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Compliance minimization of thin plates made of material with predefined Kelvin moduli. Part II. The effective boundary value problem and exemplary solutions

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The compliance minimization of transversely homogeneous plates with predefined Kelvin moduli leads to the equilibrium problem of an effective hyperelastic plate with the hyperelastic potential expressed explicitly in terms of both the membrane and bending strain measures, as derived in Part I of the present paper. The aim of this second part of the paper is to show convexity of this potential and, consequently, uniqueness of solutions of the minimum compliance problem considered. Theoretical results are illustrated by numerically calculated optimal trajectories of the eigenstate corresponding to the largest Kelvin modulus.

Key words: free material optimization, compliance minimization, anisotropic plates.

1. Introduction

The free material optimization problem put forward in the first part of the present paper; namely *minimize the compliance of a thin transversely homogeneous plate of predefined Kelvin moduli of the elasticity tensor*, means the optimal distribution of the eigenstates of the elasticity tensor to make the plate as stiff as possible. The problem has been reduced to the equilibrium problem of an effective plate of specific hyperelastic properties. While the initial plate equilibrium problem is decoupled (the in-plane and bending problems can be solved independently), the optimization process couples these deformations. The effective hyperelastic constitutive equations (3.59) in Dzierżanowski and Lewiński [3] link the stress $\mathbf{N}$ and couple resultants $\mathbf{M}$ with the in-plane $\boldsymbol{\varepsilon}$ and flexural $\boldsymbol{\varkappa}$ strain measures. We shall prove that the underlying hyperelastic potential is a homogeneous function of degree 2, bounded from both sides and strictly convex. The latter property is crucial but fairly difficult to obtain. Its proof is based on discussing of the positive definiteness of the Hessian.

The results of the first part of the present paper were based on the assumption of possible interchanging the „min" and „max" operations in (I.2.19). The correctness of this switching is substantiated in Sec. 3.4 below by utilizing the duality relations proved in CZARNECKI and LEWIŃSKI [2], thus making the present two-part paper complete.

According to the theorem by MINTY [5], the relevant constitutive equations are monotone and this property implies the uniqueness of a solution. However, the problem of its existence, linked with the regularity assumptions, will be discussed elsewhere.

Theoretical considerations are illustrated by the examples of trajectories of the second-order symmetric tensor field $\boldsymbol{\omega}_1(x)$ corresponding to the greatest value of Kelvin modulus in the optimal constitutive tensor field $\mathbf{A}(x)$, related to the given strain fields obtained with the help of the Airy stress functions in two-dimensional (plane) elasticity and the Navier and Levy's infinite series, representing the deflection function in the theory of Kirchhoff plates. Consequently, components of the in-plane strain tensor and the curvature tensor of a plate in bending are calculated analytically and this step is followed by combining both loading cases in the membrane-bending (M-B) problem. This in turn allows for setting the variable coefficients of two ordinary differential equations whose solutions determine the families of orthogonal trajectories corresponding to the eigenvalues of $\boldsymbol{\omega}_1(x)$.

2. The effective hyperelastic problem (P^*)

The problem of minimization of the plate compliance over possible eigenstates of the elasticity tensor $\mathbf{A}$, which determines both in-plane and bending stiffnesses, reduces to problem (P^*) or (I.2.22)–(I.2.24); the Roman numeral I refers to the Part I of the present paper, or to DZIERŻANOWSKI and LEWIŃSKI [3]. The effective potential (I.2.23) has been reduced to the form (I.3.13), while the constitutive equations have the form (I.3.59).

Assume that vectors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ are not co-linear and, for the purpose of this section only, set a basis (B.1), see Appendix B. Next, by making use of (I.3.50), (I.3.52) and (I.3.54), rewrite (I.3.59) in the form

$$
\begin{aligned}
\mathbf{N} &= \frac{1}{2}(\lambda_1 + \lambda_2)\left[(1 + \nu\,\phi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}))\,\boldsymbol{\varepsilon} + \nu\,\psi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})\,\boldsymbol{\kappa}\right], \\
\mathbf{K} &= \frac{1}{2}(\lambda_1 + \lambda_2)\left[\nu\,\psi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})\,\boldsymbol{\varepsilon} + (1 - \nu\,\phi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}))\,\boldsymbol{\kappa}\right].
\end{aligned} \tag{2.1}
$$

Hence, contravariant representations of $\mathbf{N}$ and $\mathbf{K}$ in basis (B.1) are given by vectors

$$\begin{bmatrix} N^1 \\ N^2 \\ N^3 \end{bmatrix} = \frac{1}{2}(\lambda_1+\lambda_2)\begin{bmatrix} 1+\nu\,\phi \\ \nu\,\psi \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} K^1 \\ K^2 \\ K^3 \end{bmatrix} = \frac{1}{2}(\lambda_1+\lambda_2)\begin{bmatrix} \nu\,\psi \\ 1-\nu\,\phi \\ 0 \end{bmatrix}. \tag{2.2}$$

Next, apply (I.3.50) in the representation of the constitutive tensor $\mathbf{A}$ in terms of G, ϕ and ψ. Rewriting (2.1) one obtains

$$\mathbf{N} = \mathbf{A}\,\boldsymbol{\varepsilon}, \qquad \mathbf{K} = \mathbf{A}\,\boldsymbol{\kappa}, \tag{2.3}$$

since it is clear that tensor $\mathbf{A}$ links both equations by $\mathbf{D} = (h^2/12)\mathbf{A}$, see (I.2.2) and (I.2.3).

Formulae (2.3) can be rearranged in a form

$$\mathbf{N} = (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\boldsymbol{\varepsilon}, \qquad \mathbf{K} = (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\boldsymbol{\kappa}, \tag{2.4}$$

thus leading to the mixed representation of tensor $\mathbf{A}$ given by

$$[A^i{}_j] = \begin{bmatrix} \frac{1}{2}(\lambda_1+\lambda_2)(1+\nu\,\phi) & \frac{1}{2}(\lambda_1+\lambda_2)\nu\,\psi & 0 \\ \frac{1}{2}(\lambda_1+\lambda_2)\nu\,\psi & \frac{1}{2}(\lambda_1+\lambda_2)(1-\nu\,\phi) & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{2.5}$$

Indeed, by introducing the basis and co-basis vectors (B.1) and (B.6) in (2.3) one obtains

$$\begin{aligned} \mathbf{A}\,\boldsymbol{\varepsilon} &= (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\boldsymbol{\varepsilon} = (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\mathbf{e}_1 = A^i{}_j\,\delta^j{}_1\,\mathbf{e}_i \\ &= A^i{}_1\,\mathbf{e}_i = A^1{}_1\,\boldsymbol{\varepsilon} + A^2{}_1\,\boldsymbol{\kappa}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathbf{A}\,\boldsymbol{\kappa} &= (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\boldsymbol{\kappa} = (A^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^j)\,\mathbf{e}_2 = A^i{}_j\,\delta^j{}_2\,\mathbf{e}_i \\ &= A^i{}_2\,\mathbf{e}_i = A^1{}_2\,\boldsymbol{\varepsilon} + A^2{}_2\,\boldsymbol{\kappa}. \end{aligned} \tag{2.7}$$

Comparing these formulae with those in (2.1) and taking into consideration that $\boldsymbol{\omega}_3$ is perpendicular to the plane spanned by $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, finally yields (2.5).

Upon solving the problem (I.2.22)–(I.2.24), with potential (I.3.13), one can find optimal eigentensors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ of the constitutive tensor $\mathbf{A}$ by using the Eqs. (I.2.11) and (I.3.45), (I.3.46). Let us note that the eigentensors of $\mathbf{A}$ can be calculated as, see [8],

$$\begin{aligned} \mathbf{P}_1 &= \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\,(\mathbf{A}-\lambda_2\mathbf{E})\,(\mathbf{A}-\lambda_3\mathbf{E})\,, \\ \mathbf{P}_2 &= \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\,(\mathbf{A}-\lambda_1\mathbf{E})\,(\mathbf{A}-\lambda_3\mathbf{E})\,, \\ \mathbf{P}_3 &= \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,(\mathbf{A}-\lambda_1\mathbf{E})\,(\mathbf{A}-\lambda_2\mathbf{E})\,, \end{aligned} \tag{2.8}$$

where $\mathbf{E}$ represents the metric tensor, see Appendix B, or explicitly in the the basis $\mathbf{e}_j \otimes \mathbf{e}^k$

$$[(\mathbf{P}_1)^j{}_k] = \begin{bmatrix} \frac{1}{2}(1+\phi) & \frac{1}{2}\psi & 0 \\ \frac{1}{2}\psi & \frac{1}{2}(1-\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$[(\mathbf{P}_2)^j{}_k] = \begin{bmatrix} \frac{1}{2}(1-\phi) & -\frac{1}{2}\psi & 0 \\ -\frac{1}{2}\psi & \frac{1}{2}(1+\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.10}$$

and

$$[(\mathbf{P}_3)^j{}_k] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.11}$$

It is a simple matter to check that $\|\mathbf{P}_i\| = 1,\ i = 1,2,3.$

The projectors $\mathbf{P}_\alpha$, $\alpha = 1,2$, can be represented by

$$\mathbf{P}_\alpha = \overset{\alpha}{\xi}_1 \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} + \overset{\alpha}{\xi}_3 (\boldsymbol{\varepsilon} \otimes \boldsymbol{\kappa} + \boldsymbol{\kappa} \otimes \boldsymbol{\varepsilon}) + \overset{\alpha}{\xi}_2 \boldsymbol{\kappa} \otimes \boldsymbol{\kappa} \tag{2.12}$$

where, for $\alpha = 1$,

$$\overset{1}{\xi}_1 = \left(\frac{\gamma_1}{\|\boldsymbol{\varepsilon}\|}\right)^2, \qquad \overset{1}{\xi}_2 = \left(\frac{\gamma_2}{\|\boldsymbol{\kappa}\|}\right)^2, \qquad \overset{1}{\xi}_3 = \sqrt{\overset{1}{\xi}_1 \ \overset{1}{\xi}_2} \tag{2.13}$$

and γ_1, γ_2, are given by (I.3.40), or, alternatively,

$$\begin{aligned} (\gamma_1)^2 &= \frac{1}{2\sin^2\alpha}\left(1 + \phi - \psi \frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|^2}\right), \\ (\gamma_2)^2 &= \frac{1}{2\sin^2\alpha}\left(1 - \phi - \psi \frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2}\right) \end{aligned} \tag{2.14}$$

with

$$\sin^2\alpha = 1 - \left(\frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|\,\|\boldsymbol{\kappa}\|}\right)^2. \tag{2.15}$$

For $\alpha = 2$, the coefficients $\overset{2}{\xi}_j$ are given by (I.2.10) with γ_α replaced by δ_α.

If referred to the basis in $\mathbb{R}^3$ in the form (I.2.7), the tensors appearing in (2.12) are represented by the matrices

$$\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} = \begin{bmatrix} (\varepsilon_{11})^2 & \varepsilon_{11}\varepsilon_{22} & \sqrt{2}\varepsilon_{11}\varepsilon_{12} \\ \varepsilon_{22}\varepsilon_{11} & (\varepsilon_{22})^2 & \sqrt{2}\varepsilon_{22}\varepsilon_{12} \\ \sqrt{2}\varepsilon_{12}\varepsilon_{11} & \sqrt{2}\varepsilon_{12}\varepsilon_{22} & 2\,(\varepsilon_{12})^2 \end{bmatrix}, \tag{2.16}$$

$$\boldsymbol{\varepsilon} \otimes \boldsymbol{\kappa} = \begin{bmatrix} \varepsilon_{11}\kappa_{11} & \varepsilon_{11}\kappa_{22} & \sqrt{2}\varepsilon_{11}\kappa_{12} \\ \varepsilon_{22}\kappa_{11} & \varepsilon_{22}\kappa_{22} & \sqrt{2}\varepsilon_{22}\kappa_{12} \\ \sqrt{2}\varepsilon_{12}\kappa_{11} & \sqrt{2}\varepsilon_{12}\kappa_{22} & 2\,\varepsilon_{12}\kappa_{12} \end{bmatrix}, \tag{2.17}$$

$$\boldsymbol{\kappa} \otimes \boldsymbol{\kappa} = \begin{bmatrix} (\kappa_{11})^2 & \kappa_{11}\kappa_{22} & \sqrt{2}\kappa_{11}\kappa_{12} \\ \kappa_{22}\kappa_{11} & (\kappa_{22})^2 & \sqrt{2}\kappa_{22}\kappa_{12} \\ \sqrt{2}\kappa_{12}\kappa_{11} & \sqrt{2}\kappa_{12}\kappa_{22} & 2\,(\kappa_{12})^2 \end{bmatrix}, \tag{2.18}$$

with $\boldsymbol{\varepsilon} \otimes \boldsymbol{\kappa} = (\boldsymbol{\kappa} \otimes \boldsymbol{\varepsilon})^T$.

Let us now write the incremental form of the constitutive equations (2.1)

$$\begin{aligned} \Delta\mathbf{N} &= \frac{\partial\mathbf{N}}{\partial\boldsymbol{\varepsilon}}\Delta\boldsymbol{\varepsilon} + \frac{\partial\mathbf{N}}{\partial\boldsymbol{\kappa}}\Delta\boldsymbol{\kappa}, \\ \Delta\mathbf{K} &= \frac{\partial\mathbf{K}}{\partial\boldsymbol{\varepsilon}}\Delta\boldsymbol{\varepsilon} + \frac{\partial\mathbf{K}}{\partial\boldsymbol{\kappa}}\Delta\boldsymbol{\kappa}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \frac{\partial\mathbf{N}}{\partial\boldsymbol{\varepsilon}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\varepsilon} + \nu\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\kappa} + (1+\nu\,\phi)\,\mathbf{I}_4\right], \\ \frac{\partial\mathbf{N}}{\partial\boldsymbol{\kappa}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\phi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\varepsilon} + \nu\,\frac{\partial\psi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\kappa} + \nu\,\psi\,\mathbf{I}_4\right], \\ \frac{\partial\mathbf{K}}{\partial\boldsymbol{\varepsilon}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\varepsilon} - \nu\,\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\otimes\boldsymbol{\kappa} + \nu\,\psi\,\mathbf{I}_4\right], \\ \frac{\partial\mathbf{K}}{\partial\boldsymbol{\kappa}} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\nu\,\frac{\partial\psi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\varepsilon} - \nu\,\frac{\partial\phi}{\partial\boldsymbol{\kappa}}\otimes\boldsymbol{\kappa} + (1-\nu\,\phi)\,\mathbf{I}_4\right]. \end{aligned} \tag{2.20}$$

Here $\mathbf{I}_4$ stands for the unit tensor in $\mathbb{E}_s^4$ and

$$\begin{aligned} \frac{\partial\phi}{\partial\boldsymbol{\varepsilon}} &= 2\frac{\psi}{G}\left(\psi\,\boldsymbol{\varepsilon} - \phi\,\boldsymbol{\kappa}\right), & \frac{\partial\phi}{\partial\boldsymbol{\kappa}} &= -2\frac{\psi}{G}\left(\phi\,\boldsymbol{\varepsilon} + \psi\,\boldsymbol{\kappa}\right), \\ \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} &= -2\frac{\phi}{G}\left(\psi\,\boldsymbol{\varepsilon} - \phi\,\boldsymbol{\kappa}\right), & \frac{\partial\psi}{\partial\boldsymbol{\kappa}} &= 2\frac{\phi}{G}\left(\phi\,\boldsymbol{\varepsilon} + \psi\,\boldsymbol{\kappa}\right). \end{aligned} \tag{2.21}$$

3. Properties of (P^*) problem

3.1. Strict convexity of W_λ

The aim of the present section is to show the strict convexity of potential W_λ which was defined in (I.3.3), (I.3.4) and expressed explicitly in (I.3.13).

Note first that W_λ is a homogeneous function of degree 2, or

$$W_\lambda(\alpha\boldsymbol{\varepsilon}, \alpha\boldsymbol{\varkappa}) = \alpha^2 W_\lambda(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) \tag{3.1}$$

for $\alpha \geq 0$. According to [7], the pointwise supremum of an arbitrary collection of convex functions is also convex, but this theorem cannot be applied to (I.3.5), because tensors $\boldsymbol{\omega}_i$ are restricted by orthogonality conditions. It folows that convexity of W_λ cannot be inferred from (I.3.23), it must be deduced by examining the specific properties of the explicit expression (I.3.13) instead.

It is sufficient to prove convexity of $\tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ in (I.3.52) with respect to both arguments. The proof of convexity consists in checking positive semidefiniteness of the Hessian matrix, see [7],

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}^{11} & \mathbf{F}^{12} \\ \mathbf{F}^{21} & \mathbf{F}^{22} \end{bmatrix} \tag{3.2}$$

with

$$\begin{aligned} \mathbf{F}^{11} &= \frac{\partial^2 \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}, \qquad \mathbf{F}^{12} = \frac{\partial^2 \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\kappa}}, \\ \mathbf{F}^{21} &= \frac{\partial^2 \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\kappa} \otimes \partial \boldsymbol{\varepsilon}}, \qquad \mathbf{F}^{22} = \frac{\partial^2 \tilde{U}^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})}{\partial \boldsymbol{\kappa} \otimes \partial \boldsymbol{\kappa}}. \end{aligned} \tag{3.3}$$

In the sequel it is proved that (3.2) is positive definite, which implies strict convexity of $\tilde{U}^*$. From (I.3.58), (2.19), (2.20) and (2.21) one may deduce that

$$\mathbf{F}^{11} = 2\left[(1+\nu\phi)\mathbf{I}_4 + \frac{2\,\nu}{G}(\psi\boldsymbol{\varepsilon} - \phi\boldsymbol{\kappa}) \otimes (\psi\boldsymbol{\varepsilon} - \phi\boldsymbol{\kappa})\right], \tag{3.4}$$

$$\mathbf{F}^{12} = 2\left[\nu\psi\mathbf{I}_4 - \frac{2\,\nu}{G}(\psi\boldsymbol{\varepsilon} - \phi\boldsymbol{\kappa}) \otimes (\phi\boldsymbol{\varepsilon} + \psi\boldsymbol{\kappa})\right], \tag{3.5}$$

$$\mathbf{F}^{22} = 2\left[(1-\nu\phi)\mathbf{I}_4 + \frac{2\,\nu}{G}(\phi\boldsymbol{\varepsilon} + \psi\boldsymbol{\kappa}) \otimes (\phi\boldsymbol{\varepsilon} + \psi\boldsymbol{\kappa})\right], \tag{3.6}$$

where $\mathbf{I}_4$ is the unit tensor in $\mathbb{E}^4_s$, $G = G(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$, see (I.3.50), and $\mathbf{F}^{21} = (\mathbf{F}^{12})^{\mathrm{T}}$.

The task is now to prove that the following quadratic form:

$$2X(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot (\mathbf{F}^{11}\mathbf{a}) + \mathbf{a} \cdot (\mathbf{F}^{12}\mathbf{b}) + \mathbf{b} \cdot (\mathbf{F}^{21}\mathbf{a}) + \mathbf{b} \cdot (\mathbf{F}^{22}\mathbf{b}) \tag{3.7}$$

is positive for arbitrary $\mathbf{a}$, $\mathbf{b} \in \mathbb{E}^2_s$ provided that $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \neq 0$.

The first two terms on the r.h.s. of (3.7) can be expressed as

$$\frac{1}{2}\mathbf{a}\cdot(\mathbf{F}^{11}\mathbf{a}) = \|\mathbf{a}\|^2 + \frac{\nu}{G}\left[\left(\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2\right)\|\mathbf{a}\|^2 + 2(\mathbf{a}\cdot\boldsymbol{\varepsilon})^2 + 2(\mathbf{a}\cdot\boldsymbol{\kappa})^2\right] - \frac{2\nu}{G^3}\left[\left(\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2\right)(\mathbf{a}\cdot\boldsymbol{\varepsilon}) + 2(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})(\mathbf{a}\cdot\boldsymbol{\kappa})\right]^2, \tag{3.8}$$

$$\frac{1}{2}\mathbf{a}\cdot(\mathbf{F}^{12}\mathbf{b}) = \frac{2\nu}{G}\Big\{\left[(\mathbf{a}\cdot\boldsymbol{\kappa})(\mathbf{b}\cdot\boldsymbol{\varepsilon}) - (\mathbf{a}\cdot\boldsymbol{\varepsilon})(\mathbf{b}\cdot\boldsymbol{\kappa}) + (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})(\mathbf{a}\cdot\mathbf{b})\right] + \frac{1}{G^2}\left[\left(\|\boldsymbol{\varepsilon}\|^2 - \|\boldsymbol{\kappa}\|^2\right)^2(\mathbf{b}\cdot\boldsymbol{\kappa})(\mathbf{a}\cdot\boldsymbol{\varepsilon}) - 4(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2(\mathbf{b}\cdot\boldsymbol{\varepsilon})(\mathbf{a}\cdot\boldsymbol{\kappa})\right] + \frac{2}{G^2}\left(\|\boldsymbol{\kappa}\|^2 - \|\boldsymbol{\varepsilon}\|^2\right)(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})\left[(\mathbf{a}\cdot\boldsymbol{\varepsilon})(\mathbf{b}\cdot\boldsymbol{\varepsilon}) - (\mathbf{a}\cdot\boldsymbol{\kappa})(\mathbf{b}\cdot\boldsymbol{\kappa})\right]\Big\}. \tag{3.9}$$

The third term in (3.7) is exactly the same as (3.9), while the fourth one is similar to (3.8), the only difference results in replacing $\mathbf{a}$ with $\mathbf{b}$, $\boldsymbol{\varepsilon}$ with $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}$ with $\boldsymbol{\varepsilon}$.

Substituting all terms in (3.7) one finds

$$X(\mathbf{a},\mathbf{b}) = X_0(\mathbf{a},\mathbf{b}) + \Delta(\mathbf{a},\mathbf{b}), \tag{3.10}$$

where

$$X_0(\mathbf{a},\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \nu\left[\phi(\boldsymbol{\varepsilon},\boldsymbol{\kappa})\left(\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2\right) + 2\psi(\boldsymbol{\varepsilon},\boldsymbol{\kappa})(\mathbf{a}\cdot\mathbf{b})\right] \tag{3.11}$$

and

$$\Delta(\mathbf{a},\mathbf{b}) = 2\nu\, G(\boldsymbol{\varepsilon},\boldsymbol{\kappa})\ \left[\phi(\boldsymbol{\varepsilon},\boldsymbol{\kappa})(\mathbf{a}\cdot\boldsymbol{\kappa} + \mathbf{b}\cdot\boldsymbol{\varepsilon}) - \psi(\boldsymbol{\varepsilon},\boldsymbol{\kappa})(\mathbf{a}\cdot\boldsymbol{\varepsilon} - \mathbf{b}\cdot\boldsymbol{\kappa})\right]^2. \tag{3.12}$$

By virtue of $\nu > 0$, one can estimate

$$X(\mathbf{a},\mathbf{b}) \geq X_0(\mathbf{a},\mathbf{b}) \tag{3.13}$$

and the assumption $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \neq 0$ leads to

$$X(\mathbf{a},\mathbf{b}) \geq \left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right) X_1(\mathbf{a},\mathbf{b}) \tag{3.14}$$

with

$$X_1(\mathbf{a},\mathbf{b}) = 1 + \nu X_2(\mathbf{a},\mathbf{b}) \tag{3.15}$$

and

$$X_2(\mathbf{a},\mathbf{b}) = \frac{\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}\,\phi(\boldsymbol{\varepsilon},\boldsymbol{\kappa}) + \frac{2(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}\,\psi(\boldsymbol{\varepsilon},\boldsymbol{\kappa}). \tag{3.16}$$

It remains to prove that

$$X_1(\mathbf{a},\mathbf{b}) \geq 1 - \nu. \tag{3.17}$$

To this end, set the representations of $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ from $\mathbb{E}_s^2$ similarly to (I.2.8); express $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ as

$$\|\mathbf{a}\| = R\cos\vartheta, \qquad \|\mathbf{b}\| = R\sin\vartheta, \tag{3.18}$$

and let $\delta = \sphericalangle(\mathbf{a}, \mathbf{b})$ or

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\delta = R^2\cos\vartheta\sin\vartheta\cos\delta. \tag{3.19}$$

Next, by making use of the fact that $\phi^2 + \psi^2 = 1$, see (I.3.50), express ϕ and ψ in terms of a certain angle

$$\phi = \cos\varphi, \qquad \psi = \sin\varphi \tag{3.20}$$

and rewrite X_2 in terms of ϑ and φ thus obtaining

$$X_2 = \cos 2\vartheta\cos\varphi + \sin 2\vartheta\sin\varphi\cos\delta. \tag{3.21}$$

It is easily seen that

$$\min\{X_2 \mid \delta\in\mathbb{R}\} = \min\{\cos(2\vartheta-\varphi), \cos(2\vartheta+\varphi)\}, \tag{3.22}$$

hence,

$$\min\{X_2 \mid \delta\in\mathbb{R}, \quad \vartheta\in\mathbb{R}\} = -1 \tag{3.23}$$

and the inequality $X_2 \geq -1$ implies (3.17).

Estimates $1-\nu > 0$ and (3.14) prove that $X(\mathbf{a}, \mathbf{b}) > 0$ if $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \neq 0$, thus completing the proof of strict convexity of the function (I.3.52) with respect to both arguments.

It is worth pointing out that U_λ^* is bounded, i.e.

$$\lambda_3\left(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2\right) \leq 4U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \leq \lambda_1\left(\|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\kappa}\|^2\right) \tag{3.24}$$

which follows from (I.3.5). To this end, one has to recall the estimates: $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and make use of (I.3.6).

3.2. Properties of the constitutive equations

Since W_λ is a homogeneous function of degree 2, see (3.1), the Euler theorem applies and makes it possible to express this potential in terms of strains and stress resultants by the equation

$$2\cdot(2U_\lambda^*)(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \mathbf{N}\cdot\boldsymbol{\varepsilon} + \mathbf{K}\cdot\boldsymbol{\kappa}. \tag{3.25}$$

This property can be checked by using (I.3.59). Indeed, by computing the scalar products

$$\begin{aligned} \mathbf{N}\cdot\boldsymbol{\varepsilon} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\|\boldsymbol{\varepsilon}\|^2+\nu\mathbf{L}(\boldsymbol{\varepsilon},\boldsymbol{\kappa})\cdot\boldsymbol{\varepsilon}\right], \\ \mathbf{K}\cdot\boldsymbol{\kappa} &= \frac{1}{2}(\lambda_1+\lambda_2)\left[\|\boldsymbol{\kappa}\|^2+\nu\mathbf{L}(\boldsymbol{\kappa},\boldsymbol{\varepsilon})\cdot\boldsymbol{\kappa}\right] \end{aligned} \tag{3.26}$$

and making use of (I.3.57), one may note that

$$\mathbf{L}(\boldsymbol{\varepsilon},\boldsymbol{\kappa})\cdot\boldsymbol{\varepsilon}+\mathbf{L}(\boldsymbol{\kappa},\boldsymbol{\varepsilon})\cdot\boldsymbol{\kappa} = G(\boldsymbol{\varepsilon},\boldsymbol{\kappa}) \tag{3.27}$$

and this equality confirms (3.25). Since U^*_λ is strictly convex, the constitutive equations are strictly monotone, or

$$\left[\mathbf{N}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})-\mathbf{N}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\right]\cdot(\overset{1}{\boldsymbol{\varepsilon}}-\overset{2}{\boldsymbol{\varepsilon}})+\left[\mathbf{K}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})-\mathbf{K}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\right]\cdot(\overset{1}{\boldsymbol{\kappa}}-\overset{2}{\boldsymbol{\kappa}}) \geq 0 \tag{3.28}$$

and the equality holds if and only if $\overset{1}{\boldsymbol{\varepsilon}}=\overset{2}{\boldsymbol{\varepsilon}}$, $\overset{1}{\boldsymbol{\kappa}}=\overset{2}{\boldsymbol{\kappa}}$.

This property is due to Minty [5], see also Ekeland and Temam [4]. It implies the uniqueness of solutions to the problem (P^*) or (I.2.24). Indeed, assume that two pairs $(\overset{1}{\mathbf{u}},\overset{1}{w})$, $(\overset{2}{\mathbf{u}},\overset{2}{w})$ satisfy (I.2.6)

$$\begin{aligned} &\int_\Omega \left[\mathbf{N}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})\cdot\boldsymbol{\varepsilon}(\mathbf{v})+\mathbf{K}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})\cdot\boldsymbol{\kappa}(v)\right]dx = f(\mathbf{v},v), \\ &\int_\Omega \left[\mathbf{N}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\cdot\boldsymbol{\varepsilon}(\mathbf{v})+\mathbf{K}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\cdot\boldsymbol{\kappa}(v)\right]dx = f(\mathbf{v},v) \end{aligned} \tag{3.29}$$

where $\overset{\alpha}{\boldsymbol{\varepsilon}}=\boldsymbol{\varepsilon}(\overset{\alpha}{\mathbf{u}})$, $\overset{\alpha}{\boldsymbol{\kappa}}=\boldsymbol{\kappa}(\overset{\alpha}{\mathbf{w}})$. Suppose that $(\mathbf{v},v)$ are common for both equations and subtract them to find

$$\begin{aligned} &\int_\Omega \left[\left(\mathbf{N}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})-\mathbf{N}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\right)\cdot\boldsymbol{\varepsilon}(\mathbf{v})\right]dx \\ &\qquad + \int_\Omega \left[\left(\mathbf{K}(\overset{1}{\boldsymbol{\varepsilon}},\overset{1}{\boldsymbol{\kappa}})-\mathbf{K}(\overset{2}{\boldsymbol{\varepsilon}},\overset{2}{\boldsymbol{\kappa}})\right)\cdot\boldsymbol{\kappa}(v)\right]dx = 0. \end{aligned} \tag{3.30}$$

Next, choose $\mathbf{v}=\overset{1}{\mathbf{u}}-\overset{2}{\mathbf{u}}$, $v=\overset{1}{w}-\overset{2}{w}$. Then

$$\boldsymbol{\varepsilon}(\mathbf{v})=\overset{1}{\boldsymbol{\varepsilon}}-\overset{2}{\boldsymbol{\varepsilon}}, \qquad \boldsymbol{\kappa}(v)=\overset{1}{\boldsymbol{\kappa}}-\overset{2}{\boldsymbol{\kappa}}. \tag{3.31}$$

By the strict monotonicity, see (3.28), the equality (3.30) can only be fulfilled if $\overset{1}{\boldsymbol{\varepsilon}} = \overset{2}{\boldsymbol{\varepsilon}}$ and $\overset{1}{\boldsymbol{\kappa}} = \overset{2}{\boldsymbol{\kappa}}$ which means that $(\overset{1}{\mathbf{u}}, \overset{1}{w})$, $(\overset{2}{\mathbf{u}}, \overset{2}{w})$ can differ only in the terms describing a rigid body motion (with infinitesimal rotations). If both $(\overset{\alpha}{\mathbf{u}}, \overset{\alpha}{w})$ satisfy appropriate kinematic boundary conditions, these terms vanish leading to the identities: $\overset{1}{\mathbf{u}} = \overset{2}{\mathbf{u}}$, $\overset{1}{w} = \overset{2}{w}$. This proves the uniqueness of solution to the problem (P^*), however the problem of its existence is not dealt with in the present paper.

3.3. The variational formulation of (P^*)

Note that

$$\mathbf{N} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \mathbf{M} \cdot \boldsymbol{\varkappa}(v) = \mathbf{N} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \mathbf{K} \cdot \boldsymbol{\kappa}(v). \tag{3.32}$$

Substitution of (I.3.59) into (I.2.6) gives

$$\begin{aligned} \frac{1}{2}\int_{\Omega} (\lambda_1 + \lambda_2) \left[\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \boldsymbol{\kappa}(w) \cdot \boldsymbol{\kappa}(v)\right] dx \\ + \frac{1}{2}\int_{\Omega} (\lambda_1 - \lambda_2) \left[\mathbf{L}\left(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\kappa}(w)\right) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \mathbf{L}\left(\boldsymbol{\kappa}(w), \boldsymbol{\varepsilon}(\mathbf{u})\right) \cdot \boldsymbol{\kappa}(v)\right] dx \\ = f(\mathbf{v}, v) \quad \forall\ (\mathbf{v}, v) \in V. \end{aligned} \tag{3.33}$$

This will be called a variational formulation of problem (P^*).

Consider the case of pure bending. Assume that the plate is subject only to the transverse loading, i.e. $f(\mathbf{v}, v) = f(v)$. The goal is to check whether $\mathbf{u} = \mathbf{0}$. Setting $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$ in (3.33) and taking into account that

$$\mathbf{L}(\boldsymbol{\kappa}, \mathbf{0}) = \boldsymbol{\kappa}, \qquad \mathbf{L}(\mathbf{0}, \boldsymbol{\kappa}) = \mathbf{0} \tag{3.34}$$

gives

$$\mathbf{N} = \mathbf{0}, \qquad \mathbf{K} = \frac{1}{2}(\lambda_1 + \lambda_2)(\boldsymbol{\kappa} + \nu\boldsymbol{\kappa}) = \lambda_1 \boldsymbol{\kappa}, \tag{3.35}$$

thus Eq. (3.33) reduces to

$$\int_{\Omega} \lambda_1 \boldsymbol{\kappa}(w) \cdot \boldsymbol{\kappa}(v) dx = f(v) \tag{3.36}$$

and the conclusion that the pair $(\mathbf{u} = \mathbf{0}, w)$, w being solution to (3.36), solves the problem (I.2.6), (I.2.25) follows. Tensor $\mathbf{A}$ has the following representation:

$$\mathbf{A} = \lambda_1 \hat{\boldsymbol{\kappa}} \otimes \hat{\boldsymbol{\kappa}} + \lambda_2 \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2 + \lambda_3 \boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3 \tag{3.37}$$

with $\boldsymbol{\omega}_2 \perp \hat{\boldsymbol{\kappa}}$, $\boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$, $\hat{\boldsymbol{\kappa}}$ given by (I.3.14) and the potential U_λ^* reduces to the expression given in (I.3.64). According to (3.28), the relevant solution $(\mathbf{u} = \mathbf{0}, w)$ is unique.

Next, assume that the plate is subjected to the in-plane forces only (membrane case), i.e. $f(\mathbf{v}, v) = f(\mathbf{v})$. By similar arguments one can show that $w = 0$ in this case while $\mathbf{u}$ is the solution to the problem

$$\int_\Omega \lambda_1 \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx = f(\mathbf{v}). \tag{3.38}$$

Tensor $\mathbf{A}$ has the representation (3.37) with $\hat{\boldsymbol{\kappa}}$ replaced by $\hat{\boldsymbol{\varepsilon}}$. Potential $2U_\lambda^*$ reduces to $\frac{1}{2}\lambda_1 \|\boldsymbol{\varepsilon}\|^2$.

3.4. The primal formulation

All results of the paper are based on the equivalence of two expressions: (I.2.19) and (I.2.20). This equivalence property will be inferred by proving that the solution $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$ of the problem (P^*) or (I.2.24) is proportional to the solution $(\breve{\mathbf{u}}, \breve{w})$ of the problem (P) or – to the problem primal for (P^*). We shall draw upon the results published in Czarnecki and Lewiński [2].

Let us introduce the function of arguments $\mathbf{N}, \mathbf{K} \in \mathbb{E}_s^2$

$$U_\lambda(\mathbf{N}, \mathbf{K}) = \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\left(\|\mathbf{N}\|^2 + \|\mathbf{K}\|^2\right) - \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) G(\mathbf{N}, \mathbf{K}), \tag{3.39}$$

where $G(\cdot,\cdot)$ is defined by (I.3.50). According to (7.8) in [2], the potential (I.3.11) can be represented as

$$U_\lambda^*(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \sup\left\{ \mathbf{N} \cdot \boldsymbol{\varepsilon} + \mathbf{K} \cdot \boldsymbol{\kappa} - U_\lambda(\mathbf{N}, \mathbf{K}) \mid \ (\mathbf{N}, \mathbf{K}) \in \mathbb{E}_s^2 \right\}. \tag{3.40}$$

Therefore, the potential U_λ^* is dual to U_λ.

Let $\sum(\Omega)$ denote the set of fields $(\mathbf{N}, \mathbf{M})$ given on Ω, satisfying the variational equation (I.2.6). According to Castigliano's theorem, the compliance is expressed by

$$C(\mathbf{A}) = \min_{(\tilde{\mathbf{N}}, \tilde{\mathbf{M}}) \in \Sigma(\Omega)} \int_\Omega \left[\tilde{\mathbf{N}} \cdot (\mathbf{A}^{-1}\tilde{\mathbf{N}}) + \tilde{\mathbf{M}} \cdot (\mathbf{D}^{-1}\tilde{\mathbf{M}})\right] dx, \tag{3.41}$$

where $\mathbf{A}$, $\mathbf{D}$ are given by (I.2.2).

Substitution of (3.41) into (I.2.15) leads to a new expression $\breve{C}_0$ for the optimal compliance. Minimization over $\mathbf{A} \in \mathcal{T}_\lambda(\Omega)$ can be performed analytically. By using the analogy with problem (3.2) in [2], we obtain the problem (P):

$$\breve{C}_0 = \min_{(\tilde{\mathbf{N}}, \tilde{\mathbf{K}}) \in \Sigma(\Omega)} \int_\Omega U_\lambda(\tilde{\mathbf{N}}, \tilde{\mathbf{K}}) dx, \tag{3.42}$$

where $U_\lambda(\tilde{\mathbf{N}},\tilde{\mathbf{K}})$ is given by (3.39), see (4.48) in [2]. Note that $\breve{C}_0 = C_0$, C_0 being defined by (I.2.15). We shall prove that the problem dual to (P) is just (P^*) or (I.2.24).

Assume that (3.42) is solvable and one of the solutions is denoted by $(\breve{\mathbf{N}},\breve{\mathbf{M}})$. Let us rewrite (3.42) as follows

$$\breve{C}_0 = \min_{(\tilde{\mathbf{N}},\tilde{\mathbf{K}})\in\Sigma(\Omega)} \max_{(\tilde{\mathbf{u}},\tilde{w})\in V} \left[\int_\Omega U_\lambda(\tilde{\mathbf{N}},\tilde{\mathbf{K}})dx + f(\tilde{\mathbf{u}},\tilde{w}) - \int_\Omega (\tilde{\mathbf{N}}\cdot\tilde{\boldsymbol{\varepsilon}} + \tilde{\mathbf{K}}\cdot\tilde{\boldsymbol{\kappa}})dx \right], \tag{3.43}$$

where $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$, $\tilde{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\tilde{w})$ and $\tilde{\mathbf{N}}$, $\tilde{\mathbf{K}}$ defined on Ω are viewed as appropriately regular.

The stationarity conditions of (3.43) lead to the relations linking the unknown fields $\tilde{\mathbf{N}}$, $\tilde{\mathbf{K}}$ with the unknown multipliers $\breve{\boldsymbol{\varepsilon}}$, $\breve{\boldsymbol{\kappa}}$:

$$\begin{aligned} \breve{\boldsymbol{\varepsilon}} &= \left.\frac{\partial U_\lambda(\mathbf{N},\mathbf{K})}{\partial \mathbf{N}}\right|_{\mathbf{N}=\breve{\mathbf{N}},\,\mathbf{K}=\breve{\mathbf{K}}} = \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)(\breve{\mathbf{N}} - \nu\mathbf{L}(\breve{\mathbf{N}},\breve{\mathbf{K}})), \\ \breve{\boldsymbol{\kappa}} &= \left.\frac{\partial U_\lambda(\mathbf{N},\mathbf{K})}{\partial \mathbf{K}}\right|_{\mathbf{N}=\breve{\mathbf{N}},\,\mathbf{K}=\breve{\mathbf{K}}} = \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)(\breve{\mathbf{K}} - \nu\mathbf{L}(\breve{\mathbf{K}},\breve{\mathbf{N}})). \end{aligned} \tag{3.44}$$

Their inversions read

$$\begin{aligned} \breve{\mathbf{N}} &= \left.\frac{\partial U^*_\lambda(\boldsymbol{\varepsilon},\boldsymbol{\kappa})}{\partial \boldsymbol{\varepsilon}}\right|_{\boldsymbol{\varepsilon}=\breve{\boldsymbol{\varepsilon}},\,\boldsymbol{\kappa}=\breve{\boldsymbol{\kappa}}}, \\ \breve{\mathbf{K}} &= \left.\frac{\partial U^*_\lambda(\boldsymbol{\varepsilon},\boldsymbol{\kappa})}{\partial \boldsymbol{\kappa}}\right|_{\boldsymbol{\varepsilon}=\breve{\boldsymbol{\varepsilon}},\,\boldsymbol{\kappa}=\breve{\boldsymbol{\kappa}}}, \end{aligned} \tag{3.45}$$

and, using (I.3.56), one finds

$$\begin{aligned} \breve{\mathbf{N}} &= \frac{1}{4}(\lambda_1+\lambda_2)(\breve{\boldsymbol{\varepsilon}} + \nu\mathbf{L}(\breve{\boldsymbol{\varepsilon}},\,\breve{\boldsymbol{\kappa}})), \\ \breve{\mathbf{K}} &= \frac{1}{4}(\lambda_1+\lambda_2)(\breve{\boldsymbol{\kappa}} + \nu\mathbf{L}(\breve{\boldsymbol{\kappa}},\,\breve{\boldsymbol{\varepsilon}})). \end{aligned} \tag{3.46}$$

The fields $\breve{\mathbf{u}}$, $\breve{w}$, $\breve{\boldsymbol{\varepsilon}}$, $\breve{\boldsymbol{\kappa}}$, $\breve{\mathbf{N}}$, $\breve{\mathbf{K}}$ satisfy the constitutive equations (3.46) and the equations of equilibrium (I.2.6) (upon the change (I.3.2) and $\mathbf{K} = (\sqrt{12}/h)\mathbf{M}$) along with appropriate kinematic boundary conditions. We note that this set of equations is almost the same as the set of local equations of problem (P^*) that can be inferred from (3.33). It has been proved that the solution $(\hat{\mathbf{u}},\hat{w},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\kappa}},\hat{\mathbf{N}},\hat{\mathbf{M}})$ to the problem (P^*) is unique, provided it exists. The equations of problems (P) and (P^*) differ in the coefficients of the constitutive equations, cf. (3.46) and

(2.1). We note that both the problems are uniquely solvable. The solution $(\breve{\mathbf{u}}, \breve{w})$ of problem (P) is linked with the solution $(\hat{\mathbf{u}}, \hat{w})$ of problem (P^*) by $\breve{\mathbf{u}} = 2\hat{\mathbf{u}}$, $\breve{w} = 2\hat{w}$, while $\breve{\mathbf{N}} = \hat{\mathbf{N}}$, $\breve{\mathbf{K}} = \hat{\mathbf{K}}$, since the moduli in (3.46) are two times smaller than the moduli in (2.1). Since $(\breve{\mathbf{N}}, \breve{\mathbf{K}})$ is the minimizer of (3.42), one may write

$$\breve{C}_0 = \int_\Omega U_\lambda(\breve{\mathbf{N}}, \breve{\mathbf{K}})dx \tag{3.47}$$

or

$$\breve{C}_0 = \int_\Omega U_\lambda(\hat{\mathbf{N}}, \hat{\mathbf{K}})dx. \tag{3.48}$$

The equilibrium equation implies the identity

$$f(\hat{\mathbf{u}}, \hat{w}) = \int_\Omega (\hat{\mathbf{N}} \cdot \hat{\boldsymbol{\varepsilon}} + \hat{\mathbf{K}} \cdot \hat{\boldsymbol{\kappa}})dx \tag{3.49}$$

which makes it possible to rearrange (3.48) to the form

$$\breve{C}_0 = \int_\Omega \Big[U_\lambda(\hat{\mathbf{N}}, \hat{\mathbf{K}}) - (\hat{\mathbf{N}} \cdot \hat{\boldsymbol{\varepsilon}} + \hat{\mathbf{K}} \cdot \hat{\boldsymbol{\kappa}})\Big]\, dx + f(\hat{\mathbf{u}}, \hat{w}). \tag{3.50}$$

By using (3.44) and (3.27), we compute

$$\begin{aligned} &\hat{\mathbf{N}} \cdot \hat{\boldsymbol{\varepsilon}} + \hat{\mathbf{K}} \cdot \hat{\boldsymbol{\kappa}} \\ &\quad = \breve{\mathbf{N}} \cdot \left(\frac{1}{2}\breve{\boldsymbol{\varepsilon}}\right) + \breve{\mathbf{K}} \cdot \left(\frac{1}{2}\breve{\boldsymbol{\kappa}}\right) \\ &\quad = \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\Big[\breve{\mathbf{N}} \cdot \Big(\breve{\mathbf{N}} - \nu\mathbf{L}(\breve{\mathbf{N}}, \breve{\mathbf{K}})\Big) + \breve{\mathbf{K}} \cdot \Big(\breve{\mathbf{K}} - \nu\mathbf{L}(\breve{\mathbf{K}}, \breve{\mathbf{N}})\Big)\Big] \\ &\quad = U_\lambda(\breve{\mathbf{N}}, \breve{\mathbf{K}}) = U_\lambda(\hat{\mathbf{N}}, \hat{\mathbf{K}}), \end{aligned} \tag{3.51}$$

which gives $\breve{C}_0 = f(\hat{\mathbf{u}}, \hat{w})$.

Let us rearrange (I.2.24). The minimum is attained for $(\mathbf{v}, v) = (\hat{\mathbf{u}}, \hat{w})$, hence

$$\hat{C}_0 = -2J_\lambda(\hat{\mathbf{u}}, \hat{w}) \tag{3.52}$$

or

$$\hat{C}_0 = 2f(\hat{\mathbf{u}}, \hat{w}) - \int_\Omega 2W_{\lambda(x)}\,(\varepsilon(\hat{\mathbf{u}}), \boldsymbol{\varkappa}(\hat{w}))\, dx. \tag{3.53}$$

The stationarity condition of $J_\lambda(\mathbf{v}, v)$ assumes the form of the equilibrium equation (I.2.6) in which $\mathbf{N} = \hat{\mathbf{N}}$, $\mathbf{M} = \hat{\mathbf{M}}$, given by (I.2.25). Since

$$\hat{\mathbf{N}} \cdot \boldsymbol{\varepsilon}(\hat{\mathbf{u}}) + \hat{\mathbf{M}} \cdot \boldsymbol{\varkappa}(\hat{w}) = 2W_{\lambda(x)}\left(\boldsymbol{\varepsilon}(\hat{\mathbf{u}}), \boldsymbol{\varkappa}(\hat{w})\right), \tag{3.54}$$

see (3.25), we conclude that

$$f(\hat{\mathbf{u}}, \hat{w}) = \int_\Omega 2W_{\lambda(x)}\left(\boldsymbol{\varepsilon}(\hat{\mathbf{u}}), \boldsymbol{\varkappa}(\hat{w})\right) dx \tag{3.55}$$

which gives $\hat{C}_0 = f(\hat{\mathbf{u}}, \hat{w})$.

Thus we have arrived at

$$\breve{C}_0 = \hat{C}_0, \tag{3.56}$$

which ends the proof of possibility of switching the „min" and „max" operations in (3.43). Let us rewrite (3.43) as

$$\breve{C}_0 = \max_{(\tilde{\mathbf{u}},\tilde{w})\in V}\left[\int_\Omega \min_{\tilde{\mathbf{N}},\tilde{\mathbf{K}}\in\mathbb{E}^2_s}\left[U_\lambda(\tilde{\mathbf{N}}, \tilde{\mathbf{K}}) - (\tilde{\mathbf{N}} \cdot \tilde{\boldsymbol{\varepsilon}} + \tilde{\mathbf{K}} \cdot \tilde{\boldsymbol{\kappa}})\right] dx + f(\tilde{\mathbf{u}}, \tilde{w})\right], \tag{3.57}$$

hence

$$\breve{C}_0 = \max_{(\tilde{\mathbf{u}},\tilde{w})\in V}\left[-\int_\Omega \max_{\tilde{\mathbf{N}},\tilde{\mathbf{K}}\in\mathbb{E}^2_s}\left[\tilde{\mathbf{N}} \cdot \tilde{\boldsymbol{\varepsilon}} + \tilde{\mathbf{K}} \cdot \tilde{\boldsymbol{\kappa}} - U_\lambda(\tilde{\mathbf{N}}, \tilde{\mathbf{K}})\right] dx + f(\tilde{\mathbf{u}}, \tilde{w})\right]. \tag{3.58}$$

We use (3.40) to find

$$\breve{C}_0 = \max_{(\tilde{\mathbf{u}},\tilde{w})\in V}\left[-\int_\Omega U^*_\lambda(\tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\kappa}}) dx + f(\tilde{\mathbf{u}}, \tilde{w})\right]. \tag{3.59}$$

By homogeneity of U^*_λ and linearity of $f(\cdot,\cdot)$ we write

$$\breve{C}_0 = 2\max_{(\tilde{\mathbf{u}},\tilde{w})\in V}\left[-\int_\Omega 2U^*_\lambda\left(\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}, \frac{1}{2}\tilde{\boldsymbol{\kappa}}\right) dx + f\left(\frac{1}{2}\tilde{\mathbf{u}}, \frac{1}{2}\tilde{w}\right)\right], \tag{3.60}$$

hence

$$\breve{C}_0 = -2\min_{(\tilde{\mathbf{u}},\tilde{w})\in V}\left[2\int_\Omega U^*_\lambda\left(\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}, \frac{1}{2}\tilde{\boldsymbol{\kappa}}\right) dx - f\left(\frac{1}{2}\tilde{\mathbf{u}}, \frac{1}{2}\tilde{w}\right)\right]. \tag{3.61}$$

We introduce the notation (I.3.3):

$$\breve{C}_0 = -2 \min_{(\tilde{\mathbf{u}},\tilde{w})\in V} \left[\int_\Omega W_{\lambda(x)} \left(\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}, \frac{1}{2}\tilde{\boldsymbol{\kappa}} \right) dx - f\left(\frac{1}{2}\tilde{\mathbf{u}}, \frac{1}{2}\tilde{w} \right) \right] \tag{3.62}$$

and now we use the notation (I.2.22) to obtain

$$\breve{C}_0 = -2 \min_{(\tilde{\mathbf{u}},\tilde{w})\in V} J_\lambda \left(\frac{1}{2}\tilde{\mathbf{u}}, \frac{1}{2}\tilde{w} \right), \tag{3.63}$$

which can be written as, see (I.2.21),

$$\breve{C}_0 = -2 \min_{(\tilde{\mathbf{u}},\tilde{w})\in V} \max_{\mathbf{A}\in\mathcal{T}_\lambda(\Omega)} J\left(\mathbf{A}, \frac{1}{2}\tilde{\mathbf{u}}, \frac{1}{2}\tilde{w} \right). \tag{3.64}$$

Let us recall (I.2.19):

$$\breve{C}_0 = -2 \max_{\mathbf{A}\in\mathcal{T}_\lambda(\Omega)} \min_{(\tilde{\mathbf{u}},\tilde{w})\in V} J\left(\mathbf{A}, \tilde{\mathbf{u}}, \tilde{w} \right). \tag{3.65}$$

Since $C_0 = \breve{C}_0$ we confirm (3.56). Let $(\tilde{\mathbf{u}}, \tilde{v}) = (\breve{\mathbf{u}}, \breve{w})$ be the minimizer of (3.64) and $(\hat{\mathbf{u}}, \hat{w})$ be the minimizer of (3.65). We confirm once again that $\frac{1}{2}\breve{\mathbf{u}} = \hat{\mathbf{u}}$, $\frac{1}{2}\breve{w} = \hat{w}$. Thus the passage from (3.43) to (3.57) is justified.

4. Examples of optimal designs

4.1. Trajectories of the symmetric second-order tensor eigenvalues

Any symmetric second-order tensor $\mathbf{a} \in \mathbb{E}^2_s$ admits a spectral decomposition

$$\mathbf{a} = \alpha_1 \, \mathbf{d} \otimes \mathbf{d} + \alpha_2 \, \mathbf{d}^\perp \otimes \mathbf{d}^\perp \tag{4.1}$$

where $\{\mathbf{d}, \mathbf{d}^\perp\}$ denotes the eigenbasis of $\mathbf{a}$ and forms an orthonormal (Cartesian) basis in $\mathbb{R}^2$. The unit tensor in $\mathbb{E}^2_s$ is given by

$$\mathbf{I}_2 = \mathbf{d} \otimes \mathbf{d} + \mathbf{d}^\perp \otimes \mathbf{d}^\perp, \tag{4.2}$$

thus we may re-write (4.1) in a form

$$\mathbf{a} = \frac{1}{2}(\alpha_1 + \alpha_2)\mathbf{I}_2 + \mathbf{t}, \qquad \mathbf{t} = \frac{1}{2}(\alpha_1 - \alpha_2)(2\,\mathbf{d}\otimes\mathbf{d} - \mathbf{I}_2) \tag{4.3}$$

and it is a matter of straighforward calculations that $\mathbf{I}_2 \cdot \mathbf{t} = 0$. Equation (4.3) determines the isotropic decomposition of $\mathbf{a}$, see e.g. [1], and $\mathbf{t}$ stands for its deviatoric (pure shear) part.

Let us introduce an angle φ as linking the arbitrary orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2\}$ with the eigenbasis $\{\mathbf{d}, \mathbf{d}^{\perp}\}$ by

$$\mathbf{d} = \cos\varphi\,\mathbf{i}_1 + \sin\varphi\,\mathbf{i}_2, \qquad \mathbf{d}^{\perp} = \sin\varphi\,\mathbf{i}_1 - \cos\varphi\,\mathbf{i}_2. \tag{4.4}$$

Substituting (4.4) in (4.3) and making use of (I.2.7) leads to

$$\mathbf{t} = \frac{1}{2}(\alpha_1 - \alpha_2)\Big[\cos(2\,\varphi)(\mathbf{B}_1 - \mathbf{B}_2) + \sqrt{2}\sin(2\,\varphi)\mathbf{B}_3\Big] \tag{4.5}$$

or

$$\mathbf{t} = \frac{1}{2}(a_{11} - a_{22})(\mathbf{B}_1 - \mathbf{B}_2) + \sqrt{2}\,a_{12}\mathbf{B}_3, \tag{4.6}$$

see (I.2.8), by rotational invariance of the isotropic decomposition.

Components of $\mathbf{a}$ depend on the spatial variables $(x_1, x_2) \equiv (x, y)$, hence comparing (4.5) with (4.6) allows for setting

$$\tan\left(2\varphi(x,y)\right) = 2\,\frac{a_{12}(x,y)}{a_{11}(x,y) - a_{22}(x,y)}, \tag{4.7}$$

thus establishing an equation determining the family of curves $y = y(x)$. Consequently, $\mathbf{d}$ denotes a vector locally (i.e. at given x) tangent to a curve $y(x)$ and $\varphi = \sphericalangle(\mathbf{d}, \mathbf{i}_1)$, hence $dy/dx = \tan\varphi$. Making use of the trigonometric identity

$$\tan(2\varphi) = \frac{2\tan\varphi}{1 - \tan^2\varphi}, \tag{4.8}$$

we may re-write (4.7) in a form

$$\left(\frac{dy}{dx}\right)^2 + \frac{a_{11} - a_{22}}{a_{12}}\left(\frac{dy}{dx}\right) - 1 = 0 \tag{4.9}$$

whose solutions, known as trajectories of eigenvalues of a given tensor $\mathbf{a}$, are given by

$$\frac{dy}{dx} + \frac{a_{11} - a_{22}}{2\,a_{12}} \pm \left(\left(\frac{a_{11} - a_{22}}{2\,a_{12}}\right)^2 + 1\right)^{1/2} = 0. \tag{4.10}$$

The latter equations determine two families of curves which are orthogonal for every $(x, y) \in \Omega$. In most cases, functions $y = y(x)$, solving the differential equations in (4.10), cannot be computed analytically, thus in the sequel we make use of the numerical algorithm in MAPLE.

4.2. Numerical examples

In the following we will find the trajectories of $\boldsymbol{\omega}_1$, i.e. the eigenstate of the optimally oriented constitutive tensor $\mathbf{A}$ corresponding to the greatest Kelvin modulus λ_1 for given strain fields in some examples of membrane-bending load cases. For this purpose, we will make use of the formula in (I.3.45). The results of calculations will be compared with those obtained for pure in-plane or bending loadings. All numerical data for which the results in the sequel were obtained are shown as measureless, but they correspond to a certain consistent system of units.

EXAMPLE 1. In the first example, let us consider a rectangular plate with the middle plane Ω whose dimensions $2a = 80$ and $2b = 50$ are shown in Fig. 1. The thickness of the plate is uniform and we set $h = 1$. For simplicity of the calculations, we assume that the plate is made of the isotropic material such that $E = 205000$ and $\nu = 0.3$. Obviously, in such case $\lambda_1 > \lambda_2 = \lambda_3$ and the optimal tensor $\mathbf{A}$ admits the form

$$\mathbf{A} = (\lambda_1 - \lambda_2)\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \lambda_2 \mathbf{I}_4, \tag{4.11}$$

see (I.2.10) and (I.2.13). Assume that in-plane displacements $u_1 = u_2 = 0$ along $A - B$ and the transversal displacement $w = 0$ along all the boundary of the middle plane Ω. The plate is subjected to the in-plane tractions along $C - D$ and the magnitude of its resultant equals $P = -50$. The transversal load is uniformly applied to Ω and its intensity $q = 1$.

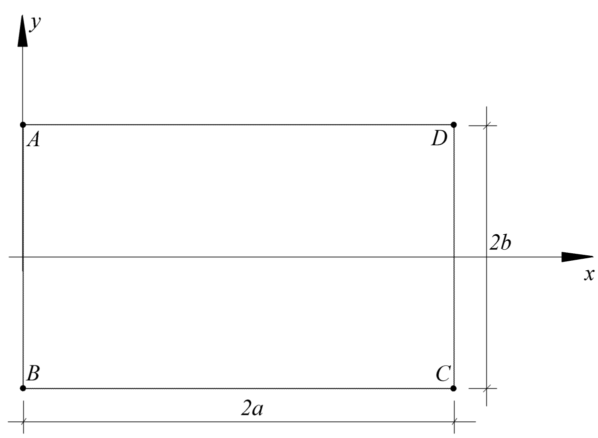

FIG. 1. Middle plane Ω of a plate.

For calculating the components of the plane strain tensor $\boldsymbol{\varepsilon}$, let us make use of the Airy stress function

$$F(x,y) = -\frac{3}{2} P\,x\,\frac{y}{2b} - 2\,P\,(2a-x)\left(\frac{y}{2b}\right)^3, \tag{4.12}$$

determining the components of the membrane force tensor $\mathbf{N}$

$$N_{11} = \frac{\partial^2 F}{\partial (x_2)^2}, \qquad N_{22} = \frac{\partial^2 F}{\partial (x_1)^2}, \qquad N_{12} = -\frac{\partial^2 F}{\partial x_1\,\partial x_2} \tag{4.13}$$

satisfying the boundary conditions

$$N_{12}(x,\pm b) = 0, \qquad N_{22}(x,\pm b) = 0, \qquad \int_{-b}^{b} N_{12}(2a,y)\,dy = P \tag{4.14}$$

and finally apply the constitutive relation

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \frac{1}{E\,h} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix}. \tag{4.15}$$

Components of the curvature tensor $\boldsymbol{\kappa}$ can be derived by assuming in (I.2.1) and (I.3.2) that

$$w(x,y) = \frac{16\,q}{D\,\pi^6}\,f(x,y), \tag{4.16}$$

where

$$D = \frac{E\,h^3}{12(1-\nu^2)} \tag{4.17}$$

and

$$f(x,y) = \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} \frac{1}{mn\left(\dfrac{m^2}{(2a)^2} + \dfrac{n^2}{(2b)^2}\right)^2} \sin\frac{m\pi x}{2a} \sin\frac{n\pi(y+b)}{2b}. \tag{4.18}$$

Next, we calculate the components of $\boldsymbol{\omega}_1$ by (I.3.45) and we substitute $a_{ij} = (\boldsymbol{\omega}_1)_{ij}$, $i,j = 1,2$, in (4.10), thus obtaining the formula for the trajectories corresponding to the eigenvalues of $\boldsymbol{\omega}_1$, see Fig. 2.

Comparing the optimal trajectories in Fig. 2 with those obtained for membrane and bending loadings acting independently, see Fig. 3 and Fig. 4 respectively, we may conclude that their layout strongly depends on the values of the function $\xi(\boldsymbol{\varepsilon},\boldsymbol{\kappa}) = \|\boldsymbol{\kappa}\|^2/\|\boldsymbol{\varepsilon}\|^2$ whose contours are shown in Fig. 5. Indeed, the optimal angle $x_0 = \sphericalangle(\hat{\boldsymbol{\varepsilon}},\boldsymbol{\omega}_1)$ rapidly tends to 0 if $\xi \to 0$ and $x_0 \to \hat{\alpha} = \sphericalangle(\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\kappa}})$

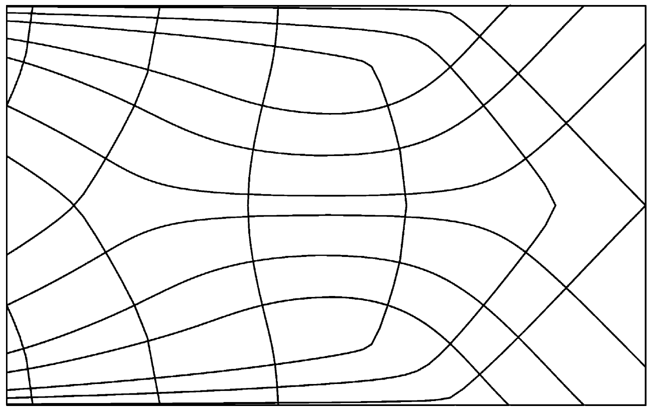

Fig. 2. Trajectories of the optimal field $\boldsymbol{\omega}_1(x, y)$ eigenvalues (Ex. 1).

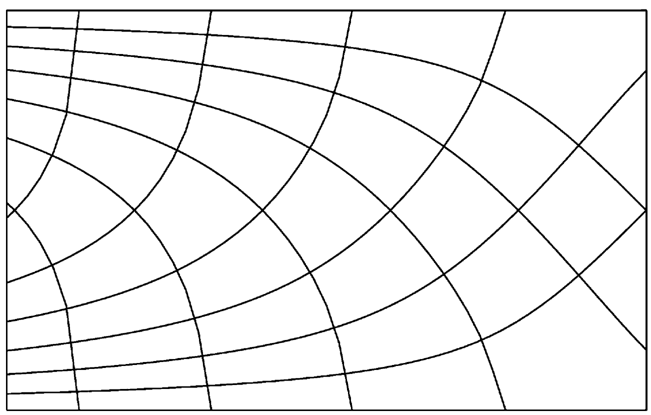

Fig. 3. Trajectories of the field $\boldsymbol{\varepsilon}(x, y)$ eigenvalues (Ex. 1).

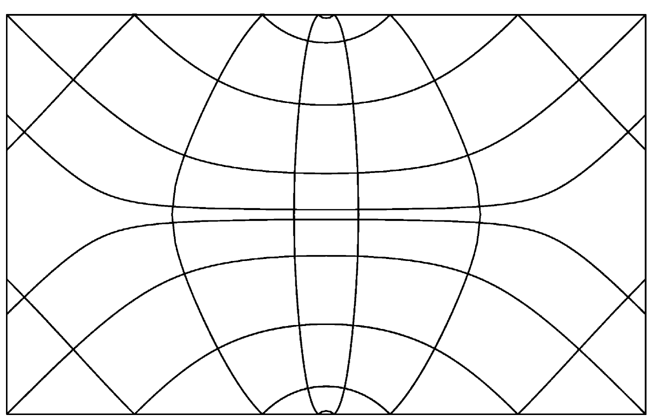

Fig. 4. Trajectories of the field $\boldsymbol{\kappa}(x, y)$ eigenvalues (Ex. 1).

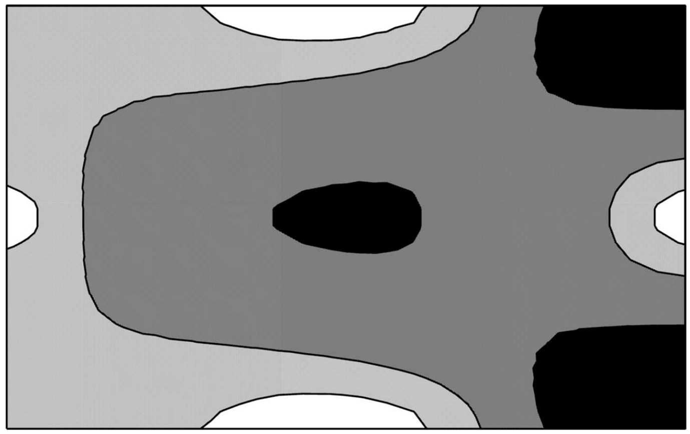

FIG. 5. Contours $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 0.2$, $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 1$ and $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 5$, with $\xi < 0.2$ and $\xi > 5$, corresponding to white and black respectively (Ex. 1).

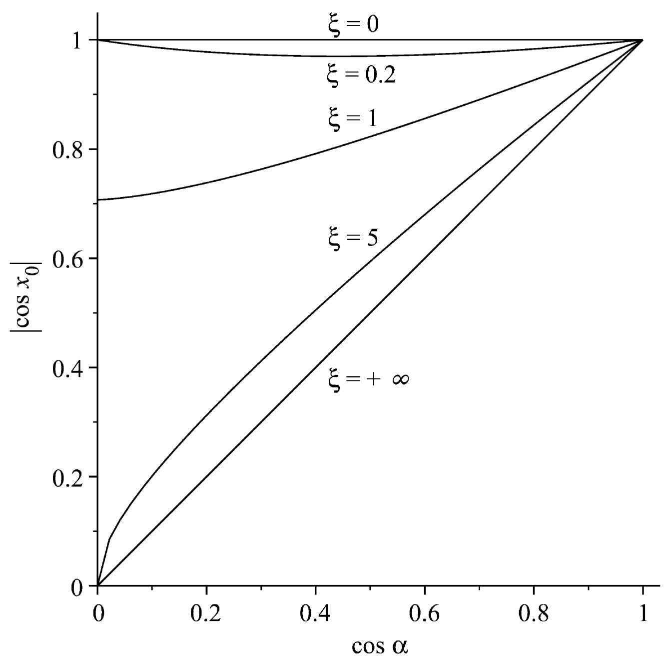

FIG. 6. Family of functions determining $|\cos x_0|$ for varying values of ξ.

if $\xi \rightarrow +\infty$, see Fig. 6, where the lines corresponding to the varying value $\xi \in [0, +\infty)$ determine a set of plots

$$|\cos x_0| = \frac{\sqrt{2}}{2}\big(1 + \tilde{\phi}(\xi, t) + \tilde{\psi}(\xi, t)\sqrt{\xi}\,t\big)^{1/2} \tag{4.19}$$

where $t = \cos\hat{\alpha}$ and

$$\tilde{\phi}(\xi,t) = (1-\xi)\big((1-\xi)^2 + 4\xi t^2\big)^{-1/2}, \tag{4.20}$$

$$\tilde{\psi}(\xi,t) = 2\sqrt{\xi}\,t\big((1-\xi)^2 + 4\xi t^2\big)^{-1/2}, \tag{4.21}$$

see (I.3.50).

EXAMPLE 2. Next, assume that the plate in Fig. 1 is subjected to the in-plane boundary conditions $u_1(0,0) = u_2(0,0) = 0$ and $u_2(2a,0) = 0$. The transversal displacement $w = 0$ along all the boundary of the middle plane and $\partial w/\partial \mathbf{n} = 0$ along the edges $A - D$ and $B - C$. Let p_2 and q respectively denote the in-plane and transversal loadings uniformly applied to Ω. In what follows we set $p_2 = q = -10$.

The Airy stress function assumed as

$$F(x,y) = -\frac{1}{20}\,\frac{p_2\,y}{b^2}\Big(5\,b^2\,(x-a)^2 - 5\,y^2 x\,(x-2\,a) - y^2\left(2\,b^2 - y^2\right)\Big) \tag{4.22}$$

satisfies the boundary conditions

$$\begin{aligned}
&\int_{-b}^{b} N_{11}(0,y)\,dy = 0, && \int_{-b}^{b} N_{11}(0,y)\,dy = 0,\\
&\int_{-b}^{b} yN_{11}(0,y)\,dy = 0, && \int_{-b}^{b} yN_{11}(2a,y)\,dy = 0,\\
&\int_{-b}^{b} N_{12}(0,y)\,dy = -2p_2ab, && \int_{-b}^{b} N_{12}(2a,y)\,dy = 2p_2ab,\\
&N_{12}(x,\pm b) = 0, && N_{22}(x,\pm b) = 0,
\end{aligned} \tag{4.23}$$

and the components of a strain tensor $\boldsymbol{\varepsilon}$ are determined by making use of (4.13) and (4.15).

Next we calculate the curvature tensor representation by (I.2.1), (I.3.2) and

$$w(x,y) = \frac{2qa^4}{3\,D}\left(\left(\frac{x}{2a}\right) - 2\left(\frac{x}{2a}\right)^3 + \left(\frac{x}{2a}\right)^4\right) + f(x,y), \tag{4.24}$$

where

$$\begin{aligned}
f(x,y) = \frac{2\,q}{D\,a}\sum_{m=1,3,\ldots}\frac{1}{(\alpha_m)^5}\Bigg[&\frac{2\sinh(\alpha_m b)}{\sinh(2\alpha_m b) + 2\alpha_m b}\,\alpha_m y\,\sinh(\alpha_m y)\\
&-\left(1 + \frac{2\alpha_m b\,\sinh^2(\alpha_m b)}{\sinh(2\alpha_m b) + 2\alpha_m b}\right)\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m b)}\Bigg]
\end{aligned} \tag{4.25}$$

and $\alpha_m = (m\pi)/(2a)$.

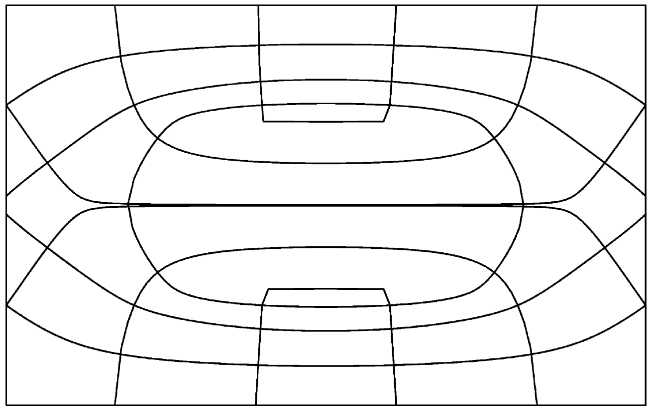

Fig. 7. Trajectories of the optimal field $\boldsymbol{\omega}_1(x, y)$ eigenvalues (Ex. 2).

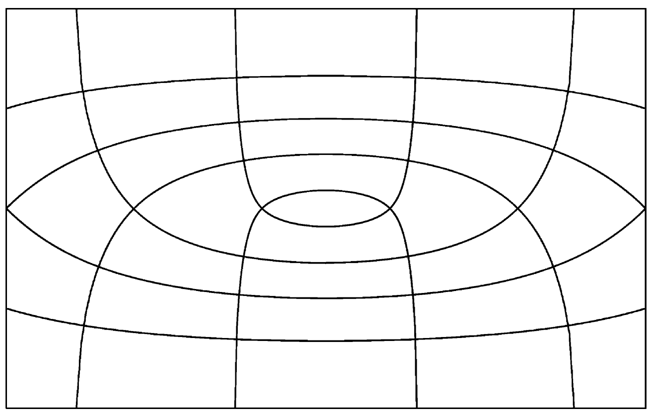

Fig. 8. Trajectories of the field $\boldsymbol{\varepsilon}(x, y)$ eigenvalues (Ex. 2).

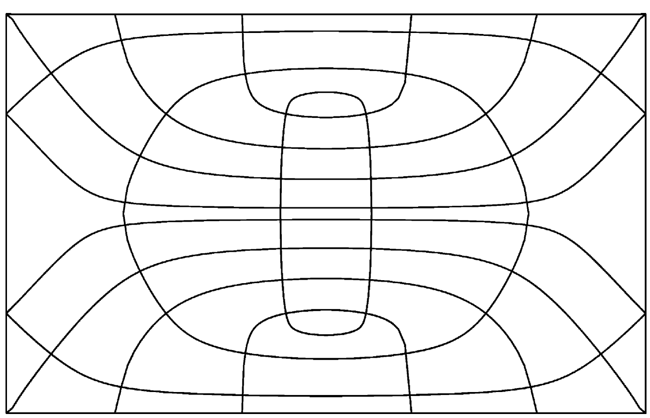

Fig. 9. Trajectories of the field $\boldsymbol{\kappa}(x, y)$ eigenvalues (Ex. 2).

Proceeding in the same fashion as in the previous example, we finally obtain the optimal trajectories of $\boldsymbol{\omega}_1$, see Fig. 7, and these calculated for the in-plane and bending cases treated separately, see Fig. 8 and Fig. 9 respectively. The contour map of function $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ is shown in Fig. 10.

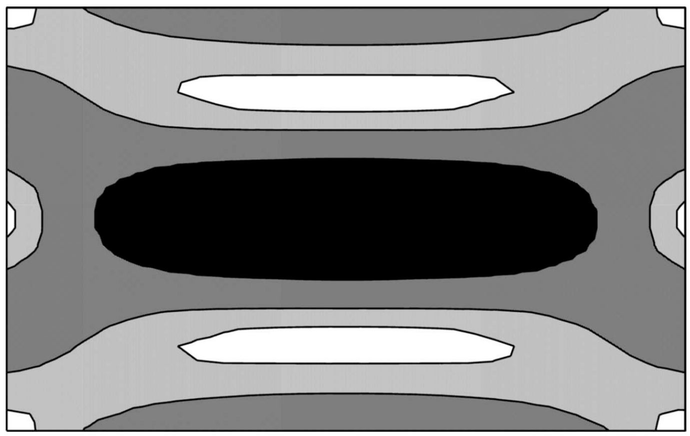

FIG. 10. Contours $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 0.2$, $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 1$ and $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = 5$ with $\xi < 0.2$ and $\xi > 5$, corresponding to white and black respectively (Ex. 2).

Similarly to Example 1, one may observe that the trajectories corresponding to the eigenvalues of the optimal proper tensor field $\boldsymbol{\omega}_1(x\,y)$ in Fig. 7 depend on the values of function $\xi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ and follow the pattern of $\boldsymbol{\kappa}(x, y)$ trajectories, see Fig. 9, if $\|\boldsymbol{\varepsilon}\| \approx 0$ or the one of $\boldsymbol{\varepsilon}(x, y)$, see Fig. 8, if $\|\boldsymbol{\kappa}\| \approx 0$.

5. Final remarks

The analysis and examples provided in Sec. 4 show an exceptional sensitivity of the potential W_λ with respect to small changes of the parameter $\xi = (\|\boldsymbol{\kappa}\|/\|\boldsymbol{\varepsilon}\|)^2$. Namely, if ξ is small, the structural response is almost in-plane, while for bigger values of ξ, the optimal structure switches to the bending behaviour.

The results of the present paper extend to the optimum design of thin shells within Love's first approximation, since in this model the constitutive equations (I.2.3) are valid and remain decoupled, see NAGHDI [6], while the form of the strain-displacement relations does not affect the final results. Thus the optimum design problem of shells reduces to the equilibrium problem of an effective thin hyperelastic shell endowed with the constitutive equations being both coupled and nonlinear.

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Appendix B

Assume that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, treated as vectors in $\mathbb{R}^3$ are not colinear. Then introduce a basis

$$\begin{aligned} \mathbf{e}_1 &= \boldsymbol{\varepsilon}, \\ \mathbf{e}_2 &= \boldsymbol{\kappa}, \\ \mathbf{e}_3 &= \frac{\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon} \times \boldsymbol{\kappa}\|}, \end{aligned} \tag{B.1}$$

such that

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{e}_3 &= 0, \\ \mathbf{e}_2 \cdot \mathbf{e}_3 &= 0, \\ \|\mathbf{e}_3\| &= 1. \end{aligned} \tag{B.2}$$

Next, calculate the covariant components $E_{ij} = E_{ji}$ of a metric tensor $\mathbf{E} = E_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$

$$\begin{aligned} E_{11} &= \|\boldsymbol{\varepsilon}\|^2, & E_{13} &= E_{23} = 0, \\ E_{12} &= \boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}, & E_{33} &= 1, \\ E_{22} &= \|\boldsymbol{\kappa}\|^2, \end{aligned} \tag{B.3}$$

and recall that mixed components of $\mathbf{E}$ are given by formula $E^i{}_j = E^j{}_i = \delta^i{}_j$, where

$$\delta^i{}_j = \mathbf{e}^i \cdot \mathbf{e}_j = E^{ik}\mathbf{e}_k \cdot \mathbf{e}_j = E^{ik}E_{kj}. \tag{B.4}$$

Making use of (B.3) and (B.4) allows for the calculation of contravariant components $E^{ij} = E^{ji}$

$$\begin{aligned} E^{11} &= \frac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2 \|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, & E^{13} &= E^{23} = 0, \\ E^{12} &= -\frac{\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2 \|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, & E^{33} &= 1, \\ E^{22} &= \frac{\|\boldsymbol{\varepsilon}\|^2}{\|\boldsymbol{\varepsilon}\|^2 \|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa})^2}, \end{aligned} \tag{B.5}$$

and co-basis vectors $\mathbf{e}^i = E^{ij}\,\mathbf{e}_j$

$$\begin{aligned}
\mathbf{e}^1 &= E^{11}\,\mathbf{e}_1 + E^{12}\,\mathbf{e}_2 \\
&= \frac{\|\boldsymbol{\kappa}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2}\,\boldsymbol{\varepsilon} - \frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2}\,\boldsymbol{\kappa}, \\
\mathbf{e}^2 &= E^{21}\,\mathbf{e}_1 + E^{22}\,\mathbf{e}_2 \\
&= -\frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2}\,\boldsymbol{\varepsilon} + \frac{\|\boldsymbol{\varepsilon}\|^2}{\|\boldsymbol{\varepsilon}\|^2\,\|\boldsymbol{\kappa}\|^2 - (\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2}\,\boldsymbol{\kappa}, \\
\mathbf{e}^3 &= E^{33}\,\mathbf{e}_3 = \frac{\boldsymbol{\varepsilon}\times\boldsymbol{\kappa}}{\|\boldsymbol{\varepsilon}\times\boldsymbol{\kappa}\|}.
\end{aligned}$$

In this notation, the mixed representation of $\mathbf{E}$ can be expressed by

$$\mathbf{E} = \delta^i{}_j\,\mathbf{e}_i \otimes \mathbf{e}^i. \tag{B.6}$$

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