Invariants of a Cartesian tensor of rank 3

F. AHMAD

Centre for Advanced Mathematics and Physics National University of Sciences and Technology Sector H-12, Islamabad, Pakistan e-mail: faizmath@hotmail.com

GENERAL METHODS ARE APPLIED to find complete sets of invariants of a tensor of rank 3. When the results are specialized to the piezoelectric tensor, it is found that the tensor has no linear invariant. Also under SO(2), as well as SO(3), the piezoelectric tensor has five quadratic invariants. The sets of invariants are complete.

Key words: Cartesian tensor, linear invariants, quadratic invariants, piezoelectric tensor, complete set of invariants.

Copyright © 2011 by IPPT PAN

1. Introduction

LET c_{ijkl} , i, j, k, l = 1, 2, 3, DENOTE COMPONENTS of the elasticity tensor. It is well-known that a *complete* set of linear invariants of the tensor, under SO(3), consists of the following two invariants:

$$C_1 = c_{11} + c_{22} + c_{33} + 2(c_{12} + c_{23} + c_{13}),$$

$$C_2 = c_{11} + c_{22} + c_{33} + 2(c_{44} + c_{55} + c_{66}),$$

where, in the above, the familiar two-index notation has been used. For example $c_{23} = c_{2233}$ and $c_{55} = c_{1313}$, etc. [5, 6]. Note that if the above notation is used for the stiffness tensor, then a slightly different representation has to be adopted for the compliance tensor [11, Ch. 3]. The *completeness* of the above set means that any other linear invariant of the tensor must be a linear combination of the two members of the set. Invariants of a tensor play an important role in describing the physical phenomena. For example, consider an arbitrary set of mutually orthogonal unit vectors \mathbf{n}_i , i = 1, 2, 3. Let $\mathbf{v}_j(\mathbf{n}_i)$, j = 1, 2, 3, denote the velocity of a body wave propagating in the direction specified by \mathbf{n}_i . Then

$$\sum_{j=1}^{3} (\mathbf{v}_j(\mathbf{n}_1))^2 + (\mathbf{v}_j(\mathbf{n}_2))^2 + (\mathbf{v}_j(\mathbf{n}_3))^2 = \frac{C_2}{\rho},$$

where ρ denotes the density of the material [3]. VANNUCCI [14] has given an excellent description and examples of practical usefulness of tensor invariants.

F. Ahmad

In [13], TING initiated the study of invariants of the product tensor, $c_{ijkl}c_{pqrs}$, called the quadratic invariants of the elasticity tensor, and he found two invariants of this tensor. This number was increased to four by AHMAD [1]. The question of completeness remained open until NORRIS [8] showed that the set of seven quadratic invariants comprising four of AHMAD [1] and C_1^2, C_2^2, C_1C_2 , indeed form a complete set. It should be noted that a quadratic invariant of a tensor **T** is the same as a *linear* invariant of the product tensor **T** \otimes **T**.

The task of checking the completeness of a set of invariants is made easier if one knows *a priori* the number of elements in the set. The following result of AHMAD and RASHID [2] is fundamental in this regard.

THEOREM 1. The number of linear invariants of a tensor of rank r under SO(2) or SO(3), is the same as the dimension of the space of isotropic tensors of rank r, respectively in two or three dimensions.

An immediate corollary of the above Theorem is

COROLLARY 2. A tensor of odd rank defined over a space of an even dimension has no linear invariant.

By employing the methods of group theory, [4], the following results were established in [2, 10]:

THEOREM 3. The number of independent linear invariants of a tensor of rank r under SO(2) and SO(3), respectively denoted by $I_2(r)$ and $I_3(r)$, is given by

$$I_2(r) = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \frac{r!}{((r/2)!)^2} & \text{if } r \text{ is even} \end{cases}$$
$$I_3(r) = \sum_{i=0}^{\left[\frac{r+1}{2}\right]} \frac{r!(r+1-3i)}{i!i!(r+1-2i)!},$$

where $\left[\frac{r+1}{2}\right]$ denotes the largest integer less than or equal to $\frac{r+1}{2}$.

It may be mentioned that the expression for $I_3(r)$ is equivalent to a result of RACAH [9]. The number of invariants may be less than $I_2(r)$ or $I_3(r)$, if the tensor has symmetry with respect to one or more pairs of indices.

Invariants of a tensor of rank 3 play a significant role when electromechanical coupling exists. JERPHAGNON [7] used invariants of the third-rank Cartesian tensor to describe optical phenomenon. SCHRÖDER and GROSS [12] have developed an invariant formulation of the electromechanical enthalpy function for the transversely isotropic materials. From Theorem 1, it easily follows that a Cartesian tensor of rank 3 can have only one *linear* invariant. However, it may have several quadratic invariants and VANNUCCI [14] has used the method of *polar analysis* to find invariants of the *plane* piezoelectric tensor under the special orthogonal group SO(2), i.e. the indices are constrained to take values 1 and 2 and the coordinate transformations are represented by the rotation matrix

$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.$$

He found four independent quadratic invariants and one invariant of the fourth order. We shall apply results of Theorems 1 and 2 first to find linear as well as quadratic invariants of a tensor of rank 3, not only with respect to SO(2) but also SO(3). Specialization to the piezoelectric tensor yields zero linear invariant and five quadratic invariants under both SO(2) and SO(3). The set of invariants, with respect to SO(2) i.e. the *plane* invariants, is equivalent to the results of VANNUCCI [14].

2. Invariants of a tensor of rank 3

2.1. Invariants under SO(3)

Let T_{ijk} denote components of a tensor of rank 3. First we shall find the invariants without any assumption of symmetry, later we shall specialize to the case of piezoelectric tensor which is symmetric with respect to the last two indices. Theorem 3 predicts one linear invariant. Also the permutation tensor, ϵ_{ijk} , is the only isotropic tensor of rank 3 in three dimensions. Thus the only linear invariant of **T** is the following:

(2.1)
$$L = \epsilon_{ijk} T_{ijk} = T_{123} + T_{231} + T_{312} - T_{321} - T_{132} - T_{213}$$

The piezoelectric tensor e_{ijk} has symmetry with respect to the last two indices. It is obvious that if we replace T_{ijk} by e_{ijk} in the above expression, we get L = 0. Thus there is no linear invariant of the tensor with respect to SO(3). Also, by the Corollary of Theorem 1, no linear invariant exists with respect to SO(2).

Next we consider quadratic invariants of the tensor i.e. linear invariants of $T_{ijk}T_{pqr}$, which is a tensor of rank 6 such that the first three and the last three indices can be interchanged, i.e. it has the symmetry $ijk \leftrightarrow pqr$. Applying Theorem 3, we find $I_3(6) = 15$. Also tensors of the form $\delta_{ij}\delta_{kp}\delta_{qr}$, with the indices permuted among themselves, are isotropic tensors of rank 6. Their number is $\binom{6}{2} = 15$. Since every isotropic tensor must be of this form, we conclude that this set is complete. From this set, we choose those members which have the required symmetry $ijk \leftrightarrow pqr$. This gives us the following tensors:

$$(2.2) \qquad \begin{aligned} \delta_{ij}\delta_{kr}\delta_{pq}, \quad \delta_{ik}\delta_{jq}\delta_{pr}, \quad \delta_{ip}\delta_{jk}\delta_{qr}, \quad \delta_{ip}\delta_{jq}\delta_{kr}, \quad \delta_{ip}\delta_{jr}\delta_{kq}, \quad \delta_{iq}\delta_{jp}\delta_{kr}, \\ \delta_{ir}\delta_{jq}\delta_{kp}, \quad \frac{1}{2}(\delta_{ij}\delta_{kp}\delta_{qr} + \delta_{ir}\delta_{jk}\delta_{pq}), \quad \frac{1}{2}(\delta_{ij}\delta_{kq}\delta_{pr} + \delta_{ik}\delta_{jr}\delta_{pq}), \\ \frac{1}{2}(\delta_{ik}\delta_{jp}\delta_{qr} + \delta_{iq}\delta_{jk}\delta_{pr}), \quad \frac{1}{2}(\delta_{iq}\delta_{jr}\delta_{kp} + \delta_{ir}\delta_{jp}\delta_{kq}). \end{aligned}$$

Thus we get the following complete set of quadratic invariants of T_{ijk} under SO(3)

$$T_{1} = T_{iik}T_{ppk}, \qquad T_{2} = T_{iji}T_{pjp}, \qquad T_{3} = T_{ijj}T_{iqq}, \qquad T_{4} = T_{ijk}T_{ijk}, T_{5} = T_{ijk}T_{ikj}, \qquad T_{6} = T_{ijk}T_{jik}, \qquad T_{7} = T_{ijk}T_{kji}, T_{8} = \frac{1}{2}(T_{iik}T_{kqq} + T_{ijj}T_{ppi}) = T_{iik}T_{kpp}, (2.3) \qquad T_{9} = \frac{1}{2}(T_{iik}T_{pkp} + T_{iji}T_{ppj}), T_{10} = \frac{1}{2}(T_{ijk}T_{jqq} + T_{ijj}T_{pip}), T_{11} = \frac{1}{2}(T_{ijk}T_{kij} + T_{ijk}T_{jki}).$$

Since L is a linear invariant of T_{ijk} , L^2 must be a quadratic invariant of the tensor and it must be possible to express L^2 as a linear combination of the members of (2.3). To see this, we use the identity

$$L^{2} = \epsilon_{ijk}\epsilon_{pqr}T_{ijk}T_{pqr}$$

= $(\delta_{ip}\delta_{jq}\delta_{kr} - \delta_{ir}\delta_{jq}\delta_{kp} + \delta_{iq}\delta_{jr}\delta_{kp} - \delta_{ip}\delta_{jr}\delta_{kq} + \delta_{ir}\delta_{jp}\delta_{kq} - \delta_{iq}\delta_{jp}\delta_{kr})T_{ijk}T_{pqr}$
= $T_{4} - T_{5} - T_{6} - T_{7} + 2T_{11}.$

2.2. Invariants under SO(2)

Consider the tensor T_{ijk} , i, j, k = 1, 2. The matrix representing a rotation through an angle φ is

(2.4)
$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix},$$

and the coordinate transformations form the special orthogonal group SO(2). Now we have a tensor of rank 3 in two dimensions. Such a tensor is usually called a *plane* tensor. By Corollary 2, no linear or cubic invariant of the tensor exists. We look for the linear invariants of the product tensor $T_{ijk}T_{pqr}$. It has rank 6 in two dimensions. From Theorem 3, $I_2(6) = 20$. This means that there are 20 independent isotropic tensors of the form $\delta_{ij}\delta_{kp}\delta_{qr}$ or $\delta_{ij}\delta_{kl}\epsilon_{qr}$. In three dimensions, there were 15 independent tensors of the first type. This number reduces to ten in two dimensions because of identities of the form

$$\epsilon_{ijk}\epsilon_{pqr} = \delta_{ip}\delta_{jq}\delta_{kr} - \delta_{ir}\delta_{jq}\delta_{kp} + \delta_{iq}\delta_{jr}\delta_{kp} - \delta_{ip}\delta_{jr}\delta_{kq} + \delta_{ir}\delta_{jp}\delta_{kq} - \delta_{iq}\delta_{jp}\delta_{kr},$$

and the fact that, in two dimensions, the left side vanishes.

A complete set of nine independent tensors of the first type, with the symmetry $ijk \leftrightarrow pqr$, is found to be

$$(2.5) \qquad \begin{aligned} \delta_{ij}\delta_{kr}\delta_{pq}, \quad \delta_{ik}\delta_{jq}\delta_{pr}, \quad \delta_{ip}\delta_{jk}\delta_{qr}, \quad \delta_{ip}\delta_{jr}\delta_{kq}, \quad \delta_{ir}\delta_{jq}\delta_{kp}, \quad \delta_{iq}\delta_{jp}\delta_{kr}, \\ \frac{1}{2}(\delta_{iq}\delta_{jk}\delta_{pr} + \delta_{jp}\delta_{qr}\delta_{ik}), \quad \frac{1}{2}(\delta_{iq}\delta_{jr}\delta_{kp} + \delta_{ir}\delta_{jp}\delta_{kq}), \\ \frac{1}{2}(\delta_{ir}\delta_{jk}\delta_{pq} + \delta_{ij}\delta_{kp}\delta_{qr}). \end{aligned}$$

A similar calculation with tensors of the form $\delta_{ij}\delta_{kl}\epsilon_{qr}$ produces the following set of independent tensors with desired symmetry:

(2.6)
$$\frac{\frac{1}{2}(\delta_{ik}\delta_{pq}\epsilon_{jr}+\delta_{ij}\delta_{pr}\epsilon_{kq}), \quad \frac{1}{2}(\delta_{ip}\delta_{kq}\epsilon_{jr}+\delta_{ip}\delta_{jr}\epsilon_{kq}),}{\frac{1}{2}(\delta_{iq}\delta_{kp}\epsilon_{jr}+\delta_{ir}\delta_{jp}\epsilon_{kq}), \quad \frac{1}{2}(\delta_{ir}\delta_{kq}\epsilon_{jp}+\delta_{kp}\delta_{jr}\epsilon_{iq}),}{\frac{1}{2}(\delta_{ir}\delta_{jp}\epsilon_{kq}+\delta_{kp}\delta_{iq}\epsilon_{rj}), \quad \frac{1}{2}(\delta_{ir}\delta_{jq}\epsilon_{kp}+\delta_{kp}\delta_{jq}\epsilon_{ri}),}{\frac{1}{2}(\delta_{ip}\delta_{jk}\epsilon_{qr}+\delta_{ip}\delta_{qr}\epsilon_{jk}).}$$

Combining (2.5) and (2.6), we have the following set of sixteen *plane* quadratic invariants of the tensor T_{ijk} :

$$M_{1} = T_{iik}T_{ppk}, \qquad M_{2} = T_{iji}T_{pjp}, \qquad M_{3} = T_{ijj}T_{iqq}, \\M_{4} = T_{ijk}T_{ikj}, \qquad M_{5} = T_{ijk}T_{kji}, \qquad M_{6} = T_{ijk}T_{jik}, \\M_{7} = \frac{1}{2}(T_{ijj}T_{pip} + T_{iji}T_{jqq}), \qquad M_{8} = \frac{1}{2}(T_{ijk}T_{kij} + T_{ijk}T_{jki}), \\(2.7) \qquad M_{9} = \frac{1}{2}(T_{ijj}T_{ppi} + T_{iik}T_{kqq}), \qquad M_{10} = \frac{1}{2}(\epsilon_{jr}T_{iji}T_{ppr} + \epsilon_{kq}T_{iik}T_{pqp}), \\M_{11} = \frac{1}{2}(\epsilon_{jr}T_{ijk}T_{ikr} + \epsilon_{kq}T_{ijk}T_{iqj}), \qquad M_{12} = \frac{1}{2}(\epsilon_{jr}T_{ijk}T_{kir} + \epsilon_{kq}T_{ijk}T_{jqi}), \\M_{13} = \frac{1}{2}(\epsilon_{jp}T_{ijk}T_{pki} + \epsilon_{iq}T_{ijk}T_{kqj}), \qquad M_{14} = \frac{1}{2}(\epsilon_{kq}T_{ijk}T_{jqi} + \epsilon_{rj}T_{ijk}T_{kir}), \\M_{15} = \frac{1}{2}(\epsilon_{kp}T_{ijk}T_{pji} + \epsilon_{ri}T_{ijk}T_{kjr}), \qquad M_{16} = \frac{1}{2}(\epsilon_{qr}T_{ijj}T_{iqr} + \epsilon_{jk}T_{ijk}T_{iqq}). \end{cases}$$

In the above, it is understood that a repeated index is summed over. The tensor ϵ_{ij} is the *permutation* tensor in two dimensions with $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$.

Also note that, although in T_i , i = 1, ..., 11 and M_i , i = 1, ..., 16 a pair of expressions may appear identical, but they represent distinct invariants due to different ranges over which the indices take their values. For example,

$$T_1 = T_{iik}T_{ppk} = (T_{111} + T_{221} + T_{331})^2 + (T_{112} + T_{222} + T_{332})^2 + (T_{113} + T_{223} + T_{333})^2,$$

but

$$M_1 = T_{iik}T_{ppk} = (T_{111} + T_{221})^2 + (T_{112} + T_{222})^2.$$

3. Quadratic invariants of the piezoelectric tensor

The piezoelectric tensor e_{ijk} is symmetric with respect to the last two indices i.e. $e_{ijk} = e_{ikj}$. In this Section, we shall find complete sets of quadratic invariants of e_{ijk} , both under SO(3), as well as SO(2), by simply replacing T_{ijk} by e_{ijk} in the results of the last section and using its symmetry to simplify the expressions.

3.1. Invariants under SO(3)

Quadratic invariants of T_{ijk} , without any assumption of symmetry, are given in (2.3). When we replace T_{ijk} by e_{ijk} , we find that, due to symmetry in the last two indices, the following relations exist among invariants:

$$T_2 = T_1, T_5 = T_4, T_9 = T_1,$$

$$T_6 = T_7, T_{10} = T_8, T_{11} = \frac{1}{2}(T_6 + T_7) = T_6,$$

leaving only five of them to be linearly independent. Thus a complete set of quadratic invariants of the piezoelectric tensor consists of the following five members, corresponding to T_1, T_3, T_4, T_6 and T_8 .

$$\begin{split} E_1 &= e_{iik}e_{ppk} = e_{11}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{24}^2 + e_{26}^2 + e_{33}^2 + e_{34}^2 + e_{35}^2 \\ &\quad + 2e_{16}e_{22} + 2e_{15}e_{24} + 2e_{11}e_{26} + 2e_{15}e_{33} + 2e_{24}e_{33} + 2e_{16}e_{34} \\ &\quad + 2e_{22}e_{34} + 2e_{11}e_{35} + 2e_{26}e_{35}, \end{split}$$

$$\begin{split} E_2 &= e_{ijj}e_{iqq} = e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{21}^2 + e_{22}^2 + e_{23}^2 + e_{31}^2 + e_{32}^2 + e_{33}^2 \\ &\quad + 2e_{11}e_{12} + 2e_{11}e_{13} + 2e_{12}e_{13} + 2e_{21}e_{22} + 2e_{21}e_{23} + 2e_{22}e_{23} \\ &\quad + 2e_{31}e_{32} + 2e_{31}e_{33} + 2e_{32}e_{33}, \end{split}$$

$$\begin{split} E_3 &= e_{ijk}e_{ijk} = e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{21}^2 + e_{22}^2 + e_{23}^2 + e_{31}^2 + e_{32}^2 + e_{33}^2 \\ &\quad + 2(e_{14}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{23}^2 + e_{33}^2 + e_{33}^2 + e_{33}^2 + 2e_{36}^2), \end{split}$$

$$\begin{split} E_4 &= e_{ijk}e_{jik} = e_{11}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{24}^2 + e_{26}^2 + e_{33}^2 + e_{35}^2 + e_{36}^2), \cr$$

$$\begin{split} E_4 &= e_{ijk}e_{jik} = e_{11}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{24}^2 + e_{26}^2 + e_{33}^2 + e_{35}^2 + e_{36}^2), \cr$$

$$\begin{split} E_4 &= e_{ijk}e_{jik} = e_{11}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{24}^2 + e_{26}^2 + e_{33}^2 + e_{35}^2 + e_{36}^2), \cr$$

$$\begin{split} E_4 &= e_{ijk}e_{jik} = e_{11}^2 + e_{15}^2 + e_{16}^2 + e_{22}^2 + e_{24}^2 + e_{26}^2 + e_{33}^2 + e_{35}^2 + e_{36}^2), \cr$$

$$E_{5} = e_{iik}e_{kpp} = e_{11}^{2} + e_{22}^{2} + e_{33}^{2} + e_{11}e_{12} + e_{11}e_{13} + e_{16}e_{21} + e_{16}e_{22}$$
$$+ e_{21}e_{22} + e_{16}e_{23} + e_{22}e_{23} + e_{11}e_{26} + e_{12}e_{26} + e_{13}e_{26} + e_{15}e_{31}$$
$$+ e_{24}e_{31} + e_{15}e_{32} + e_{24}e_{32} + e_{15}e_{33} + e_{24}e_{33} + e_{31}e_{33} + e_{32}e_{33}$$
$$+ e_{21}e_{34} + e_{22}e_{34} + e_{23}e_{34} + e_{11}e_{35} + e_{12}e_{35} + e_{13}e_{35}.$$

In the above, we have used the commonly used two index notation. Thus $e_{35} = e_{313} = e_{331}, e_{26} = e_{212} = e_{221}$ etc.

3.2. Independence of the invariants

The set of invariants $\{E_1, ..., E_5\}$ will be linearly independent if the condition

$$(3.1) a_1 E_1 + \dots + a_5 E_5 = 0,$$

holds if and only if $a_1 = a_2 = \cdots = a_5 = 0$, for arbitrary values of parameters $e_{i\alpha}$, $i = 1, \ldots, 3$, $\alpha = 1, \ldots, 6$. In (3.1), let $e_{36} = 1$ and let all other parameters vanish. This leads to $a_3 = 0$. Subsequently, when we fix one or more parameters, we shall understand that all other parameters have been made to vanish without actually mentioning it. Letting $e_{12} = 1$, this gives $a_2 = 0$. Thus (3.1) reduces to

$$(3.2) a_1 E_1 + a_4 E_4 + a_5 E_5 = 0.$$

In (3.2) let $e_{15} = 1$. This leads to

$$a_1 + a_4 = 0.$$

Let $e_{21} = e_{22} = 1$. This gives

$$a_1 + a_4 + 2a_5 = 0.$$

Finally, $e_{11} = e_{26} = 1$ produces the equation

$$4a_1 + 2a_4 + 2a_5 = 0$$

Thus $a_1 = a_2 = \cdots = a_5 = 0$, hence the set $\{E_1, \ldots, E_5\}$ is linearly independent.

3.3. Invariants under SO(2)

In M_i , i = 1, ..., 16, replace T_{ijk} by e_{ijk} . Symmetry of the tensor leads to the following relations

$$M_2 = M_1, \qquad M_5 = M_6, \qquad M_7 = \frac{1}{2}(M_1 + M_3 - M_4 + M_5),$$

$$M_8 = \frac{1}{2}(M_5 + M_6) = M_5, \qquad M_9 = M_7,$$

$$M_{10} = M_{11} = M_{12} = M_{13} = M_{14} = M_{16} = 0.$$

Thus a complete set of invariants consists of five invariants corresponding to M_1, M_3, M_4, M_5 and M_{15} . Using the two index notations, the expressions for these invariants become

$$F_{1} = e_{iik}e_{ppk} = e_{11}^{2} + e_{22}^{2} + e_{16}^{2} + e_{26}^{2} + 2e_{16}e_{22} + 2e_{11}e_{26},$$

$$F_{2} = e_{ijj}e_{iqq} = e_{11}^{2} + e_{12}^{2} + e_{21}^{2} + e_{22}^{2} + 2e_{11}e_{12} + 2e_{21}e_{22},$$

$$F_{3} = e_{ijk}e_{ijk} = e_{11}^{2} + e_{12}^{2} + e_{21}^{2} + e_{22}^{2} + 2(e_{16}^{2} + e_{26}^{2}),$$

$$F_{4} = e_{ijk}e_{kij} = e_{11}^{2} + e_{22}^{2} + e_{16}^{2} + e_{26}^{2} + 2e_{16}e_{21} + 2e_{12}e_{26},$$

$$F_{5} = \frac{1}{2}(\epsilon_{kp}e_{ijk}e_{pji} + \epsilon_{ri}e_{ijk}e_{kjr}) = 2(e_{11}e_{21} - e_{11}e_{16} - e_{12}e_{16} - e_{12}e_{26}),$$

$$-e_{12}e_{22} + e_{21}e_{26} + e_{22}e_{26}.$$

Linear independence of the set $\{F_1, \ldots, F_5\}$ can be established in a manner similar to the one used for $\{E_1, \ldots, E_5\}$.

In [14] VANNUCCI used polar analysis to derive five plane quadratic invariants, D_1, \ldots, D_5 . Ignoring the factor of 1/8, his invariants, in the present notation, are as follows:

$$\begin{split} D_1 &= e_{11}^2 + 2e_{11}e_{26} + 2e_{21}e_{16} + e_{21}^2 + 2e_{22}e_{16} + e_{22}^2 - e_{12}^2 + 2e_{12}e_{26}, \\ D_2 &= 2(e_{11}e_{21} - e_{11}e_{16} - e_{12}e_{16} - e_{12}e_{22} + e_{21}e_{26} + e_{22}e_{26}), \\ D_3 &= e_{11}^2 + 2e_{11}e_{12} + e_{21}^2 + 2e_{22}e_{21} + e_{22}^2 + e_{12}^2, \\ D_4 &= e_{11}^2 - 2e_{11}e_{12} + 4e_{11}e_{26} - 4e_{21}e_{16} + e_{21}^2 - 2e_{21}e_{22} + 4e_{16}e_{22} \\ &\quad + e_{22}^2 + e_{12}^2 - 4e_{12}e_{26} + 4e_{26}^2 + 4e_{16}^2, \\ D_5 &= e_{11}^2 - 2e_{11}e_{12} - 4e_{11}e_{16} + 4e_{21}e_{16} + e_{21}^2 - 2e_{21}e_{22} - 4e_{16}e_{22} \\ &\quad + e_{22}^2 + e_{12}^2 + 4e_{12}e_{26} + 4e_{26}^2 + 4e_{16}^2. \end{split}$$

Each member of the above set can be expressed in terms of F_1, \ldots, F_5 in the following manner:

$$D_1 = F_1 - F_3 + F_4,$$

$$D_2 = F_5, \quad D_3 = F_2,$$

$$D_4 = 2F_1 - F_2 + 2F_3 - 2F_4,$$

$$D_5 = -2F_1 - F_2 + 2F_3 + 2F_4$$

Because of the relation $D_1^2 + D_2^2 = D_3 D_4$ [14], it is possible to express any one of them in terms of the other three. This means that only four of the plane invariants are *independent*. This led to Vannucci's dropping of D_2 from his list of invariants.

On the other hand, since none of the D_i , i = 1, ..., 5 or F_i , i = 1, ..., 5, can be expressed as a *linear* combination of the other four, each of the sets $\{D_1, ..., D_5\}$ and $\{F_1, ..., F_5\}$ is *linearly independent*. Both sets are *complete* in the sense that any quadratic invariant of the piezoelectric tensor e_{ijk} must be a linear combination of members of the set. However, the set $\{F_1, ..., F_5\}$ is simpler in appearance since each member contains only six terms, compared to twelve of D_4 and D_5 .

Acknowledgement

The author is grateful to the Higher Education Commission of Pakistan for financial support. Helpful comments of reviewers are also acknowledged.

References

- F. AHMAD, Invariants and structural invariants of the anisotropic elasticity tensor, Q. Jl. Mech. Appl. Math., 55, 597–606, 2002.
- F. AHMAD, M.A. RASHID, Linear invariants of a Cartesian tensor, Q. Jl. Mech. Appl. Math., 62, 31–38, 2009.
- V.I. ALSHITS, J. LOTHE, Some basic properties of bulk elastic waves in anisotropic media, Wave Motion 40, 297–313, 2004.
- 4. M. HAMERMESH, Group Theory and its Applications to Physical Problems, Addison Wesley, Reading 1964.
- 5. R.F.S. HEARMON, Crystal Physics, Benjamin, Menlo Park 1974.
- 6. J.P. HIRTH, J. LOTHE, Theory of Dislocations, McGraw-Hill, New York 1968.
- J. JERPHAGNON, Invariants of the third-rank Cartesian tensor: Optical nonlinear susceptibilities, Phys. Rev. B, 2, 1091–1098, 1970.
- A.N. NORRIS, Quadratic invariants of elastic moduli, Q. Jl. Mech. Appl. Math., 60, 367– 389, 2007.
- 9. G. RACAH, Determinazione del numero dei tensori isotropi independenti di rango n, ATTI-Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat., Ser. 11, **12**, 386–389, 1933.
- M.A. RASHID, F. AHMAD, N. AMIR, Linear invariants of a Cartesian tensor under SO(2), SO(3) and SO(4), International Journal of Theoretical Physics, 50, 479–487, 2011.
- 11. D. ROYER AND E. DIEULESAINT, *Elastic Waves in Solids I: Free and Guided Propagation*, Springer Verlag, Berlin 2000.
- J. SCHRÖDER, D. GROSS, Invariant formulation of the electromechanical enthalpy function of transversely isotropic piezoelectric materials, Archives of Applied Mechanics, 73, 533–552, 2004.

- T.C.T. TING, Invariants of anisotropic elastic constants, Q. Jl. Mech. Appl. Math., 40, 431–448, 1987.
- 14. P. VANNUCCI, The polar analysis of a third order piezoelectricity-like plane tensor, International Journal of Solids and Structures, 44, 7803–7815, 2007.

Received November 18, 2010; revised version April 27, 2011.