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Parametric method for unsteady two-dimensional MHD boundary-layer on a body for which temperature varies with time

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This paper concerns unsteady two-dimensional laminar temperature magnetohydrodynamic (MHD) boundary-layer of incompressible fluid. The present magnetic field is homogenous and perpendicular to the body surface along which the boundary-layer is developing. Body temperature varies with time. Outer electric filed is neglected and magnetic Reynolds number is significantly lower than one, i.e. the considered problem is in induction-less approximation. In order to solve the described problem, multiparametric (generalized similarity) method is used and so-called universal equations are obtained. The obtained universal equations are solved numerically in appropriate approximation and a part of the obtained results is given in the form of figures and the corresponding conclusions.

Key words: MHD, magnetic field, electroconductivity, temperature, similarity parameters, universal equations.

Notations

B	magnetic induction,
c_p	specific heat capacity,
D	standardization constant,
Ec	Eckert number,
F	characteristic function $F = U\partial z/\partial t$,
$f_{k,n}$	dynamical parameters,
g	time derivative of characteristic function z,
$g_{k,n}$	magnetic parameters,
h	characteristic linear scale of transversal coordinate,
H	characteristic function $H = \delta^*/\delta^{**}$,
H^*	characteristic function $H^* = \delta^*/h$,
H^{**}	characteristic function $H^{**} = \delta^{**}/h$,
$l_{k,n}$	temperature parameters,
N	characteristic function $N = \sigma B^2/\rho$,
q	temperature difference between body surface and free stream,

q_w heat flux,
t time,
T temperature,
u, v longitudinal and transversal velocity in boundary-layer respectively,
U free stream velocity,
x, y longitudinal and transversal coordinate respectively,
z characteristic function $z = h^2/\nu$.

Greek symbols:
δ^* displacement thickness,
δ^{**} momentum thickness,
Φ dimensionless stream function,
η dimensionless transversal coordinate,
λ thermal conductivity,
μ viscosity,
ν kinematic viscosity,
Θ dimensionless temperature difference,
ρ fluid density,
σ conductivity,
τ shear stress,
Ψ stream function,
ξ characteristic function $\xi = \tau_w h/(\mu U)$,
ξ_t characteristic function.

Subscripts:
0 initial time moment $t = t_0$,
1 boundary-layer cross-section $x = x_0$,
∞ free stream,
w body surface.

1. Introduction

THE PROBLEM OF BOUNDARY-LAYER SEPARATION and control has attracted considerable attention over several decades because of the fundamental flow physics and technological applications. Some of the essential ideas related to boundary-layer separation and the need to prevent its occurring have been addressed by Prandtl [1]. For a long time, the following methods were used for boundary-layer control: admit the body motion in streamwise direction, increasing of the boundary-layer velocity, boundary-layer suction, second gas injection, profile laminarization, body cooling. The interest in the outer magnetic field effect on heat-physical processes appeared sixty years ago [2]. The research in MHD flows was stimulated by two problems: the protection of space vehicles from aerodynamic overheating and destruction during the passage through the dense layers of the atmosphere; the enhancement of the operational ability of the constructive elements of high temperature MHD generators for direct transformation of heat energy into electric. The first problem showed that the influence of a magnetic field on ionized gases was a convenient control method for mass, heat and hydrodynamic processes. Solutions of the mentioned problems were followed by rapid increase of analytical papers and experimental procedures concerning heat transfer in MHD boundary-layer [3, 4]. The MHD research extended gradually to new applied problems and nowadays, the research in visco-elastic fluids [5], magneto-biological processes and in medicine, are present [6].

Flow of an incompressible viscous fluid over a surface has an important influence on several technological applications in the field of metallurgy and chemical engineering. For example, during extrusion of a polymer in a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a sheet, which is then solidified through quenching or gradual cooling by direct contact with cooling fluid. In these cases, the properties of the final product depend to a great extent on the rate of cooling which is governed by the conditions in the boundary-layer. In this paper, in view of the mentioned research, mathematical model of unsteady temperature, two-dimensional laminar MHD boundary-layer of incompressible fluid is studied, which is directly related to the two mentioned physical models. Magnetic field is a function of longitudinal coordinate and perpendicular to the body surface, along which a boundary-layer is developing. Furthermore it is assumed that the magnetic Reynolds number is significantly lower than one, i.e. the considered problem is in induction-less approximation and electric field is neglected. Velocity of flow is considered to be much lower than speed of light and usual assumption in temperature boundary-layer calculation that temperature difference is small (under 50°C) is used, i.e. characteristic properties of fluid are constant (viscosity, heat conduction, electro-conductivity, magnetic permeability, mass heat capacity ...). Body surface temperature is a time function. The obtained system of partial differential equations can be solved for every particular case using modern numerical methods and a computer.

In this paper, quite different approach is used, based on ideas proposed in papers [7–10] which is extended in papers [11–13]. Essence of this approach is in introducing adequate transformations and sets of parameters in starting equations of laminar two-dimensional unsteady temperature MHD boundary-layer of incompressible fluid, which transform the system of equations and corresponding boundary conditions into a form unique for all particular problems and this form is considered to be universal.

2. Mathematical model

The described two-dimensional problem of MHD unsteady temperature boundary-layer in inductioneless approximation is mathematically presented by equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}\left(u-U\right), \tag{2.2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B^2}{\rho c_p}\left(u-U\right)^2, \tag{2.3}$$

and the corresponding boundary and initial conditions:

$$\begin{aligned}
&u=0, \quad v=0, \quad T=T_w(t) && \text{for } y=0,\\
&u\to U(x,t), \quad T\to T_\infty && \text{for } y\to\infty,\\
&u=u_0(x,y), \quad T=T_0(x,y) && \text{for } t=t_0,\\
&u=u_1(t,y), \quad T=T_1(t,y) && \text{for } x=x_0.
\end{aligned} \tag{2.4}$$

In the previous equations and the boundary conditions, the notations common in the boundary-layer theory are used for different physical values. Here, x,y are longitudinal and transversal coordinates respectively; t – time, u,v – longitudinal and transversal velocity in boundary-layer respectively; $U(x,t)$ – free stream velocity; ν – kinematic viscosity; σ – electro-conductivity; ρ – fluid density; B – magnetic induction; T – temperature; c_p – specific heat capacity; μ – viscosity; $T_w(t)$ – body surface temperature; T_∞ – free stream temperature; $u_0(x,y)$, $T_0(x,y)$ – longitudinal velocity and fluid temperature in time $t=t_0$ respectively; $u_1(t,y)$, $T_1(t,y)$ – longitudinal velocity and fluid temperatures in cross-section $x=x_0$ respectively.

For further consideration, the stream function $\Psi(x,y,t)$ is introduced with the following relations:

$$\frac{\partial \Psi}{\partial x}=-v, \qquad \frac{\partial \Psi}{\partial y}=u, \tag{2.5}$$

which satisfies Eq. (2.1) identically and transform the momentum equation (2.2) into equation:

$$\frac{\partial^2 \Psi}{\partial t\partial y}+\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x\partial y}-\frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^3 \Psi}{\partial y^3}-\frac{\sigma B^2}{\rho}\left(\frac{\partial \Psi}{\partial y}-U\right), \tag{2.6}$$

and the energy equation (2.3) into equation:

$$\frac{\partial T}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial T}{\partial y}=\frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial^2 \Psi}{\partial y^2}\right)^2+\frac{\sigma B^2}{\rho c_p}\left(\frac{\partial \Psi}{\partial y}-U\right)^2. \tag{2.7}$$

Boundary and initial conditions (2.4) are transformed into conditions:

$$
\begin{aligned}
&\Psi = 0, \quad \frac{\partial \Psi}{\partial y} = 0, \quad T = T_w(t) && \text{for } y = 0,\\
&\frac{\partial \Psi}{\partial y} \to U(x,t), \quad T \to T_\infty && \text{for } y \to \infty,\\
&\frac{\partial \Psi}{\partial y} = u_0(x,y), \quad T = T_0(x,y) && \text{for } t = t_0,\\
&\frac{\partial \Psi}{\partial y} = u_1(t,y), \quad T = T_1(t,y) && \text{for } x = x_0.
\end{aligned}
\tag{2.8}
$$

Momentum equation (2.6) is decoupled from the energy equation (2.7) and it can be solved independently. Solution of Eq. (2.6) is used for solving of the Eq. (2.7).

3. Universal equations

In order to analyze the described flow problem, following new variables are introduced:

$$
\begin{aligned}
&x = x, \quad t = t, \quad \eta = \frac{Dy}{h(x,t)},\\
&\Phi(x,t,\eta) = \frac{D\Psi(x,y,t)}{U(x,t)h(x,t)},\\
&\Theta(x,t,\eta) = \frac{T_w - T}{T_w - T_\infty},
\end{aligned}
\tag{3.1}
$$

where D is a normalizing constant, η – dimensionless transversal coordinate, $h(x,t)$ is the characteristic linear scale of transversal coordinate in boundary layer, $\Phi(x,y,\eta)$ – dimensionless stream function and $\Theta(x,t,\eta)$ – dimensionless temperature difference. According to the introduced variables, Eqs. (2.6) and (2.7) are transformed into the following system:

$$
\begin{aligned}
&D^2\frac{\partial^3\Phi}{\partial\eta^3} + f_{1,0}\left(\Phi\frac{\partial^2\Phi}{\partial\eta^2} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2 + 1\right) + (f_{0,1} + g_{1,0})\left(1 - \frac{\partial\Phi}{\partial\eta}\right)\\
&\qquad + \frac{1}{2}(F\Phi + \eta g)\frac{\partial^2\Phi}{\partial\eta^2} = z\frac{\partial^2\Phi}{\partial t\partial\eta} + UzX(\eta;x),\\
&\frac{D^2}{\mathrm{Pr}}\frac{\partial^2\Theta}{\partial\eta^2} - D^2\mathrm{Ec}\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2 - \mathrm{Ec}g_{1,0}\left(1 - \frac{\partial\Phi}{\partial\eta}\right)^2 + (1-\Theta)\,l_1\\
&\qquad + \frac{1}{2}\eta g\frac{\partial\Theta}{\partial\eta} + \frac{1}{2}(F + 2f_{1,0})\Phi\frac{\partial\Theta}{\partial\eta} = z\frac{\partial\Theta}{\partial t} - UzY(x;\eta),
\end{aligned}
\tag{3.2}
$$

where, for the sake of a shorter expression, the notations are introduced:

$$
\begin{aligned}
&z=\frac{h^2}{\nu},\quad g=\frac{\partial z}{\partial t},\quad N=\frac{\sigma B^2}{\rho},\quad g_{1,0}=Nz,\quad F=U\frac{\partial z}{\partial x},\\
&f_{1,0}=z\frac{\partial U}{\partial x},\qquad f_{0,1}=\frac{z}{U}\frac{\partial U}{\partial t},\qquad l_1=\frac{z}{T_w-T_\infty}\frac{dT_w}{dt},\\
&X(x_1;x_2)=\frac{\partial \Phi}{\partial x_1}\frac{\partial^2\Phi}{\partial\eta\partial x_2}-\frac{\partial \Phi}{\partial x_2}\frac{\partial^2\Phi}{\partial x_1\partial\eta},\\
&Y(x_1;x_2)=\frac{\partial \Phi}{\partial x_1}\frac{\partial \Theta}{\partial x_2}-\frac{\partial \Phi}{\partial x_2}\frac{\partial \Theta}{\partial x_1},\\
&\mathrm{Pr}=\frac{\nu\rho c_p}{\lambda}\ \text{– Prandtl number},\\
&\mathrm{Ec}=\frac{U^2}{c_p(T_w-T_\infty)}\ \text{– Eckert number.}
\end{aligned}
\tag{3.3}
$$

Now we introduce the sets of parameters:

– dynamical

$$f_{k,n}=U^{k-1}\frac{\partial^{k+n}U}{\partial x^k\partial t^n}z^{k+n}\qquad (k,n=0,1,2,\dots,\ k\vee n\neq 0),\tag{3.4}$$

– magnetic

$$g_{k,n}=U^{k-1}\frac{\partial^{k-1+n}N}{\partial x^{k-1}\partial t^n}z^{k+n}\qquad (k,n=0,1,2,\dots,\ k\neq 0),\tag{3.5}$$

– temperature

$$l_k=\frac{1}{q}\frac{d^kq}{dt^k}z^k\qquad (k=1,2,\dots),\tag{3.6}$$

where

$$q=T_w-T_\infty,$$

and constant parameter:

$$g=\frac{\partial z}{\partial t}=\mathrm{const},\tag{3.7}$$

which can have various values. It can be noticed that the first parameters are already given in the Eqs. (3.3). Introduced sets of parameters reflect the nature of the change of free stream velocity, alteration characteristic of variable N and the change of body surface temperature, and a part of that, in the integral form (by means of z and $(\partial z/\partial t)$), pre-history of flow in boundary-layer.

Further, using the parameters (3.4)–(3.6) as new independent variables and differentiation operators for x and t:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty} \frac{\partial f_{k,n}}{\partial x}\frac{\partial}{\partial f_{k,n}} + \left\{\begin{matrix} 0, & \text{for } \mathit{\Phi} \\ \sum\limits_{k=1}^{\infty} \dfrac{\partial l_k}{\partial x}\dfrac{\partial}{\partial l_k}, & \text{for } \mathit{\Theta} \end{matrix}\right\} + \sum_{\substack{k=1\\ n=0}}^{\infty} \frac{\partial g_{k,n}}{\partial x}\frac{\partial}{\partial g_{k,n}},\\
\frac{\partial}{\partial t} &= \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty} \frac{\partial f_{k,n}}{\partial t}\frac{\partial}{\partial f_{k,n}} + \left\{\begin{matrix} 0, & \text{for } \mathit{\Phi} \\ \sum\limits_{k=1}^{\infty} \dfrac{\partial l_k}{\partial t}\dfrac{\partial}{\partial l_k}, & \text{for } \mathit{\Theta} \end{matrix}\right\} + \sum_{\substack{k=1\\ n=0}}^{\infty} \frac{\partial g_{k,n}}{\partial t}\frac{\partial}{\partial g_{k,n}},
\end{aligned}
\tag{3.8}
$$

respectively, where parameter derivatives along coordinate x and time t are obtained by differentiation of Eqs. (3.4)–(3.6):

$$
\begin{aligned}
\frac{\partial f_{k,n}}{\partial x} &= \frac{1}{Uz}\left\{\left[(k-1)\,f_{1,0} + (k+n)\,F\right] f_{k,n} + f_{k+1,n}\right\} = \frac{1}{Uz} Q_{k,n},\\
\frac{\partial f_{k,n}}{\partial t} &= \frac{1}{z}\left\{\left[(k-1)\,f_{0,1} + (k+n)\,g\right] f_{k,n} + f_{k,n+1}\right\} = \frac{1}{z} E_{k,n},\\
\frac{\partial g_{k,n}}{\partial x} &= \frac{1}{Uz}\left\{\left[(k-1)\,f_{1,0} + (k+n)\,F\right] g_{k,n} + g_{k+1,n}\right\} = \frac{1}{Uz} K_{k,n},\\
\frac{\partial g_{k,n}}{\partial t} &= \frac{1}{z}\left\{\left[(k-1)\,f_{0,1} + (k+n)\,g\right] g_{k,n} + g_{k,n+1}\right\} = \frac{1}{z} L_{k,n},\\
\frac{\partial l_k}{\partial x} &= \frac{1}{Uz}\left\{kFl_k\right\} = \frac{1}{Uz} M_k,\\
\frac{\partial l_k}{\partial t} &= \frac{1}{z}\left\{(kg - l_1)\,l_k + l_{k+1}\right\} = \frac{1}{z} N_k,
\end{aligned}
\tag{3.9}
$$

where $Q_{k,n}$, $E_{k,n}$, $K_{k,n}$, $L_{k,n}$, M_k, N_k are terms in curly brackets in the obtained equations. It is important to notice that $Q_{k,n}$, $K_{k,n}$, M_k besides the parameters depend on the value $U\partial z/\partial x = F$.

Using parameters (3.4)–(3.6) as new independent variables instead of x and t, operators (3.8) and terms (3.9), system of Eqs. (3.2) is transformed into the system:

$$
\begin{aligned}
\Im_1 = &\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty} \left[E_{k,n}\frac{\partial^2 \mathit{\Phi}}{\partial\eta\,\partial f_{k,n}} + Q_{k,n} X\left(\eta; f_{k,n}\right)\right]\\
&+ \sum_{\substack{k=1\\ n=0}}^{\infty} \left[L_{k,n}\frac{\partial^2 \mathit{\Phi}}{\partial\eta\,\partial g_{k,n}} + K_{k,n} X\left(\eta; g_{k,n}\right)\right],
\end{aligned}
\tag{3.10}
$$

$$\Im_2 = \sum_{\substack{k,n=0 \\ k\vee n\neq 0}}^{\infty} \left[E_{k,n} \frac{\partial \Theta}{\partial f_{k,n}} + Q_{k,n} Y(\eta; f_{k,n}) \right] + \sum_{\substack{k=1 \\ n=0}}^{\infty} \left[L_{k,n} \frac{\partial \Theta}{\partial g_{k,n}} + K_{k,n} Y(\eta; g_{k,n}) \right] + \sum_{k=1}^{\infty} \left[N_k \frac{\partial \Theta}{\partial l_k} + M_k Y(\eta; l_k) \right], \tag{3.10 [cont.]}$$

where the following notations have been used for shorter statement: $\Im_1$ – left side of the first equation of system (3.2), $\Im_2$ – left side of the second equation of system (3.2).

In order to make system (3.10) universal it is necessary to show that value F which appears in terms for $Q_{k,n}$, $K_{k,n}$, M_k can be expressed by means of the introduced parameters. In order to prove it, we start from impulse equation of described problem:

$$\frac{\partial}{\partial t}(U\delta^*) + \frac{\partial}{\partial x}\left(U^2\delta^{**}\right) + U\left(\frac{\partial U}{\partial x} + N\right)\delta^* - \frac{\tau_w}{\rho} = 0, \tag{3.11}$$

where:

$$\delta^*(x,t) = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \qquad \text{displacement thickness,} \tag{3.12}$$

$$\delta^{**}(x,t) = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad \text{momentum thickness,} \tag{3.13}$$

$$\tau_w(x,t) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad \text{shear stress on the body.} \tag{3.14}$$

Now we introduce dimensionless characteristic functions:

$$H^*(x,t) = \frac{\delta^*}{h}, \quad H^{**}(x,t) = \frac{\delta^{**}}{h}, \quad \xi(x,t) = \frac{\tau_w h}{\mu U}, \quad \xi_t = D\frac{\partial \Theta}{\partial \eta}\bigg|_{\eta=0}, \tag{3.15}$$

which, according to Eqs. (3.1) and (3.12)–(3.14), may be expressed in the following form:

$$\begin{gathered} H^*(x,t) = \frac{1}{D}\int_0^\infty \left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta, \quad H^{**}(x,t) = \frac{1}{D}\int_0^\infty \frac{\partial \Phi}{\partial \eta}\left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta, \\ \xi(x,t) = D\,\frac{\partial^2 \Phi}{\partial \eta^2}\bigg|_{\eta=0}. \end{gathered} \tag{3.16}$$

After transition to new independent variables (introduced parameters) in terms of (3.16) values H^*, H^{**}, ξ, ξ_t become functions only of parameters $f_{k,n}$, $g_{k,n}$, l_k, g. Now, using parameters as new independent variables and derivative operators of impulse Eq. (3.11), after simple transformation the next equation is obtained:

$$F = \frac{P}{Q}, \tag{3.17}$$

where, for the sake of shorter expression, the following notations are used:

$$\begin{aligned} P = \xi - f_{1,0}\left(2H^{**} + H^*\right) - \left(f_{0,1} + g_{1,0} + \frac{1}{2}g\right) H^* \\ - \sum_{\substack{k,n=0 \\ k\vee n\neq 0}}^{\infty} \left\{ E_{k,n}\frac{\partial H^*}{\partial f_{k,n}} + \left[(k-1)\, f_{1,0} f_{k,n} + f_{k+1,n}\right] \frac{\partial H^{**}}{\partial f_{k,n}} \right\} \\ - \sum_{\substack{k=1 \\ n=0}}^{\infty} \left\{ L_{k,n}\frac{\partial H^*}{\partial g_{k,n}} + \left[(k-1)\, f_{1,0} g_{k,n} + g_{k+1,n}\right] \frac{\partial H^{**}}{\partial g_{k,n}} \right\}, \\ Q = \frac{1}{2}H^{**} + \sum_{\substack{k,n=0 \\ k\vee n\neq 0}}^{\infty} (k+n)\, f_{k,n}\frac{\partial H^{**}}{\partial f_{k,n}} + \sum_{\substack{k=1 \\ n=0}}^{\infty} (k+n)\, g_{k,n}\frac{\partial H^{**}}{\partial g_{k,n}}. \end{aligned} \tag{3.18}$$

The last two equations define the function F so that it depends only on the introduced parameters. Equation system (3.10) is now universal system of equations of described problem. Boundary conditions, also universal, are given with terms:

$$\begin{aligned} &\Phi = 0, \quad \frac{\partial \Phi}{\partial \eta} = 0, \quad \Theta = 0 \qquad \text{for } \eta = 0, \\ &\Phi \to 1, \quad \Theta \to 1 \quad \text{for } \eta \to \infty, \\ &\Phi = \Phi_0(\eta), \quad \Theta = \Theta_0(\eta) \quad \text{for} \begin{cases} f_{k,n} = 0 & (k,n = 0,1,2,\ldots,\ k\vee n \neq 0), \\ g_{k,n} = 0 & (k,n = 0,1,2,\ldots,\ k \neq 0), \\ l_k = 0 & (k = 0,1,2,\ldots), \\ g = 0, \end{cases} \end{aligned} \tag{3.19}$$

where $\Phi_0(\eta)$ – Blasius solution for stationary boundary-layer on the plate and $\Theta_0(\eta)$ is a solution of the following equation:

$$\frac{D^2}{\mathrm{Pr}}\frac{d^2\Theta_0}{d\eta^2} - D^2\mathrm{Ec}\left(\frac{d^2\Phi_0}{d\eta^2}\right)^2 + \frac{\xi_0}{H^{**}}\Phi_0\frac{d\Theta_0}{d\eta} = 0. \tag{3.20}$$

A universal system of equations (3.10) with boundary conditions (3.19) are correct for wide class of problems where $z = At + C(x)$.

System of equations (3.10) is integrated in m-parametric approximation once for good and all. The obtained characteristic functions can be used both for the adoption of general conclusions about the development of flow in the boundary-layer and to solve particular problems.

Before integration for scale $h(x,t)$ of transversal coordinate in boundary-layer, certain characteristic value is chosen. In this case $h = \delta^{**}$ and according to Eq. (3.14) $H^{**} = 1$, $H^* = \delta^*/\delta^{**} = H$, and Eq. (3.17) now has the form:

$$F = 2\left[\xi - f_{1,0}(2+H) - \left(f_{0,1} + g_{1,0} + \frac{1}{2}g\right)H \right. \\ \left. - \sum_{\substack{k,n=0 \\ k\vee n\neq 0}}^{\infty} E_{k,n}\frac{\partial H}{\partial f_{k,n}} - \sum_{\substack{k=1 \\ n=0}}^{\infty} L_{k,n}\frac{\partial H}{\partial g_{k,n}}\right]. \tag{3.21}$$

Taking parameters $f_{k,n} = 0$, $g_{k,n} = 0$, $g = 0$, the first equation of system (3.11) is simplified into form:

$$\frac{d^3\Phi_0}{d\eta^3} + \frac{\xi_0}{D^2}\Phi_0\frac{d^2\Phi_0}{d\eta^2} = 0, \tag{3.22}$$

and if $D^2 = \xi_0$, then the previous equation becomes the well-known Blasius equation. According to previous statement for normalizing constant D, value 0.47 must be chosen. For selected value h Eq. (3.20) for determining variable Θ_0 became:

$$\frac{1}{\mathrm{Pr}}\frac{d^2\Theta_0}{d\eta^2} + \Phi_0\frac{d\Theta_0}{d\eta} - \mathrm{Ec}\left(\frac{d^2\Phi_0}{d\eta^2}\right)^2 = 0. \tag{3.23}$$

Besides the mentioned advantages of parametric method, some of its shortcomings should be mentioned. The obtained universal equation, in the case when $g = const$, is exact for a broad class of free stream velocities $U(x,t)$, for which $z = At + C(x)$, where A is an arbitrary constant and $C(x)$ is a certain function of the longitudinal coordinate. For other forms of free stream velocities, these equations are approximated universal equations.

For transversal scale of thermal boundary layer, the corresponding value characterizing the dynamic layer is used. This means that the previous history of thermal boundary layer is not taken into account. Thus, if in the problem one is required to study the development of the temperature profile given at some "initial" cross-section of the layer and the temperature field in it at the starting time, then because of the parabolic nature of the boundary-layer equation, a solution in parametric form is possible only at a certain distance from the "initial" cross section and from the starting time.

Before integration of universal equations it is necessary to choose for the scale of transversal coordinate in boundary-layer $h(x,t)$ a certain characteristic value.

Usually $h = \delta^{**}$ is chosen, but this choice affects the solution. The universal equations contain the values which are functionals of the requested solution and this fact somewhat complicates their numerical solution. The obtained universal equations contain an infinite number of variables so that they can be solved only in some appropriate approximations. In this way, practically, the influence of some factors on the development of the boundary layer is neglected.

In this paper, approximate system of Eqs. (3.10) is solved in which influence of parameters $f_{1,0}$, $f_{0,1}$, $g_{1,0}$, l_1 and g are detained and influence of parameters $f_{0,1}$,l_1 derivatives are disregarded. In this way, Eqs. (3.10) is simplified into following form:

$$\begin{aligned}\Im_1 &= g f_{1,0}\frac{\partial^2 \varPhi}{\partial\eta\partial f_{1,0}} + F f_{1,0} X(\eta; f_{1,0}) + g g_{1,0}\frac{\partial^2 \varPhi}{\partial\eta\partial g_{1,0}} + F g_{1,0} X(\eta; g_{1,0}),\\ \Im_2 &= g f_{1,0}\frac{\partial \varTheta}{\partial f_{1,0}} + F f_{1,0} Y(\eta; f_{1,0}) + g g_{1,0}\frac{\partial \varTheta}{\partial g_{1,0}} + F g_{1,0} Y(\eta; g_{1,0}),\end{aligned}\tag{3.24}$$

where function F in approximation has the from:

$$F = 2\left[\xi - f_{1,0}(2+H) - \left(f_{0,1} + g_{1,0} + \frac{1}{2}g\right)H - g f_{1,0}\frac{\partial H}{\partial f_{1,0}} - g g_{1,0}\frac{\partial H}{\partial g_{1,0}}\right].\tag{3.25}$$

Boundary conditions, which coincide with the system of equations, are the conditions:

$$\begin{aligned}&\varPhi = 0, \quad \frac{\partial \varPhi}{\partial \eta} = 0, \quad \varTheta = 0 \quad \text{for } \eta = 0,\\ &\varPhi \to 1, \quad \varTheta \to 1 \quad \text{for } \eta \to \infty,\\ &\varPhi = \varPhi_0(\eta), \quad \varTheta = \varTheta_0(\eta) \quad \text{for } f_{1,0} = 0,\ f_{0,1} = 0,\ g_{1,0} = 0,\ l_1 = 0,\ g = 0,\end{aligned}\tag{3.26}$$

which is obtained from condition (3.19), using the same simplifications as in the equations. First equation of system (3.24) is four-parametric once localized approximation and second is five-parametric twice-localized approximation of system of equations (3.10).

In this paper system of Eqs. (3.24) with appropriate boundary conditions (3.26) is solved using three-diagonal method, known in the East literature as the "progonka" method. Here we briefly give the basics of "progonka" method.

By replacing the derivatives in system of equations (3.24) with corresponding differences quotient they are reduced to general form:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i,\tag{3.27}$$

where $i = 1, 2, \ldots, N-1$ is number of numerical grid nodes.

Boundary conditions for wide class of problems can be reduced to the following form:

$$-C_0 y_0 + B_0 y_1 = -F_0 A_N y_{N-1} - C_N y_N = -F_N, \tag{3.28}$$

where C_0, B_0, F_0, A_N, C_N, F_N are given values. For $C_0 \neq 0$, $C_N \neq 0$, terms (3.28) can be reduced to the form:

$$y_0 = \chi_1 y_1 + \vartheta_1 y_N = \chi_2 y_{N-1} + \vartheta_2, \tag{3.29}$$

where: $\chi_1 = B_0/C_0$, $\vartheta_1 = F_0/C_0$, $\chi_2 = A_N/C_N$, $\vartheta_1 = F_N/C_N$.

Obtained system of equations (3.27) relates the unknown values of mesh functions in three adjacent mesh nodes. Solving such a system is possible using "progonka" method [14, 15]. The "progonka" prescribes a recursive formula:

$$y_i = \alpha_{i+1} y_{i+1} + \beta_{i+1}, \qquad i = 0, 1, \ldots, N-1, \tag{3.30}$$

where α_{i+1}, β_{i+1} are unknown "progonka" coefficients. Eliminating y_{i-1} and y_i from (3.27) and (3.31) "progonka" coefficients are obtained in the following form:

$$\begin{aligned} \alpha_{i+1} &= B_i/(C_i - A_i \alpha_i), \\ \beta_{i+1} &= (A_i \beta_i + F_i)/(C_i - A_i \alpha_i), \qquad i = 0, 1, \ldots, N-1. \end{aligned} \tag{3.31}$$

The algorithm of "progonka" method is based on two cycles. At first, the coefficients α_{i+1} and β_{i+1} are computed in each point i from the expressions (3.31), starting at the left boundary. The coefficients α_1, β_1 are computed from the left boundary condition:

$$\alpha_1 = \chi_1, \qquad \beta_1 = \vartheta_1. \tag{3.32}$$

In the second cycle, the unknowns y_i are computed in each point i according to the recurrent formula (3.30). The procedure starts at the right boundary, where the solution is given by a boundary condition:

$$y_N = (\vartheta_2 + \chi_2 \beta_N)/(1 - \chi_2 \alpha_N). \tag{3.33}$$

During the application of "progonka" method on described problem "instabilty" may occur approaching to the left ($\xi = 0$) and right ($F = 0$) boundary. This problem is solved in the paper by increasing the mesh density near the boundaries.

Obtained results of numerical integration are given in next section in the form of diagrams and conclusions.

4. Results

In this section part of results obtained with numerical integration of equation system (3.24) with boundary conditions (3.26) is given. Figure 1 presents the variations of value ξ in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$. It may be noted that with increase of magnetic parameter value ξ also increase. This remark lead to conclusion that magnetic field postpone the boundary-layer separation and greater postponement is achieved with increasing of magnetic parameter $g_{1,0}$. Figure 1 is given for the case of accelerated outer flow ($f_{0,1} = 0.01$), however the same conclusion is obtained for the case of decelerated outer flow ($f_{0,1} < 0$).

The effect of dynamic parameter $f_{1,0}$ on quantities F and H for different values of magnetic parameter $g_{1,0}$ is shown in Figs. 2 and 3. Figures present the case of accelerated outer flow ($f_{0,1} = 0.01$). It is interesting to note decreasing of

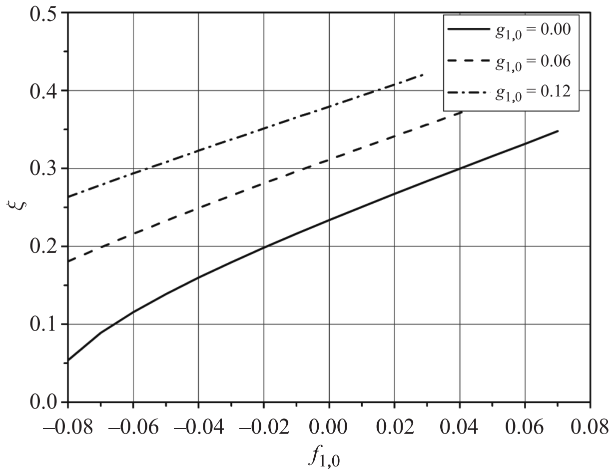

FIG. 1. Variations of quantity ξ in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$ (Pr = 1.0, Ec = 0.3, $f_{0,1} = 0.01$, $l_1 = 0.01$, $g = -0.013$).

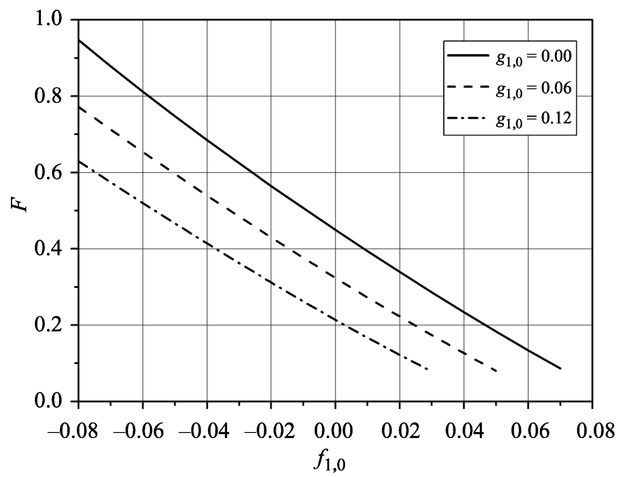

FIG. 2. Variations of quantity F in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$ (Pr = 1.0, Ec = 0.3, $f_{0,1} = 0.01$, $l_1 = 0.01$, $g = -0.013$).

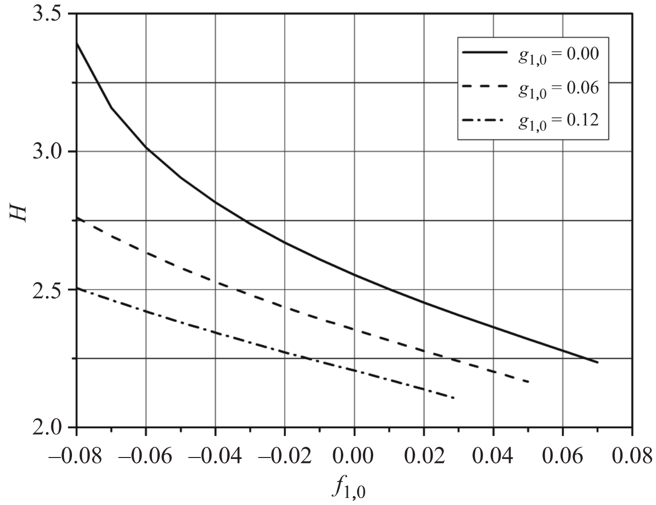

FIG. 3. Variations of quantity H in function of dynamic parameter $f_{1,0}$ for different values of magnetic parameter $g_{1,0}$ ($\mathrm{Pr} = 1.0$, $\mathrm{Ec} = 0.3$, $f_{0,1} = 0.01$, $l_1 = 0.01$, $g = -0.013$).

functions F and H with increase of magnetic parameter. Quantity H decreases also in the case when dynamic parameter increase.

Ratio of free stream velocity and velocity in boundary-layer (function Φ) is shown in the Fig. 4 in function of dimensionless transversal coordinate η for different values of magnetic parameter. From Fig. 4, we observe that with increase of magnetic parameter this ratio also increase and the minimal value is

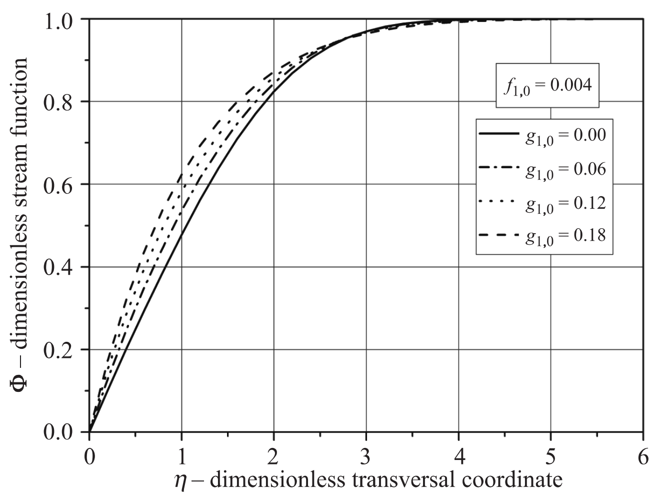

FIG. 4. Stream function for different values of magnetic parameter $g_{1,0}$ ($\mathrm{Pr} = 1.0$, $\mathrm{Ec} = 0.3$, $f_{0,1} = 0.01$, $g = -0.013$).

obtained for the case of non-conducting fluid or for the case of magnetic field absence. This analysis indicates the significant influence of magnetic field on increasing velocity in boundary-layer. The results clearly show that the magnetic field tends to delay or prevent separation. Velocity distribution is given for cross-section, which coincides to dynamic parameter $f_{1,0} = 0.004$ and accelerated outer flow ($f_{0,1} = 0.01$). Same conclusion is valid for other cross-sections of boundary-layer and also for the case of decelerated outer flow.

Momentum equation is decoupled from energy equation (but not vice versa) and all previous results are independent from Prandtl number. Figure 5 shows the effects of Prandtl number (Pr) on the dimensionless temperature for fixed values of $f_{1,0}$, l_1, $g_{1,0}$, Ec, $f_{0,1}$ and g. It is clear from Fig. 5 that the dimensionless temperature at a point increases (fluid temperature decreases) with increase in Prandtl number. The increase of Prandtl number Pr means that the thermal diffusivity decreases. So the rate of heat transfer is decreased due to the decrease of fluid temperature in the boundary-layer.

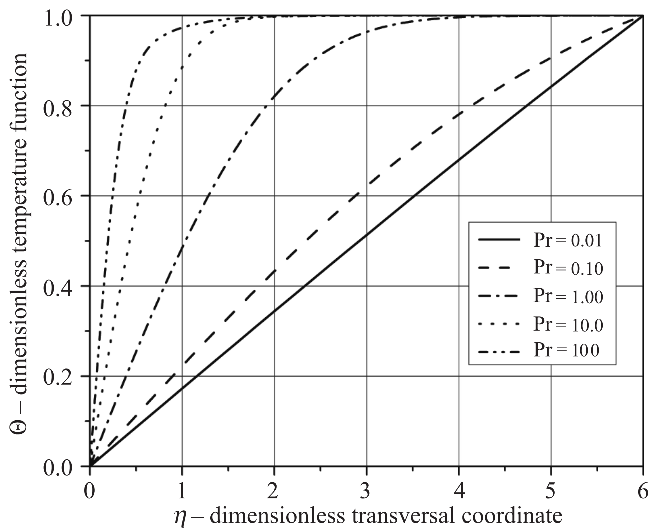

FIG. 5. Temperature function for different values of Prandtl number ($f_{1,0} = 0.01$, $l_1 = 0.02$, $g_{1,0} = 0.0$, Ec $= 1.0$, $f_{0,1} = 0.02$, $g = 0.05$).

In Fig. 6 the variation of dimensionless temperature in function of value η for different values of magnetic parameter is given. Figure presents the results obtained for cross-section which coincide to value of dynamic parameter $f_{1,0} = 0.004$. It may be noted that the highest temperature value is obtained for the case of non-conducting fluid ($g_{1,0} = 0.0$) or for the case of outer magnetic field absence, and increase of magnetic parameter results in temperature decreasing for all values of Prandtl number.

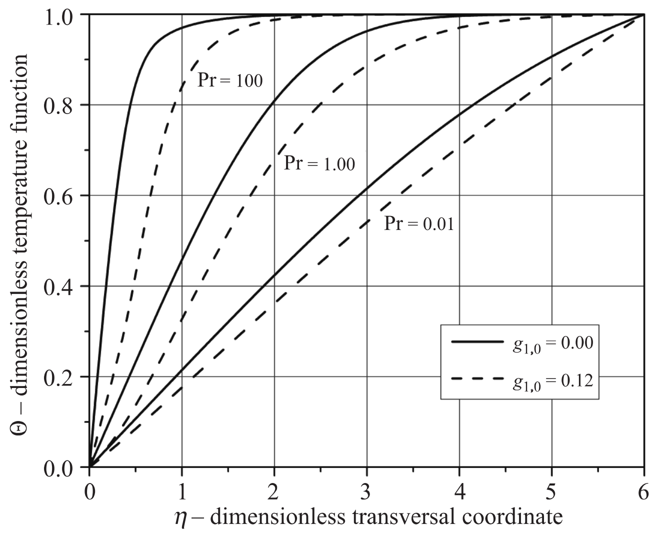

Fig. 6. Temperature function for different values of magnetic parameter $g_{1,0}$ (Ec $= 1.0$, $f_{0,1} = 0.01$, $f_{1,0} = 0.004$, $l_1 = 0.0$, $g = 0.05$).

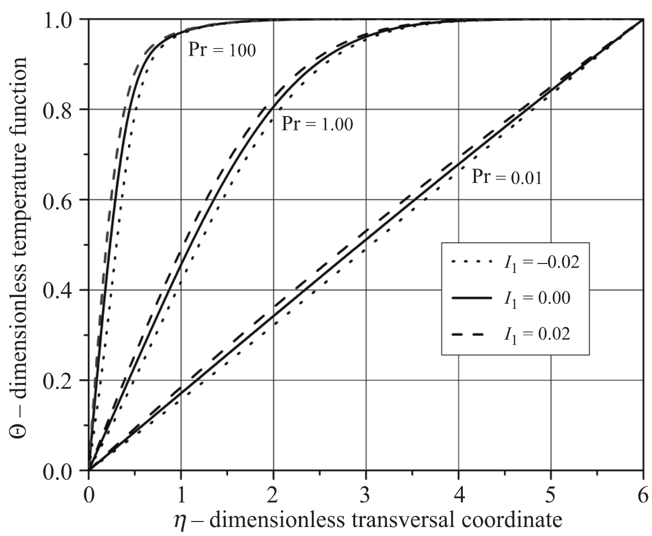

Fig. 7. Temperature function for different values of temperature parameter l_1 ($f_{1,0} = 0.01$, $g_{1,0} = 0.0$, Ec $= 1.0$, $f_{0,1} = 0.01$, $g = 0.05$).

Figures 7 describe the temperature distribution in function of dimensionless transversal coordinate η for different Prandtl numbers and temperature parameter values in boundary-layer cross-sections which coincide to value of dynamic parameters $f_{1,0} = 0.01$. Solid line presents the case of constant body surface temperature. With increasing of temperature parameter ($l_1 > 0$) dimensionless temperature also increase and in the case for body surface temperature decreasing dimensionless temperature also decrease regardless of the Prandtl number value.

Figure 8 presents the variations of value ξ_t in function of dynamic parameter $f_{1,0}$ for different values of Prandtl number. It may be noted that with increase of Prandtl number heat flux which is directly proportional to the value of ξ_t also increase.

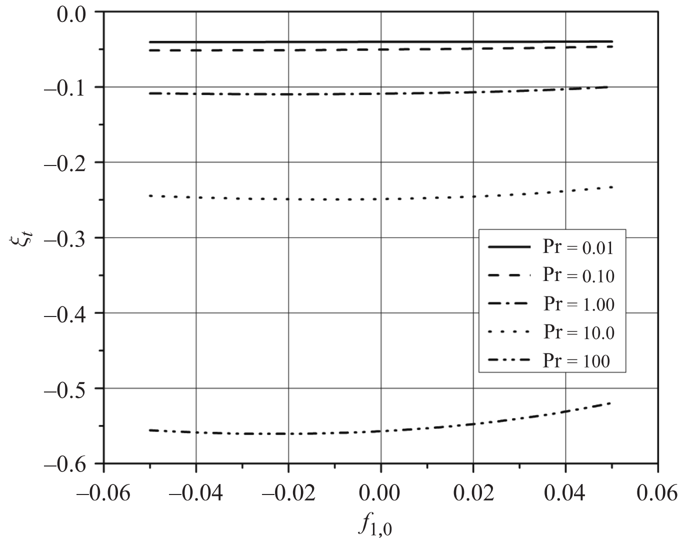

FIG. 8. Variations of quantity ξ_t in function of dynamic parameter $f_{1,0}$ for different values of Prandtl number ($f_{0,1} = 0.01$, $g_{1,0} = 0.0$, $\mathrm{Ec} = 1.0$, $l_1 = 0.0$, $g = 0.05$).

Curves of the dependence of the characteristic functions F on the parameter $f_{1,0}$ for a number of values of the magnetic parameter $g_{1,0}$ and the fixed value the parameter g are shown in Fig. 2. Similarly Fig. 8 shows the dependence of the characteristic function ξ_t on the parameter $f_{1,0}$ for a number of values of Prandtl number and the fixed value the parameter g. In the range of variations of the parameters $-0.08 \leq f_{0,1} \leq 0.07$ and $-0.2 \leq g \leq 0.2$ and of $f_{1,0}$ from the value at the leading critical point to the value at the separation point, the functional F can be approximated with a sufficient degree of accuracy by the linear dependence:

$$F = 0.4411 - 1.8827 f_{0,1} - 5.1462 f_{1,0} - 2,1989 g - 2,0782 g_{1,0}. \tag{4.1}$$

In the range of variation of the parameters $f_{1,0}$; $f_{0,1}$ and g indicated above and in the limits of variations $-0.1 \leq l_1 \leq 0.1$, the characteristic function ξ_t can be represented by the following dependences for $\mathrm{Pr} = 1$ and $\mathrm{Ec} = 1$:

$$\xi_t = -0.1099 + 0.0725 f_{0,1} + 0.045 f_{1,0} - 0.0648 g - 0,4087 l_1. \tag{4.2}$$

In the case of the absence of a magnetic field or non-conducting fluid model presented in this paper is partially reduced to the problem studied by SARAEV [16]. Partial in the sense that the momentum equations are identical and energy equations are are slightly different. As the energy and momentum equations are independent, the results obtained for the momentum equation are compared. Comparing the results it is concluded that they differ in the range of 2%. The cause of these differences is that the localization carried out by all parameters in the Saraev work, which is not the case in given paper. For this reason, here the results are more accurate.

The results of the calculations were used to solve particular problem on the unsteady thermal boundary layer flow past a semi-infinite flat plate. This problem is characterized by the conditions:

$$U(t) = \begin{cases} U & \text{for } t > 0, \\ 0 & \text{at } \ t = 0, \end{cases} \qquad T|_{y=0} = \begin{cases} T_w & \text{for } t > 0, \\ T_\infty & \text{at } \ t = 0. \end{cases} \tag{4.3}$$

Then $f_{1,0} = f_{0,1} = g = 0$ and from (4.1), seeing that $F = 0$, we obtain $g = \partial z/\partial t = 0.2006$, while for $z = \delta^{**^2}/\nu$ we find $\delta^{**} = 0.4478\sqrt{\nu t}$. In the case when $\mathrm{Pr} = 1$, $\mathrm{Ec} = 1$ and $\Delta T = T_w - T_\infty = \mathrm{const}$ using (4.2) we obtain an expression for the heat flux at the wall:

$$q_w = -\lambda \left.\frac{\partial T}{\partial y}\right|_{y=0} = -\lambda \frac{T_w - T_\infty}{h} D \left.\frac{\partial \Theta}{\partial \eta}\right|_{\eta=0} = -\lambda \frac{\Delta T}{h} \xi_t = 0.274 \frac{\Delta T}{\sqrt{\nu t}}. \tag{4.4}$$

The solution presented in paper [17] gives:

$$q_w = 0.564 \frac{\Delta T}{\sqrt{\nu t}} \left(1 - 0.5 \frac{U^2}{c_p \Delta T}\right). \tag{4.5}$$

Second term inside the brackets is obviously the Eckert number and for $\mathrm{Ec} = 1$ previous solution is reduced to a form:

$$q_w = 0.282 \frac{\Delta T}{\sqrt{\nu t}}. \tag{4.6}$$

Agreement between the obtained results is very satisfactory.

5. Conclusion

In this paper, unsteady two-dimensional MHD boundary-layer on the body of temperature varies with time is considered. This problem can be analyzed for each particular case, i.e. for a given free stream velocity. Here is used a quite different approach in order to use benefits of generalized similarity method and universal equations of the discussed problem are derived. These equations are solved numerically in some approximation and the integration results are given in the form of diagrams and conclusions. The obtained results can be used in drawing about general conclusions of boundary-layer development and in calculation of particular problems, as shown in the paper.

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