the form

$$E^4 w_{00} - (\alpha_1^2 + \alpha_2^2)E^2 w_{00} + \alpha_1^2\alpha_2^2 w_{00} = -\alpha^2\,\mathrm{Re}\,\frac{dp_{00}}{dz}, \tag{3.8}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \quad (\alpha_1^2 + \alpha_2^2) = \frac{\mu a^2}{\eta} \quad \text{and} \quad \alpha_1^2\alpha_2^2 = \frac{i\mu a^2}{\eta} R_w^2.$$

The solution of the above equation is given by

$$w_{00} = C_1(z)I_0(\alpha_1 r) + C_2(z)K_0(\alpha_1 r) + C_3(z)I_0(\alpha_2 r) + C_4(z)K_0(\alpha_2 r) - i\frac{\mathrm{Re}}{\mathrm{Rw}^2}\frac{dp_{00}}{dz}, \tag{3.9}$$

where $I_0(\alpha_i r)$ and $K_0(\alpha_i r)$ $(i = 1, 2)$ are the modified Bessel functions of the zeroth order, of the first and second kind respectively, and $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$ are arbitrary functions of z.

Substituting (3.9) in the Eq. (3.3) we get the velocity component (u_{00}) as

$$u_{00} = \frac{r}{2iR_w^2}\frac{\partial^2 p_{00}}{\partial z^2} - \frac{1}{\alpha_1}[C_1'(z)I_1(\alpha_1 r) + C_2'(z)K_1(\alpha_1 r)] - \frac{1}{\alpha_2}[C_3'(z)I_1(\alpha_2 r) - C_4'(z)K_1(\alpha_2 r)] + \frac{C_5(z)}{r}. \tag{3.10}$$

The zeroth order flux, given by

$$Q_{00} = \int_k^{r_s(z)} r w_{00} dr$$

can be obtained from (3.9) as

$$\begin{aligned} Q_{00} = {} & \frac{C_1(z)}{\alpha_1}[r_s(z)I_1(\alpha_1 r_s(z)) - kI_1(\alpha_1 k)] \\ & + \frac{C_2(z)}{\alpha_1}[r_s(z)K_1(\alpha_1 r_s(z))] - kK_1(\alpha_1 k)] \\ & + \frac{C_3(z)}{\alpha_2}[r_s(z)I_1(\alpha_2 r_s(z)) - kI_1(\alpha_2 k)] \\ & + \frac{C_4(z)}{\alpha_2}[r_s(z)K_1(\alpha_2 r_s(z))] - kK_1(\alpha_2 k)] - \frac{(r_s^2(z) - k^2)}{2iR_w^2}\frac{dp_{00}}{dz}. \end{aligned} \tag{3.11}$$

Solving the system of equations obtained by using boundary conditions (3.6) and (3.7), we get the values $C_1(z)$, $C_2(z)$, $C_3(z)$, $C_4(z)$ and dp_{00}/dz.

3.1. First order in δ (steady streaming solution)

The equations for the first order in δ are

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial w_1}{\partial z} = 0, \tag{3.12}$$

$$\frac{\partial p_1}{\partial r} = 0, \tag{3.13}$$

$$\frac{1}{4}\operatorname{Re}\bigg(\{u_{00}(r,z)e^{it}+\bar{u}_{00}(r,z)e^{-it}\}\frac{\partial}{\partial r}\{w_{00}(r,z)e^{it}+\bar{w}_{00}(r,z)e^{-it}\} + w_{00}(r,z)e^{it}+\bar{w}_{00}(r,z)e^{-it}\}\frac{\partial}{\partial z}\{w_{00}(r,z)e^{it}+\bar{w}_{00}(r,z)e^{-it}\}\bigg) = -\operatorname{Re}\frac{\partial P_1}{\partial z}+\left(\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}\right)-\frac{1}{\alpha^2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 w_1. \tag{3.14}$$

The corresponding boundary conditions are

$$w_1 = u_1 = 0 \quad \text{at } r = r_s(z) \text{ and } r = k, \tag{3.15}$$

$$\left(\frac{\partial^2 w_1}{\partial r^2}-\frac{\sigma}{r}\frac{\partial w_1}{\partial r}\right)=0 \quad \text{at } r = r_s(z) \text{ and } r = k, \tag{3.16}$$

$$Q_1 = 0 \quad \forall z. \tag{3.17}$$

We observe that the solution has both the steady and unsteady parts. Hence, we take

$$w_1 = w_s(r,z) + w_{us}(r,z,t), \tag{3.18}$$

$$u_1 = u_s(r,z) + u_{us}(r,z,t), \tag{3.19}$$

$$p_1 = p_s(r,z) + p_{us}(r,z,t), \tag{3.20}$$

where the suffix s stands for the steady part and us stands for unsteady part.

The corresponding steady state equations are

$$\frac{\partial u_s}{\partial r}+\frac{u_s}{r}+\frac{\partial w_s}{\partial z}=0, \tag{3.21}$$

$$\frac{\partial p_s}{\partial r}=0, \tag{3.22}$$

$$\frac{1}{4}\operatorname{Re}\left[u_{00}\frac{\partial \bar{w}_{00}}{\partial r}+\bar{u}_{00}\frac{\partial w_{00}}{\partial r}+w_{00}\frac{\partial \bar{w}_{00}}{\partial z}+\bar{w}_{00}\frac{\partial w_{00}}{\partial z}\right] = -\operatorname{Re}\frac{\partial P_s}{\partial z}+\left(\frac{\partial^2 w_s}{\partial r^2}+\frac{1}{r}\frac{\partial w_s}{\partial r}\right)-\frac{1}{\alpha^2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 w_s. \tag{3.23}$$

The boundary conditions in terms of w_s and u_s are:

$$w_s = u_s = 0 \quad \text{at } r = r_s(z) \text{ and } r = k, \tag{3.24}$$

$$\left(\frac{\partial^2 w_s}{\partial r^2}-\frac{\sigma}{r}\frac{\partial w_s}{\partial r}\right)=0 \quad \text{at } r = r_s(z) \text{ and } r = k, \tag{3.25}$$

$$Q_s = 0 \quad \text{at } r = r_s(z). \tag{3.26}$$

It can be noted from Eq. (3.22) that p_s is a function of z only. Eq. (3.23) can be simplified to the form

$$D^2(D^2-\alpha^2)w_s = \alpha^2\Big\{-\mathrm{Re}\,\frac{\partial p_s}{\partial z} - \frac{\mathrm{Re}}{4}\Big[u_{00}\frac{\partial \bar{w}_{00}}{\partial r} + \bar{u}_{00}\frac{\partial w_{00}}{\partial r} + w_{00}\frac{\partial \bar{w}_{00}}{\partial z} + \bar{w}_{00}\frac{\partial w_{00}}{\partial z}\Big]\Big\}. \tag{3.27}$$

The general solution of the Eq. (3.27) is

$$w_s = \frac{1}{m^2}[d_1(z)I_0(\alpha r) + d_2(z)K_0(\alpha r)] + \Big[\mathrm{Re}\,\frac{\partial p_s}{\partial z}\frac{r^2}{4} + d_3(z)\log(r)\Big] - \mathrm{Re}\,\frac{\alpha}{4}\int [I_1(\alpha r)V_1(r,z) - K_1(\alpha r)V_2(r,z)]dr + d_4(z), \tag{3.28}$$

where $d_4(z)$ is an arbitrary function of z which is to be determined, and

$$V_1(r,z) = \int_r r\Big[u_{00}\frac{\partial \bar{w}_{00}}{\partial r} + \bar{u}_{00}\frac{\partial w_{00}}{\partial r} + w_{00}\frac{\partial \bar{w}_{00}}{\partial z} + \bar{w}_{00}\frac{\partial w_{00}}{\partial z}\Big]K_1(\alpha r)dr,$$

$$V_2(r,z) = \int_r r\Big[u_{00}\frac{\partial \bar{w}_{00}}{\partial r} + \bar{u}_{00}\frac{\partial w_{00}}{\partial r} + w_{00}\frac{\partial \bar{w}_{00}}{\partial z} + \bar{w}_{00}\frac{\partial w_{00}}{\partial z}\Big]I_1(\alpha r)dr.$$

The flux (Q_s) given by

$$Q_s = \int_k^{r_s(z)} r w_s \, dr$$

can be obtained from (3.28) in the following form:

$$\begin{aligned} Q_s = {} & -\frac{1}{m^3}[d_1(z)\{r_s(z)I_1(\alpha r_s(z)) - kI_1(\alpha k)\} \\ & - d_2(z)\{r_s(z)K_1(\alpha r_s(z)) - kK_1(\alpha k)\}] + \mathrm{Re}\,\frac{\partial p_s}{\partial z}\frac{r_s^4(z)-k^4}{64} \\ & + d_3(z)/4[r_s^2(z)(2\log(r_s(z))-1) - k^2(2\log(k)-1)] \\ & - \int_k^{r_s(z)} r\Big(\int [I_1(\alpha r)V_1(r,z) - K_1(\alpha r)V_2(r,z)]dr\Big)dr \\ & + d_4(z)\frac{r_s^2(z)-k^2}{8}. \end{aligned} \tag{3.29}$$

The arbitrary functions $d_1(z)$, $d_2(z)$, $d_3(z)$ and $d_4(z)$ and the pressure gradient $\partial p_s/\partial z$ can be obtained by using the boundary conditions (3.24) and (3.26) and solving the resulting system of equations.

4. Results and discussion

The numerical calculations have been made by choosing the stenosis geometry $r_s(z)$ as

$$r_s(z) = \begin{cases} 1 - \varepsilon e^{-\pi(z-0.5)^2} & 0 \le z \le 1, \\ 1 & \text{otherwise}, \end{cases} \tag{4.1}$$

where ε is the maximum height of the stenosis.

The numerical values of arbitrary functions $C_1(z)$, $C_2(z)$, $C_3(z)$, $C_4(z)$ and $\partial p_{00}/\partial z$ are obtained solving the system of equations Eq. (3.6) and (3.7) using MATHEMATICA. Similarly, taking the polynomial expressions for the Bessel functions and using MATHEMATICA, the values of $d_1(z)$, $d_2(z)$, $d_3(z)$ and $d_4(z)$ and $\partial p_s/\partial z$ are evaluated numerically by solving the system of equations obtained from the boundary conditions (3.24) to (3.26) for various values of geometric and fluid parameters. As k represents the ratio of radii (catheter size), it varies from 0 to 1. $k = 0$ corresponds to the case when there is no catheter. The effects of the parameters k and σ on the mean pressure drop and mean shear stress, are shown graphically. The parameter σ represents the effect of couple stresses and $\sigma \le 1$. The range for the catheter velocity (w_c) and other parameters is chosen from the works of earlier authors, given in References.

Mean pressure drop

In view of the above analysis, the mean pressure drop, upto $O(\delta)$ analysis, is defined as

$$\Delta p_s(z) = \int_0^z \frac{\partial p_s}{\partial z} dz. \tag{4.2}$$

To obtain Δp_s, the calculated values of $\partial p_s/\partial z$ are integrated numerically using the trapezoidal rule. It may be noted that Δp_s is the correction of the pressure drop value corresponding to the $O(\delta)$ analysis.

The variation of pressure drop (Δp_s) for different values of the parameters is shown in Figures 2 to 4. Fig. 2 shows the effect of the catheter size k with fixed $\mathrm{Re} = 2$, $\mathrm{Rw} = 5$, $\sigma = 0.5$, $\delta = 0.5$, $w_c = 0$. It is observed that as the size of the catheter increases, pressure drop also increases. Fig. 3 depicts the effect of σ for the fixed values of $\mathrm{Re} = 2$, $\mathrm{Rw} = 5$, $\delta = 0.5$, $w_c = 0$. As the parameter σ increases, pressure drop decreases. It is important to note that the drop is much higher when compared to the Newtonian fluid as predicted by Jayaraman and Sarkar. In Fig. 4, the effect of the catheter movement on the pressure drop is presented. It is seen that as the velocity of the catheter increases, pressure drop also slightly increases.

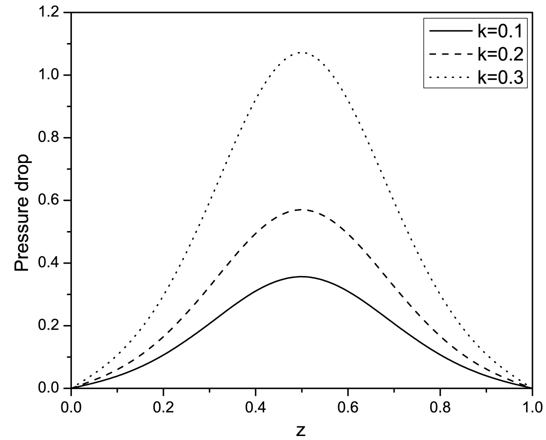

Fig. 2. Effect of k on pressure drop for $\mathrm{Re} = 2$, $\mathrm{Rw}^2 = 5$, $\sigma = 0.5$, $\delta = 0.5$, $w_c = 0$.

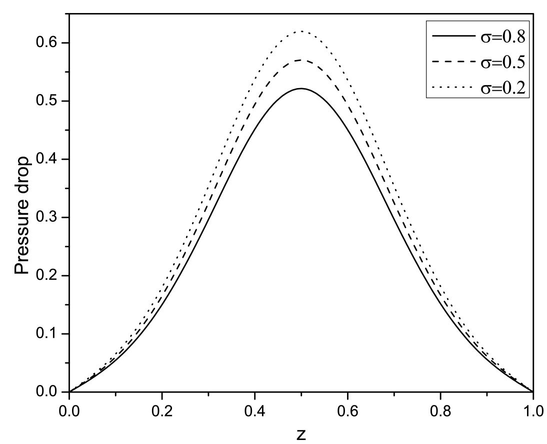

Fig. 3. Effect of σ on pressure drop with $k = 0.2$, $\mathrm{Re} = 2$, $\mathrm{Rw}^2 = 5$, $\delta = 0.5$, $w_c = 0$.

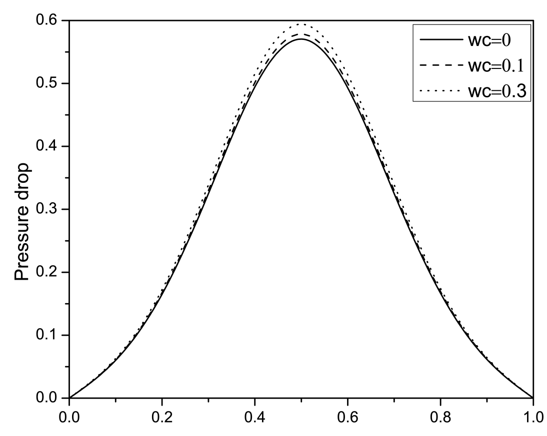

Fig. 4. Effect of w_c on pressure drop with $k = 0.2$, $\mathrm{Re} = 2$, $\mathrm{Rw}^2 = 5$, $\delta = 0.5$, $\sigma = 0.5$.

Mean wall shear stress

The mean wall shear stress up to $O(\delta)$ analysis is

$$(4.3)\qquad T_{rz} = \frac{\partial w_s}{\partial r} - \frac{1}{4\alpha^2}\frac{\partial}{\partial r}\left(\frac{\partial^2 w_s}{\partial r^2}\right) + \frac{1}{4\alpha^2 r^2}\frac{\partial w_s}{\partial r} - \frac{1}{4r\alpha^2}\frac{\partial^2 w_s}{\partial r^2}.$$

Figures 5 to 7 depict the variation of shear stress at the maximum height of the stenosis for various values of the fluid parameters, catheter size and the velocity of the catheter respectively. In Fig. 5 the effect of k on the stress for fixed

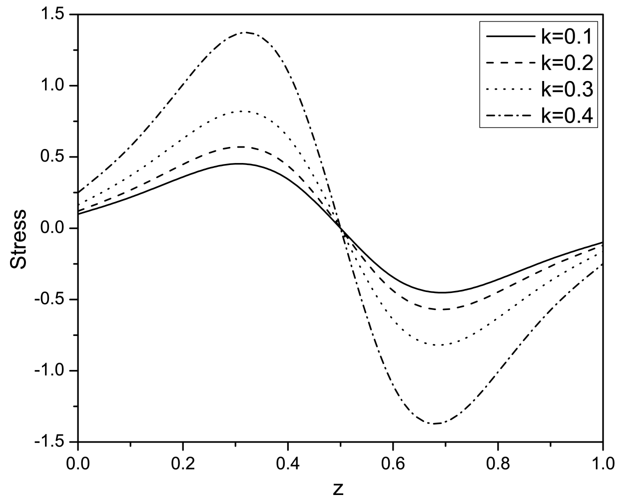

Fig. 5. Effect of k on shear stress for Re = 2, $\mathrm{Rw}^2 = 5$, $\sigma = 0.5$, $\delta = 0.5$, $w_c = 0$.

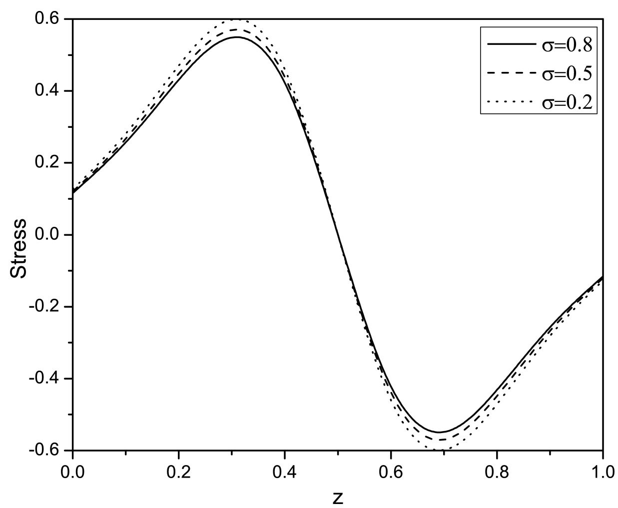

Fig. 6. Effect of σ on shear stress for $k = 0.2$, Re = 2, $\mathrm{Rw}^2 = 5$, $\delta = 0.5$, $w_c = 0$.

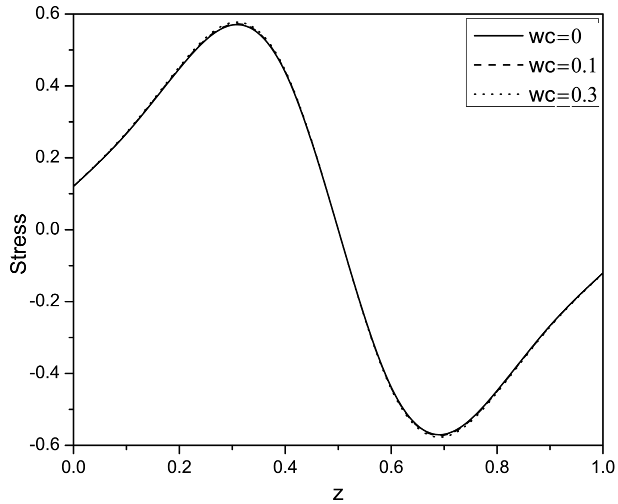

Fig. 7. Effect of w_c on shear stress with $k = 0.2$, $\mathrm{Re} = 2$, $\mathrm{Rw}^2 = 5$, $\delta = 0.5$, $\sigma = 0.5$.

values of $\mathrm{Re} = 2$, $\mathrm{Rw} = 5$, $\sigma = 0.5$, $\delta = 0.5$, $w_c = 0$ is presented. It is observed that as k increases, stress also increases. Fig. 6 explains the effect of σ on the stress. It is interesting to note that the contribution of $O(\delta)$ terms is significant for the stress and it decreases as the parameter value increases from 0.2 to 0.8, which is in agreement with that of the Newtonian case. Fig. 7 depicts the effect of the flexible nature of the catheter. It is observed that as the movement of the catheter increases, the stress also increases. However, the effect is not significant.

5. Conclusions

The contribution of the steady streaming effect on the flow of couple stress fluid, due to the oscillatory nature of the flow is studied. This configuration is intended as a simple model for studying blood flow in a stenosed artery when a catheter is inserted into it. The mean pressure drop and the mean wall shear stress, are calculated across the constricted region. The effects of the couple stress fluid parameters, size of the catheter and the velocity of the catheter on these mean quantities are discussed. It is observed that as the size of the catheter and the velocity of catheter increases, the pressure drop as well as the shear stress increases, while they decrease when the couple stress fluid parameter σ increases.

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