

Large bending deformations of pressurized curved tubes

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IN THIS PAPER, THE PROBLEM of large pure bending deformations of membrane is considered. The membrane is a sector of torus with a closed cross-section. This membrane is called a curved tube for short. We consider the homogenous, incompressible, isotropic and hyperelastic material. The external load is a constant pressure and bending couples acting at the edges. The equilibrium equations reduce to ordinary differential equations. As an example, the membrane with a constant thickness and with a circular cross-section is investigated.

Key words: membrane, curved tube, pure bending, large deformation, pressure.

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1. Introduction

IN THIS PAPER, THE PROBLEM of large pure bending deformations of a membrane is considered. The membrane is a sector of torus with a closed cross-section. This membrane is called a curved tube for short. The material of our curved tube will be assumed to be homogenous, incompressible, isotropic and hyperelastic. The external load is a constant pressure and pure couples at the edges.

The first linear study of the bending of curved tube goes back to von KÁRMÁN in 1911 [1]. The existing linear theory was extended by CLARK [2], REISSNER [2, 3], AXELRAD [4, 5] and others.

The nonlinear theory of the pure bending of cylindrical membrane was considered in [6–9]. In [6] it was assumed that the circular cylindrical membrane was first inflated into another form of a circular cylinder, which was then subjected to pure bending. It was also assumed that the displacements due to bending is small. The large bending deformations of cylindrical membrane were considered in [7–9].

In this paper, the bending problem of toroidal membrane is considered within the framework of the nonlinear membrane theory. The approach to solving of the problem is presented in [7, 8]. The approach allows us to decompose the

deformation into two parts: an in-plane deformation of the meridional cross-section, plus a rigid rotation of each of these meridional planes about a certain axis by a linearly varying angle. In this case, the equilibrium equations reduce to ordinary differential equations. These equations are derived for a membrane of a constant thickness with arbitrary closed cross-section in Section 2.

As an example, in Section 3 we consider the membrane of a circular cross-section. We investigate the dependence between the bending moment, the internal pressure and the curvature of deformed tube.

2. Equilibrium equations

Let o be the base surface corresponding to the reference configuration of a membrane (Fig. 1). The position of a point on o is determined by the position vector $\mathbf{r}(q^1, q^2)$ in the following form:

$$(2.1) \quad \begin{aligned} \mathbf{r} &= \chi_1(q^1)\mathbf{i}_1 + \chi_2(q^1)\mathbf{e}_2, \\ \mathbf{e}_2 &= \cos \beta q^2 \mathbf{i}_2 + \sin \beta q^2 \mathbf{i}_3. \end{aligned}$$

Here q^1, q^2 are Gaussian coordinates on o , vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are unit vectors of the Cartesian coordinate system.

Let the membrane be loaded by uniformly distributed pressure ξ , and let the deformed surface of membrane O be a sector of torus (Fig. 1). We refer to q^1, q^2

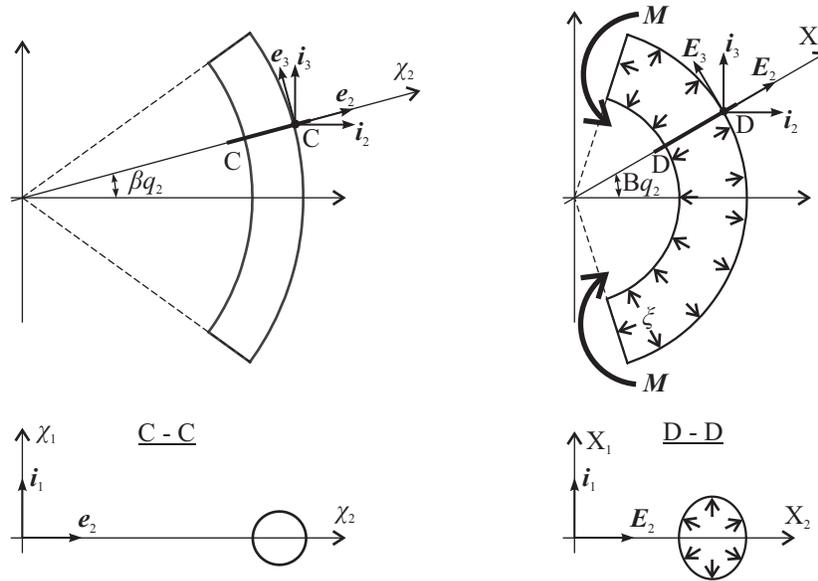


FIG. 1. Reference and deformed configurations of curved tube.

as Lagrangian coordinates of the membrane surface. The position of a point on O is determined by the position vector $\mathbf{R}(q^1, q^2)$ in the form:

$$(2.2) \quad \begin{aligned} \mathbf{R} &= X_1(q^1)\mathbf{i}_1 + X_2(q^1)\mathbf{E}_2, \\ \mathbf{E}_2 &= \cos Bq^2\mathbf{i}_2 + \sin Bq^2\mathbf{i}_3. \end{aligned}$$

It is easy to see that the components of the metric and curvature tensors of the reference and deformed surfaces are functions of q^1 only. We have [9]:

$$(2.3) \quad \begin{aligned} g_{11} &= \chi_1'^2 + \chi_2'^2, & g_{12} &= 0, & g_{22} &= \beta^2\chi_2^2, \\ G_{11} &= X_1'^2 + X_2'^2, & G_{12} &= 0, & G_{22} &= B^2X_2^2, \\ B_{11} &= \frac{X_1''X_2' - X_2''X_1'}{\sqrt{X_1'^2 + X_2'^2}}, & B_{12} &= 0, & B_{22} &= \frac{B^2X_2X_1'}{\sqrt{X_1'^2 + X_2'^2}}. \end{aligned}$$

In this case the equilibrium equations of membrane reduce to ordinary differential equations [9]. It is easy to see that the resultant force and resultant moment are independent of the coordinate q^2 at the cross-section. In addition, the resultant force is zero. The resultant moment with respect to center of mass of the cross-section Y_{2C} may be written in the form [9]:

$$(2.4) \quad \mathbf{M} = M\mathbf{i}_1 = \mathbf{i}_1 \oint \sqrt{G_{11}}G_{22}L^{22}(Y_{2C} - X_2)dq^1.$$

Here, $L^{\alpha\beta}$ ($\alpha, \beta = 1, 2$) are the components of the Cauchy-type stress resultant tensor. The constitutive equations for isotropic elastic membrane may be written in the following form [10]:

$$(2.5) \quad L^{\alpha\beta} = \frac{2h}{\eta} \sqrt{\frac{g_{11}g_{22}}{G_{11}G_{22}}} \frac{\partial W}{\partial G_{\alpha\beta}}, \quad \eta = \begin{cases} 1, & \alpha = \beta, \\ 2, & \alpha \neq \beta. \end{cases}$$

Here, h is the initial thickness of membrane, W is the strain energy density per unit surface area of membrane.

Let us introduce the functions $\lambda_1(q^1)$, $\lambda_2(q^1)$ and $\psi(q^1)$:

$$(2.6) \quad \begin{aligned} \lambda_1(q^1) &= \sqrt{\frac{G_{11}}{g_{11}}}, \\ \lambda_2(q^1) &= \sqrt{\frac{G_{22}}{g_{22}}}, \\ \text{tg } \psi(q^1) &= \frac{X_2'}{X_1'}. \end{aligned}$$

The equilibrium equations may be rewritten in the form [9]:

$$\begin{aligned}
 \frac{\partial^2 W}{\partial \lambda_1^2} \lambda_1' - \left(\frac{\partial W}{\partial \lambda_2} - \lambda_1 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \sqrt{\frac{g_{11}}{g_{22}}} B \sin \psi + \left(\frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \frac{g'_{22}}{2g_{22}} &= 0, \\
 \lambda_2' - B \sqrt{\frac{g_{11}}{g_{22}}} \lambda_1 \sin \psi + \frac{g'_{22}}{2g_{22}} \lambda_2 &= 0, \\
 \frac{\partial W}{\partial \lambda_1} \psi' - B \sqrt{\frac{g_{11}}{g_{22}}} \frac{\partial W}{\partial \lambda_2} \cos \psi - \frac{\xi}{h} \sqrt{g_{11}} \lambda_1 \lambda_2 &= 0, \\
 X_1' = \sqrt{g_{11}} \lambda_1 \cos \psi, \quad X_2' = \sqrt{g_{11}} \lambda_1 \sin \psi. &
 \end{aligned}
 \tag{2.7}$$

We consider a closed cross-section. Hence the boundary conditions are the periodicity requirements on the unknown functions. The functions $\lambda_1(q^1)$, $\lambda_2(q^1)$ and $\psi(q^1)$ are independent. If $q^1 \in [q_1^1, q_2^1]$, then the boundary conditions may be written in the following form:

$$\lambda_1(q_1^1) = \lambda_1(q_2^1), \quad \lambda_2(q_1^1) = \lambda_2(q_2^1), \quad \psi(q_1^1) = \psi(q_2^1).$$

The functions $X_1(q^1)$ and $X_2(q^1)$ may be obtained, after calculation of $\lambda_1(q^1)$, $\lambda_2(q^1)$ and $\psi(q^1)$, by solving the Eqs. (2.3) and (2.6) or the last equations (2.7). The nonlinear boundary problem (2.7) is solved by a numerical method. We use the shooting method. This method was applied to study the large bending deformations of a cylindrical membrane [9].

3. Results

We consider the membrane made of a neo-Hookean material. The strain energy function has the form

$$W = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2} - 3). \tag{3.1}$$

Let the cross-section be a circle in the reference configuration. The functions $\chi_1(q^1)$ and $\chi_2(q^1)$ are given as

$$\chi_1(q^1) = r_0 \sin q^1, \quad \chi_2(q^1) = \beta^{-1} - r_0 \cos q^1, \quad q^1 \in [0; 2\pi]. \tag{3.2}$$

We assume that

$$r_0 = 1, \quad h = 0.001, \quad \mu = 1, \quad \beta = 0.1, \quad q^2 \in [-2\pi; 2\pi]. \tag{3.3}$$

Our aim is to study the dependence of the resultant moment at the cross-section (bending moment), on the internal pressure and curvature of the de-

formed curved tube. Let us introduce the dimensionless parameters

$$(3.4) \quad \begin{aligned} M^* &= -\frac{M}{h\mu r_0^2}, & p^* &= \frac{\xi r_0}{h\mu}, & B^* &= \frac{B}{\beta}, \\ L_1^* &= \frac{G_{11}^2 L^{11} r_0}{\mu h}, & L_2^* &= \frac{G_{22}^2 L^{22} r_0}{\mu h}. \end{aligned}$$

Here, L_1^*, L_2^* are the dimensionless resultant stresses.

If $B^* \geq 1$ ($B \geq \beta$) for $p^* > 0$, we call the deformation (2.2) a direct bending. If $B^* < 1$ ($B < \beta$) for $p^* > 0$, we call the deformation (2.2) the inverse bending. The deformation (2.2) with $0 \leq B^* < 1$ is called the unbending of a curved tube.

The inflation of curved tube ($p^* > 0$) accompany the unbending ($0 \leq B^* < 1$) without application of the bending moments ($M^* = 0$) at the ends [11]. Straightening of curved tube ($B^* = 0$) does not occur only by the internal pressure ($p^* > 0$) for the neo-Hookean material. The dependence of pressure p^* on the parameter B^* is shown in Fig. 2 for different initial parameters $\beta = 0.05, 0.1, 0.2$.

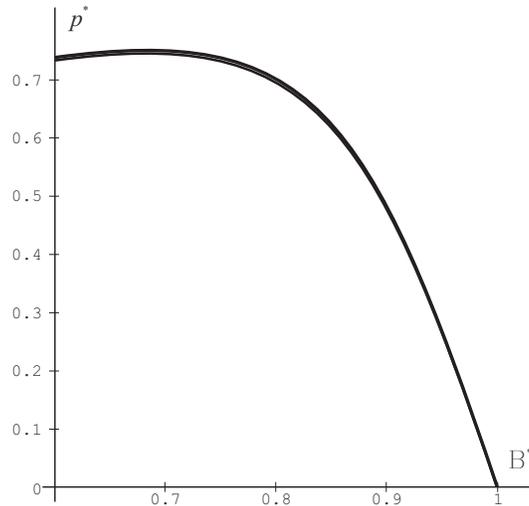


FIG. 2. The internal pressure p^* as a function of parameter B^* ($M^* = 0$).

We consider the cases in which B^* is a fixed. The dependencies of the bending moment M^* on the pressure p^* are shown in Fig. 3 for $B^* = -2, -1, 1, 2$. These dependencies have the maximum pressures. They are greatly different for positive and negative curvatures. But first an increase in the pressure p^* lead to an increase of the absolute value of the bending moment M^* in both cases. This part of the dependence of the bending moment M^* on the pressure p^* will be called the first part. The rest part of the dependence M^* on p^* will be called the second part.

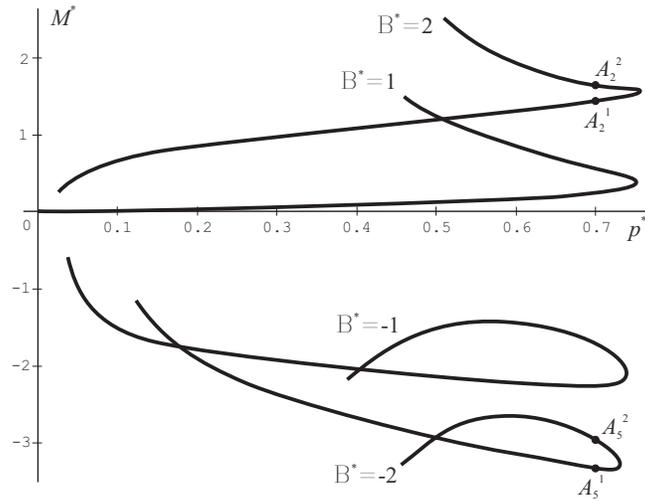


FIG. 3. Bending moment M^* as a function of internal pressure p^* .

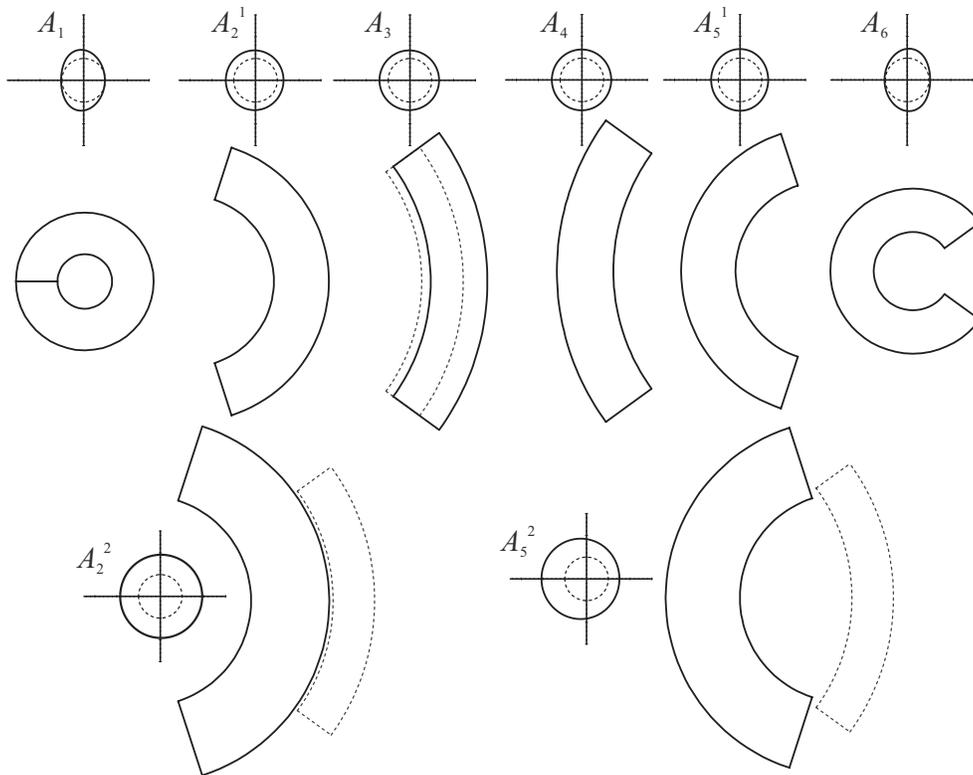


FIG. 4. The superficial and cross-sectional views of the deformed curved tube ($p^* = 0.7$). The dotted line corresponds the reference configuration.

In the second part an increase of the bending moment M^* is accompanied by a decrease in the pressure p^* for $B^* \geq 1$. From Fig. 3, we see that there are the two equilibrium states. For example, the states A_2^1 and A_2^2 (Fig. 3) have the two different bending moments for $p^* = 0.7$ and $B^* = 2$. These states differ in the shape of deformed tube (Fig. 4) and the stresses (Fig. 5).

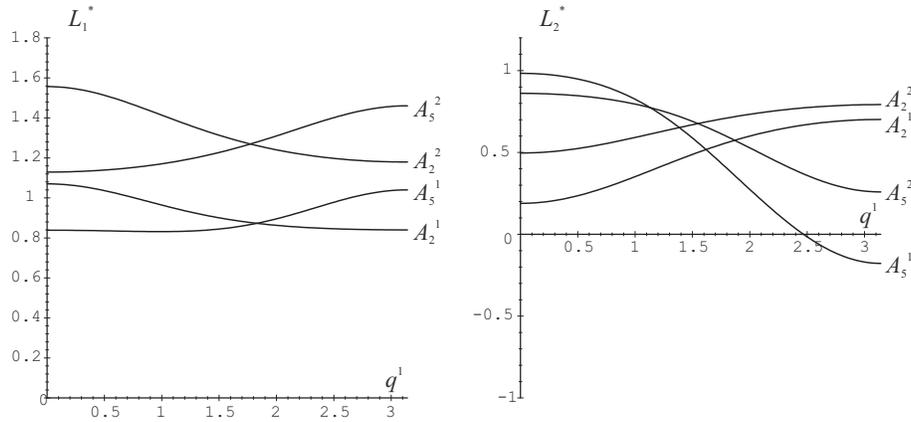


FIG. 5. The stresses ($B^* = 2$).

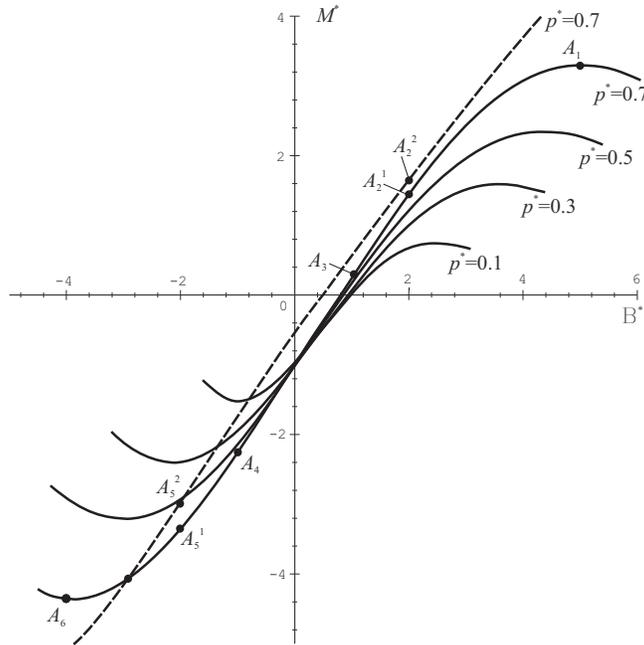


FIG. 6. The bending moment M^* as a function of the parameter B^* .

For $B^* < 0$ the dependence of the bending moment M^* on the pressure p^* is more complex in the second part (Fig. 3). In the case $B^* < 0$ there are the two equilibrium states. For example, the states A_5^1 and A_5^2 (Fig. 3) have the two different bending moments for $p^* = 0.7$ and $B^* = -2$. These states differ in the shape of deformed tube (Fig. 4) and the stresses (Fig. 5). Moreover, for $B^* < 0$ the dependence of M^* on p^* has self-intersection point, i.e., there are the two equilibrium states with the same pressure p^* , bending moment M^* and parameter B^* .

Let us now consider the cases in which the pressure p^* is fixed. The dependence between the parameter B^* and the bending moment M^* is shown in Fig. 6.

The solid lines correspond to the case, when the pressure and bending moment are given from the first part of the dependence of M^* on p^* . The dashed

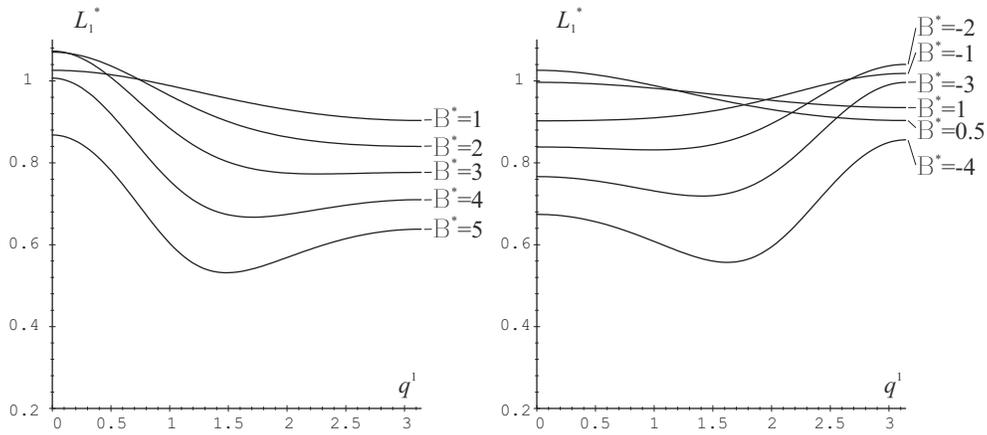


FIG. 7. The stresses L_1^* ($p^* = 0.7$).

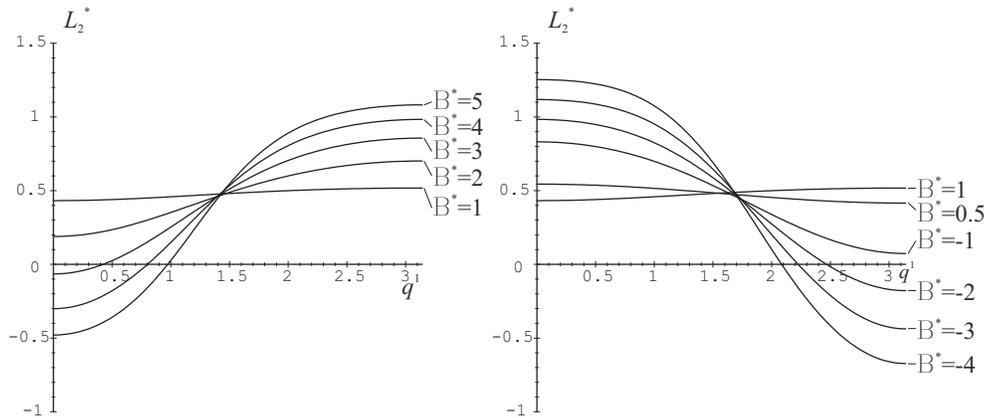


FIG. 8. The stresses L_2^* ($p^* = 0.7$).

line corresponds to the second part of the dependence M^* on p^* . The dependence of M^* on B^* is similar for different pressures. They have maximum and minimum of the bending moments. The planform and cross-sectional views of the deformed curved tube are shown in Fig. 4 for $p^* = 0.7$.

The true stresses in the membrane are shown for pressure $p^* = 0.7$ in Fig. 7 and Fig. 8. The stresses L_1^* are positive. The longitudinal stresses L_2^* can be compressive. We note that the compressive stresses arise before the extreme bending moments.

4. Conclusions

The pure bending of the curved tube subjected to an internal pressure and bending couples at the edges has been considered. The problem was investigated using the nonlinear membrane theory. The equilibrium equations were derived for a curved tube with an arbitrary cross-section, made of incompressible isotropic hyperelastic material.

We studied the deformation of the curved tube with a circular cross-section made of a neo-Hookean material. The dependence (pressure–bending moment) is presented for different fixed curvatures. The dependence (curvature–bending moment) is presented for different fixed pressures. It was obtained from numerical results that there are two equilibrium states in some cases and there are the limits of the external loads (the pressure and the bending moment).

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