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Connecting Euler and Lagrange systems as nonlocally related systems of dynamical nonlinear elasticity

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Nonlocally related systems for the Euler and Lagrange systems of two-dimensional dynamical nonlinear elasticity are constructed. Using the continuity equation, i.e., conservation of mass of the Euler system to represent the density and Eulerian velocity components as the curl of a potential vector, one obtains the Euler potential system that is nonlocally related to the Euler system. It is shown that the Euler potential system also serves as a potential system of the Lagrange system. As a consequence, a direct connection is established between the Euler and Lagrange systems within a tree of nonlocally related systems. This extends the known situation for one-dimensional dynamical nonlinear elasticity to two spatial dimensions.

Key words: dynamical nonlinear elasticity, potential variables, nonlocally related systems, symmetries, conservation laws, gauge constraints.

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Notations

Ω_0, Ω	the Lagrangian and Eulerian volume elements, respectively,
s, t	the Lagrangian and Eulerian times, respectively,
$\mathbf{y} = (y_1, y_2)^*$, $\mathbf{x} = (x_1, x_2)^* = \varphi(\mathbf{y}, s)$	the material and spatial position vectors, respectively,
$\mathbf{F} := \nabla_y \mathbf{x} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$	the first-order transformation gradient, with Jacobian $J := \det(\mathbf{F}) = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$,
$\mathbf{B} = \mathbf{F}.\mathbf{F}^*$	the left Cauchy–Green strain tensor (symmetrical),
$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \varphi(\mathbf{y}, s)}{\partial s} = \mathbf{v}(\varphi(\mathbf{y}, s), s)$	the Eulerian velocity,
ρ	the Eulerian density,
$\mathbf{f}_0$, $\mathbf{f}$	the referential and spatial body forces, respectively,
$\mathbf{T}$	the first Piola–Kirchhoff (nominal) stress,
$\boldsymbol{\sigma} = J^{-1}\mathbf{T}.\mathbf{F}$	the Cauchy stress tensor (symmetrical).

1. Introduction: Review of the one-dimensional situation and preliminary discussion of the extension to the two-dimensional situation

A major limitation in the application of symmetry methods to systems of partial differential equations (PDEs) is the inability to find useful symmetries and/or conservation laws for a problem posed for a given PDE system. Until recently, symmetry analysis was limited to the use and computation of local symmetries (Lie point symmetries, higher-order symmetries) as well as the calculation of local conservation laws. Local symmetries can be used to map solutions to other solutions, to calculate the corresponding invariant solutions; to determine whether a given PDE system can be mapped invertibly to some PDE system, belonging to a target class of PDEs that is completely characterized by its local symmetries as well as determine an explicit mapping when one exists. But often for a given PDE system of physical interest, no local symmetry exists and even if one exists, it may not be useful for a given posed problem.

In general, a symmetry of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to another solution of the same PDE system. Hence the continuous symmetry transformations (which are essentially deformations of solutions) are defined topologically and thus are not restricted to local transformations acting on the space of independent and dependent variables and their derivatives (even if this space is infinite-dimensional as is the case for the global action of higher-order symmetries). So, in principle, from this point of view any nontrivial PDE system has symmetries. The problem is to develop systematic procedures to find and use continuous symmetries beyond the local ones, obtained through a direct application of Lie's algorithm. In particular, a direct application of Lie's algorithm only allows one to calculate local symmetries whose infinitesimals depend at most on a finite number of derivatives of the dependent variables. A natural way to extend the calculation and use of continuous symmetries to include nonlocal symmetries of a given PDE system, is to embed the given PDE system in an augmented PDE system. In such an embedding, it is important that each solution of the augmented system projects onto a solution of the given PDE system and, conversely, that each solution of the given PDE system yields a solution of the augmented system. Consequently, the solution of any boundary value problem posed for the given PDE system is embedded in the solution of a boundary value problem posed for the augmented system and the converse also holds. Moreover, in order to be able to calculate further conservation laws and/or symmetries of the given PDE system, it is necessary that the relationship between the given PDE system and such an augmented PDE system is nonlocal, i.e., there is not a 1:1 local transformation connecting the solutions of the given PDE system and the augmented system. Within such a relationship, it follows that a local symmetry or local conservation law of the augmented system could yield, respectively, a nonlocal symmetry or nonlocal conservation law of the given PDE system. Conversely, a local symmetry or local conservation law of the given PDE system could yield, respectively, a nonlocal symmetry or nonlocal conservation law of the augmented system. More importantly, such nonlocal symmetries can be found through a direct application of Lie's algorithm to the calculation of local symmetries of such augmented PDE systems and, similarly, nonlocal conservation laws of a given PDE system can be calculated through any local procedure such as the direct method [2, 3, 18, 19] applied to such augmented PDE systems.

A natural way to find such nonlocally related augmented PDE systems is through the use of local conservation laws of a given PDE system. In the case of two independent variables, say x and t, a local conservation law of a given PDE system directly yields an augmented system consisting of the given system and a pair of PDEs with a potential variable w, arising from the conservation law. Satisfaction of the integrability condition $w_{xt} = w_{tx}$ leads to the relationship between the solution sets of the given and augmented PDE systems.

Another natural way of obtaining nonlocal symmetries of a given PDE system is to calculate local symmetries of a nonlocally related subsystem. An example of such a nonlocally related subsystem is the system yielding a potential system, i.e., the 'given' system is a nonlocally related subsystem of the potential system.

The situation for PDE systems with three or more independent variables is more complex. Here, in order to obtain an interesting potential system (e.g., one that yields nonlocal symmetries and/or nonlocal conservation laws) from a local conservation law of a given PDE system, one must append gauge constraints that relate the potential variables resulting from the local conservation law. To date it is not obvious, a priori, which gauge constraint is of value for a particular application. However, local symmetries of nonlocally related subsystems can yield nonlocal symmetries of a given PDE system with three or more independent variables.

Furthermore, for n local conservation laws of a given PDE system, one obtains n sets of potential variables (each set contains one potential variable in the case of two independent variables, up to $\frac{1}{2}k(k-1)$ potential variables with appended gauge constraints in the case of k independent variables). In turn, one could obtain a tree of up to 2^n nonlocally related PDE systems by considering the obtained potential systems one-by-one (n singlets – each with one set of potential variables), in pairs ($\frac{1}{2}n(n-1)$ couplets – each with two sets of potential variables), $\ldots$, and all together (one n-plet containing the n sets of potential variables). Moreover, for any PDE system contained in such a tree of nonlocally related systems, one can use its local conservation laws to obtain further potential systems and their combinations. As a consequence, one can obtain an extended tree of nonlocally related PDE systems for a given PDE system. Note that the given PDE system could be any PDE system within such an extended tree!

For details on the above, see [5, 9, 10, 16, 17] and the references therein.

Most importantly, for the situation of gas dynamics equations in $1+1$ dimensions, one can construct a tree of nonlocally related systems (including subsystems) that have the Euler and Lagrange systems as two nonlocally related subsystems, as well as other systems. Through the calculation of point symmetries of such a nonlocally related system, one systematically obtains nonlocal symmetries for both the Lagrange and Euler systems (see [5] and the references therein).

Dynamical PDE systems for one-dimensional nonlinear elasticity were considered in [6] within the framework of nonlocally related PDE systems. In particular, the equations of the Euler system

$$\mathbf{E1D}\{x,t;v,\sigma,\rho\}:\quad \begin{cases} \rho_t+(\rho v)_x=0,\\ \sigma_x+\rho f(x)=\rho(v_t+vv_x),\\ \sigma=K(\rho), \end{cases} \tag{1.1}$$

respectively, include conservation of mass, momentum and a constitutive law relating stress and density. The independent variables in **E1D** (1.1) are the absolute time t and spatial position x, while the dependent variables are the density $\rho=\rho(x,t)$, the Eulerian velocity $v=v(x,t):=x_t$, and the Cauchy stress $\sigma=\sigma(x,t)$ in the Eulerian configuration. The body force field is described by the forcing function $f(x,t)\equiv f(x)$, assumed to be conservative (hence independent of time). Since the first equation in **E1D** (1.1) is in the form of a conservation law, one can introduce a corresponding potential variable w to obtain the nonlocally related potential system:

$$\mathbf{EW1D}\{x,t;v,\sigma,\rho,w\}:\quad \begin{cases} w_t=-\rho v,\\ w_x=\rho,\\ \sigma_x+\rho f(x)=\rho(v_t+vv_x),\\ \sigma=K(\rho). \end{cases} \tag{1.2}$$

In (1.1) and (1.2), the constitutive law is given in terms of a general scalar-valued function $K(\rho)$.

As shown in [6], a local 1:1 point transformation involving an interchange of dependent and independent variables of the system **EW1D** (1.2), with $w=y$ and $t=s$ treated as independent variables, and x, v, σ, $q=1/\rho$ treated as dependent variables, yields the Lagrange system of equations given by

$$\mathbf{L1D}\{y,s;v,\sigma,q,x\}:\quad \begin{cases} v=x_s,\\ q=x_y,\\ v_s=\sigma_y+f(x),\\ \sigma=K(\rho). \end{cases} \tag{1.3}$$

Observe that the same stress measure σ is involved in both **L1D** (1.3) and **E1D** (1.1) in the 1D formulation. This point transformation relates Eulerian and Lagrangian derivatives through the correspondence

$$x_t = -x_y . y_s . \tag{1.4}$$

The systems **EW1D** (1.2) and **L1D** (1.3) are locally related to each other by this point transformation, but are nonlocally related to the Euler system **E1D** (1.1).

Through Lie's algorithm applied to such a nonlocally related system, one can systematically construct further invariant and nonclassical solutions of a given PDE system beyond those obtained through a direct application of Lie's algorithm to the given PDE system. Focusing on nonlinear elasticity, this is of significant importance since it is well-known that very few closed-form solutions of BVPs for compressible elasticity have been obtained in the literature (contrary to incompressible elasticity), due to the absence of the kinematic incompressibility constraint [12].

In [6], conservation of mass and other conservation laws were used to construct systematically potential systems for nonlinear elasticity in 1D. As discussed above, the conservation of mass equation was used to derive a potential system locally related to the Lagrange system by a pointwise transformation involving an interchange of independent and dependent variables. In particular, this system is nonlocally related to the Euler system.

The main thrust of this paper is to extend the results in [6] to include nonlocally related systems of dynamical nonlinear elasticity in higher dimensions (the 2D spatial case will be considered; results will be similar for the 3D case). In particular, the eventual aim is to find a tree of nonlocally related systems that includes the physically important Lagrange and Euler systems. As in the 1D case, through the calculation of point symmetries of such nonlocally related systems, an aim is to find the nonlocal symmetries and the corresponding new solutions for both the Lagrange and Euler systems.

As can be seen from the above discussion, for PDE systems with two independent variables (e.g., the 1D case for dynamical nonlinear elasticity), the construction, properties and uses of nonlocally related PDE systems are relatively well understood. In the case of three or more independent variables, a conservation law gives rise to a vector potential subject to *gauge freedom* through the addition of an arbitrary gradient term. The corresponding potential system is under-determined. As mentioned above, additional equations relating potential variables, i.e., *gauge constraints*, are needed to make such potential systems determined. In [1], it was shown that only determined potential systems can yield nonlocal symmetries of a given PDE system through Lie's algorithm applied to a gauge-constrained potential system.

Nonlocal symmetries have been obtained for time-dependent linear wave and Maxwell's equations (in two and three spatial dimensions), as well as nonlinear MHD equilibrium equations in three spatial dimensions for particular gauge constraints [1, 4, 8, 11]. For a comprehensive discussion and further examples, see [9, 10].

As will be shown, unlike the situation for the Euler system of dynamical nonlinear elasticity equations in one spatial dimension, that in two spatial dimensional (2D) situation, the Euler potential system arising from conservation of mass for the Euler system has more dependent variables than the Lagrange system. It will be shown that this Euler potential system is also a potential system for the Lagrange system. In particular, both the Euler and Lagrange systems are nonlocally related subsystems of the Euler potential system. Consequently, a direct application of Lie's algorithm to the Euler potential system cannot yield nonlocal symmetries of either the Euler or Lagrange systems, without appending appropriate gauge constraints to the underdetermined Euler potential system. However, since a point symmetry of either the Euler or Lagrange systems could yield a nonlocal symmetry of the Euler potential system, it follows that one might obtain a nonlocal symmetry of the Euler (Lagrange) system from a point symmetry of the Lagrange (Euler) system through their relationship within a tree of nonlocally related systems. As a consequence, one can build a tree of nonlocally related systems for the dynamical nonlinear elasticity equations in 2D that includes the Euler and Lagrange systems, as previously shown in the one spatial dimensional (1D) situation.

The rest of this paper is organized as follows. In Sec. 2, for systems of nonlinear elasticity in two spatial dimensions, we establish the Euler system, the Euler potential system arising from conservation of mass, and the Lagrange system. The set of conservation laws is obtained for a Mooney–Rivlin material. In Sec. 3, we show the relationship between the Euler potential system and the Lagrange system, and illustrate the construction of the potential functions in the case of simple shear of a Mooney–Rivlin material. It will be seen that the one-dimensional situation arises as a subcase. In Sec. 4, an extended tree of nonlocally related systems is exhibited which shows an extension when the spatial body force is constant. The presented work is summarized and future directions are discussed in Sec. 5.

Regarding the notations, vectors and tensors are denoted by boldface symbols. A comma or subscript denotes a derivative, so that for instance

$$\mathbf{g}_{,x} \equiv \frac{\partial \mathbf{g}(x)}{\partial x}$$

represents the derivative of the vector-valued function $\mathbf{g}$ with respect to x. The transpose (adjoint) of any tensor $\mathbf{A}$ is written as $\mathbf{A}^*$.

2. Two-dimensional systems of nonlinear elasticity

In order to set the stage, a few words regarding the kinematics are in order. We consider the point mapping from the reference (Lagrangian) configuration Ω_0 with coordinates $\mathbf{y} = (y_1, y_2)^*$ and Lagrangian time s, to the actual (Eulerian) configuration Ω with Eulerian coordinates $\mathbf{x} = (x_1, x_2)^*$ and time t, chosen equal to the Lagrangian time, hence $t = s$. Let $\mathbf{x} = \varphi(\mathbf{y}, s)$ denote the point mapping relating both coordinate systems, with tangent mapping $\mathbf{F} = \mathbf{F}(\mathbf{y}, s) := \nabla_y \mathbf{x}$, represented as a two-by-two matrix in Euclidean space. The Jacobian of this transformation is the scalar $J := \det(\mathbf{F})$.

2.1. Lagrange, Euler and Euler potential systems for 2D elastodynamics

We now consider the (2+1)-dimensional Euler, Euler potential and Lagrange systems for nonlinear elasticity.

Here the Euler system is given by

$$\text{(2.1)} \qquad \mathbf{E2D}\{x_1, x_2, t; v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{12}, \rho\} : \quad \begin{cases} \rho_t + \nabla_x.(\rho \mathbf{v}) = 0, \\ \nabla_x.\boldsymbol{\sigma} + \rho \mathbf{f}(x_1, x_2) = \rho(\mathbf{v}_t + (\mathbf{v}.\nabla)\mathbf{v}), \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{B}). \end{cases}$$

The Eulerian velocity field is defined in terms of the point mapping $\varphi(\mathbf{y}, s)$ by the vector

$$\text{(2.2)} \qquad \mathbf{v}(\mathbf{x}, t) := \left.\frac{\partial \varphi(\mathbf{y}, s)}{\partial t}\right|_y = \frac{\partial \varphi(\mathbf{y}, s)}{\partial s} = \mathbf{v}(\varphi(\mathbf{y}, s), s).$$

The last equation in **E2D** (2.1), expressing the constitutive law with $\mathbf{B} = \mathbf{F}.\mathbf{F}^*$ (the left Cauchy–Green strain tensor), involves (due to the symmetry of the Cauchy stress) three independent constitutive functions for the three independent stress components σ_{11}, σ_{22}, $\sigma_{12} = \sigma_{21}$.

Each conservation law of **E2D** (2.1) yields a potential system. In particular, since the first equation of **E2D** (2.1) is written as a conservation law (conservation of mass), in terms of the curl of the vector potential function $\mathbf{w} = (w_0, w_1, w_2)$ one obtains the Euler potential system given by

$$\text{(2.3)} \qquad \mathbf{EW2D}\{x_1, x_2, t; v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, \rho, w_0, w_1, w_2\} : \quad \begin{cases} v_1 = \left(\dfrac{w_{0,2} - w_{2,0}}{w_{2,1} - w_{1,2}}\right); \quad v_2 = \left(\dfrac{w_{1,0} - w_{0,1}}{w_{2,1} - w_{1,2}}\right), \\ \rho = w_{2,1} - w_{1,2}, \\ \nabla_x.\sigma + \rho \mathbf{f} = \rho(\mathbf{v}_t + \mathbf{v}.\nabla_x \mathbf{v}), \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{B}), \end{cases}$$

where $w_{1,0} = \partial w_1/\partial t$ and $w_{2,0} = \partial w_2/\partial t$. The Euler potential system **EW2D** (2.3) is nonlocally related and equivalent to the Euler system **E2D** (2.1). In particular, it is easy to see that each solution of **EW2D** (2.3) projects onto a solution of **E2D** (2.1); conversely, each solution of **E2D** (2.1) yields a solution of **EW2D** (2.3) with the gauge freedom $\mathbf{w} \to \mathbf{w} + (\Phi_{,0}, \Phi_{,1}, \Phi_{,2})$ for any differentiable function $\Phi(x_1, x_2, t)$. This gauge freedom is intrinsic to the chosen representation of the kinematic variables in **EW2D** (2.3). Consequently, gauge fixing conditions (i.e., gauge constraints) for the potential variables can be additionally imposed.

On the other hand, the $(2+1)$-dimensional Lagrange system is given by

$$\mathbf{L2D}\{y_1, y_2, s; v_1, v_2, T_{11}, T_{22}, T_{12}, T_{21}, q, x_1, x_2\} : \tag{2.4}$$

$$\begin{cases} v_1 = x_{1,s}, \ v_2 = x_{2,s}, \\ q = x_{1,1}x_{2,2} - x_{2,1}x_{1,2}, \\ v_{1,s} = T_{1,y_1} + f_{01}(\mathbf{y}), \ v_{2,s} = T_{2,y_2} + f_{02}(\mathbf{y}), \\ \mathbf{T}.\mathbf{F}^* = \mathbf{F}.\mathbf{T}^*, \end{cases}$$

with material variables y_1, y_2 and time s as independent variables. Recalling that $q = 1/\rho$ in (2.4), the Lagrangian stress is the first Piola-Kirchhoff stress $\mathbf{T}$ which is a solution of the static equilibrium equation $\nabla_y.\mathbf{T}+\mathbf{f}_0 = \mathbf{v}_s$ (initial mass density can be set to unity), written in index form in **L2D** (2.4). The Lagrangian body force vector in **L2D** (2.4) is given by $\mathbf{f}_0(\mathbf{y}) = (f_{01}(y_1, y_2), f_{02}(y_1, y_2))^*$, such that $\mathbf{f}_0(\mathbf{y}) = \mathbf{f}(\mathbf{x})$, due to the equalities

$$\forall \Omega_0, \quad \int_{\Omega_0} \rho \mathbf{f} J d\Omega_0 = \int_{\Omega_0} \rho_0 \mathbf{f}_0 d\Omega_0 \qquad \text{and} \qquad J = 1/\rho$$

which are valid for an arbitrary initial density and hold for all regions Ω_0. The last equation in **L2D** (2.4) results from the Cauchy stress being symmetrical in **E2D** (2.1). It expresses the coaxiality of $\mathbf{T}$ and $\mathbf{F}$ [15].

The conservation of momentum in **E2D** (2.1) is obtained from its Lagrangian counterpart in **L2D** (2.4), using the derivative rule

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_x.$$

In particular, from the third set of equations in (2.4), one obtains

$$\begin{aligned} &\nabla_y.\mathbf{T} + \mathbf{f}_0 = \mathbf{v}_s \Rightarrow \nabla_x.\mathbf{T}.\mathbf{F} + \mathbf{f}_0 = (\mathbf{v}_t + \mathbf{v}.\nabla_x\mathbf{v}) \Rightarrow \\ &\quad J^{-1}\nabla_x.\mathbf{T}.\mathbf{F} + \rho\mathbf{f} = \rho(\mathbf{v}_t + \mathbf{v}.\nabla_x\mathbf{v}) \Rightarrow \nabla_x(J^{-1}\mathbf{T}.\mathbf{F}) + \rho\mathbf{f} = \rho(\mathbf{v}_t + \mathbf{v}.\nabla_x\mathbf{v}), \end{aligned}$$

using the classical identity $\nabla_y.(J^{-1}.\mathbf{F}) = \mathbf{0}$. Next, since the Cauchy stress is related to the Lagrangian stress by $\boldsymbol{\sigma} = J^{-1}\mathbf{T}.\mathbf{F}$, one obtains the second set of equations (conservation of momentum) in the Euler system **E2D** (2.1). Conversely, starting from the conservation of mass and momentum in the Eulerian framework as expressed in **EW2D** (2.2) or **E2D** (2.1), yields the Lagrangian counterpart of these conservation laws, using the same relationships in reversed order.

The Lagrangian stress $\mathbf{T}$ may be expressed from the setting up of a strain energy density function $W(\mathbf{F})$ in the framework of hyperelasticity [15], depending upon the transformation gradient, the variable $\mathbf{F}$ (omitting here, for the sake of simplicity and as representative of homogeneous media, the possible explicit dependence of W upon the Lagrangian variable), as $\mathbf{T} := \partial W(\mathbf{F})/\partial(\mathbf{F})$.

Further conservation laws can be constructed from specific conservation law multipliers, either for an arbitrary constitutive law or for specific choices of the constitutive function. Conservation laws are of particular interest in fracture mechanics, since they lead to path-independent contour integrals that characterize the singularity of the stress field around the crack tip. Conservation laws have been expressed in the 1D case for specific constitutive functions $W(\mathbf{F})$ in [7]. Mathematically, conservation laws can be systematically calculated, both for variational problems (Noether's theorem) and, more generally, for non-variational problems (see [2, 3] and references therein). If a field equation is already in the form of a conservation law, the introduction of potential variables is straightforward. One can obtain a tree of nonlocally related systems of nonlinear elasticity equations through finding additional conservation laws and considering potential systems of the related PDE systems. Since the resulting potential systems have additional dependent variables, in principle one can find further conservation law multipliers that yield additional conservation laws. These additional conservation laws in turn yield further potential variables and consequently, additional potential systems in an extended tree of nonlocally related systems. But as mentioned in the Introduction, in the case of three or more independent variables, in order to find further symmetries of the Euler system or Lagrange system through the computation of local symmetries of such a nonlocally related system, one must append gauge constraints relating the potential variables (or consider the subsystems).

As an example, conservation laws of system **L2D** (2.4) are computed, adopting a Mooney–Rivlin constitutive behavior given by the strain energy density function

$$W = a(I_1 - 3) + b(I_2 - 3). \tag{2.5}$$

W is dependent on the two strain invariants, the scalar-valued functions of the two-dimensional right Cauchy Green tensor $\mathbf{B} = \mathbf{F}.\mathbf{F}^t$, given by

$$I_1 = \mathrm{Tr}(\mathbf{B}) = F^i_{.K} F^i_{.K}; \qquad I_2 := \frac{1}{2}(\mathrm{Tr}(\mathbf{B})^2 - \mathrm{Tr}(\mathbf{B}^2)) = \frac{1}{2}(I_1^2 - B^{ik}B^{ki}). \tag{2.6}$$

The nominal two-dimensional stress is then evaluated as

$$\mathbf{T} := \frac{\partial W}{\partial \mathbf{F}} = 2(a+b)\mathbf{F} + 2bJ\mathbf{C}. \tag{2.7}$$

We seek conservation law multipliers of the form $\Lambda_i = \Lambda_i(t, \mathbf{y}, \mathbf{x}, \mathbf{x}_t, \mathbf{F})$, $i = 1, 2$ and the corresponding conservation laws involving fluxes $\boldsymbol{\Psi}[\mathbf{x}]$, $\Phi^1[\mathbf{x}]$, $\Phi^2[\mathbf{x}]$ (see [5, 11]) that satisfy

$$\frac{\partial \Psi[\mathbf{x}]}{\partial t} + \frac{\partial \Phi^1[\mathbf{x}]}{\partial y^1} + \frac{\partial \Phi^2[\mathbf{x}]}{\partial y^2} = 0.$$

The fluxes depend on the variables $(t, \mathbf{y}, \mathbf{x}, \mathbf{x}_t, \mathbf{x}_y)$.

From the obtained multipliers

$$\begin{cases} \Lambda_1 = C_1 t - C_2 x_2 + C_3 x_{1,t} + C_4, \\ \Lambda_2 = C_5 t + C_2 x_1 + C_3 x_{2,t} + C_6, \end{cases}$$

with $\{C_i\}$, $i = 1, \ldots, 6$ arbitrary constants, the following six conservation laws are obtained using the GeM software [12].

1. Conservation of momentum: The equilibrium equations in absence of the body forces are conservation laws as they stand, corresponding to the two multiplier pairs $(\Lambda_1, \Lambda_2) = (1, 0)$ and $(\Lambda_1, \Lambda_2) = (0, 1)$:

$$\begin{cases} v_{1,s} = T_{11,y_1} + T_{12,y_2}, \\ v_{2,s} = T_{21,y_1} + T_{22,y_2}. \end{cases}$$

2. Conservation of energy arises from the multiplier pair $(\Lambda_1, \Lambda_2) = (x_t^1, x_t^2)$ with conserved flux density

$$\Psi[\mathbf{x}] = \frac{1}{2}((x_t^1)^2 + (x_t^2)^2).$$

3. Conservation of angular momentum arises from the multiplier pair $(\Lambda_1, \Lambda_2) = (-x^2, x^1)$, with conserved flux density $\Psi[\mathbf{x}] = x^1 x_t^2 - x^2 x_t^1$, corresponding to the projection on the y^3 axis of the conservation of angular momentum vector $\mathbf{M} = \mathbf{x} \wedge \mathbf{x}_t$.

4. Conservation of average velocity: For the two multiplier pairs $(\Lambda_1, \Lambda_2) = (t, 0)$ and $(\Lambda_1, \Lambda_2) = (0, t)$, a kind of time-translation invariance of $\mathbf{x}$ involving an average velocity is obtained. This yields the fluxes successively given by

$$\Psi[\mathbf{x}] = t x_t^1 - x^1; \quad \Phi^1[\mathbf{x}] = -2x^1 y^1 t(a+b); \quad \Phi^2[\mathbf{x}] = -2x^1 y^2 t(a+b), \tag{2.8}$$

$$\Psi[\mathbf{x}] = t x_t^2 - x^2; \quad \Phi^1[\mathbf{x}] = -2x^2 y^1 t(a+b); \quad \Phi^2[\mathbf{x}] = -2x^2 y^2 t(a+b). \tag{2.9}$$

Next, an analytical solution of a problem for **L2D** (2.4) is constructed.

2.2. Example: analytical solution for the Lagrange system L2D (2.4)

Consider a solid body occupying the square domain $y_1 \in [0,1]$, $y_2 \in [0,1]$ in the reference configuration. We adopt a neo-Hookean constitutive behavior given by the strain energy density function

$$W = \frac{1}{2}(I_1 - 3). \tag{2.10}$$

This is a specific case for the Mooney–Rivlin function in (2.5) with the choice $b = 0$, $a = 1/2$. The nominal two-dimensional stress is then evaluated as

$$\mathbf{T} := \frac{\partial W}{\partial \mathbf{F}} = \mathbf{F}. \tag{2.11}$$

The initial conditions for the particle positions and velocities are selected as follows:

$$(\mathrm{IC}): \begin{cases} x_1(y_1, y_2, 0) = y_1, \\ v_1(y_1, y_2, 0) = \sqrt{C_2}\sin(\sqrt{C_2}y_2), \;\; C_2 > 0, \\ x_2(y_1, y_2, 0) = y_2, \\ v_2(y_1, y_2, 0) = \sqrt{C_2}\sin(\sqrt{C_2}y_1). \end{cases}$$

Conservation of momentum is written in terms of the following set of decoupled PDEs for the displacement functions $x_1(y_1, y_2, t)$, $x_2(y_1, y_2, t)$:

$$(\mathrm{S}): \begin{cases} x_{1,tt} - (x_{1,y_1y_1} + x_{1,y_2y_2}) = 0, \\ x_{2,tt} - (x_{2,y_1y_1} + x_{2,y_2y_2}) = 0. \end{cases}$$

We seek a displacement field solution of (S) of the form

$$\begin{cases} u_1(y_1, y_2, t) := x_1(y_1, y_2, t) - y_1 = h_1(y_2)p_1(t), \\ u_2(y_1, y_2, t) := x_2(y_1, y_2, t) - y_2 = h_2(y_1)p_2(t). \end{cases}$$

A specific solution of system (S) satisfying the given set of initial conditions (IC) is obtained as

$$\begin{cases} x_1(y_1, y_2, t) = y_1 + \sin(\sqrt{C_2}y_2)\sin(\sqrt{C_2}t), \\ x_2(y_1, y_2, t) = y_2 + \sin(\sqrt{C_2}y_1)\sin(\sqrt{C_2}t). \end{cases} \tag{2.12}$$

The transformation gradient is evaluated as

$$\mathbf{F}(y_1, y_2, t) = \begin{pmatrix} 1 & \sqrt{C_2}\cos(\sqrt{C_2}y_2)\sin(\sqrt{C_2}t) \\ \sqrt{C_2}\cos(\sqrt{C_2}y_1)\sin(\sqrt{C_2}t) & 1 \end{pmatrix}. \tag{2.13}$$

For this specific constitutive law, the nomimal stress is simply given by $\mathbf{T}(y_1, y_2, t) = \mathbf{F}(y_1, y_2, t)$, and the Cauchy stress by

$$\boldsymbol{\sigma}(x_1, x_2, t) = J^{-1}\mathbf{F}^2 \tag{2.14}$$

with the Jacobian of the transformation given by

$$J(y_1, y_2, t) = C_2 \cos(\sqrt{C_2}y_2)\cos(\sqrt{C_2}y_1)\sin^2(\sqrt{C_2}t). \tag{2.15}$$

Since the Cauchy stress is an Eulerian field, it has to be expressed in terms of the spatial coordinates (x_1, x_2, t). Hence one has to substitute the inverse kinematic relations $y_1(x_1, x_2, t)$, $y_2(x_1, x_2, t)$, obtained by inverting the solution from (2.12), into the right-hand side of (2.14).

3. Relationship between the Euler potential system EW2D (2.3) and the Lagrange system L2D (2.4)

Solutions of a boundary value problem (BVP) in nonlinear elasticity exist for suitable boundary conditions (uniqueness is not always ensured). The Lagrange system **L2D** (2.4) and the Euler potential system **EW2D** (2.3), for suitable boundary conditions, have solutions of the same BVP, but expressed in terms of different independent variables (s, y_1, y_2 and t, x_1, x_2, respectively). However, the Euler potential system **EW2D** (2.3) has one more dependent variable than the Lagrange system **L2D** (2.4), unlike the situation in the 1D case, where both systems have the same number of dependent variables.

3.1. Relationship between systems EW2D and L2D

Now consider an invertible point transformation (hodograph-type) transformation of the Lagrange system **L2D** (2.4) that interchanges its independent variables y_1, y_2 with its dependent variables x_1, x_2. The transformed Lagrange system **L2D** (2.4) now has s, x_1, x_2 as its independent variables with y_1, y_2, playing the role of two of its dependent variables.

In order to relate the Euler potential system **EW2D** and the Lagrange system **L2D**, we now show that the Euler potential system is also a potential system for the transformed Lagrange system **L2D** (2.4). In particular, first note that

$$dx_i = v_i dt + \frac{\partial x_i}{\partial y_j} dy_j = v_i dt + \frac{\partial x_i}{\partial y_j}\left[\frac{\partial y_j}{\partial t}dt + \frac{\partial y_j}{\partial x_k}dx_k\right]. \tag{3.1}$$

From (3.1), it follows that, in terms of the Kronecker symbol δ_{ik}, one obtains

$$\frac{\partial x_i}{\partial y_j}\frac{\partial y_j}{\partial x_k} = x_{i,j}y_{j,k} = \delta_{ik}, \tag{3.2}$$

and

$$v_i = -\frac{\partial x_i}{\partial y_j}\frac{\partial y_j}{\partial t} = -x_{i,j}y_{j,0}. \tag{3.3}$$

From (3.2) and (3.3), it follows that each solution of the Lagrange system **L2D** (2.4) yields a solution of the Euler potential system **EW2D** (2.3) provided that the three potential variables w_0, w_1, w_2 of **EW2D** solve the system

$$\begin{cases} w_{2,1} - w_{1,2} = y_{1,1}y_{2,2} - y_{2,1}y_{1,2}, \\ w_{0,2} - w_{2,0} = y_{2,0}y_{1,2} - y_{1,0}y_{2,2}, \\ w_{1,0} - w_{0,1} = y_{1,0}y_{2,1} - y_{2,0}y_{1,1}. \end{cases} \tag{3.4}$$

It is easy to check that the identity

$$(y_{1,1}y_{2,2} - y_{2,1}y_{1,2})_{,0} + (y_{2,0}y_{1,2} - y_{1,0}y_{2,2})_{,1} + (y_{1,0}y_{2,1} - y_{2,0}y_{1,1})_{,2} \equiv 0 \tag{3.5}$$

holds for any functions $y_1(x_1, x_2, t), y_2(x_1, x_2, t)$. Consequently, it follows that:

1. Each solution of the Lagrange system **L2D** (2.4) yields a solution of the Euler potential system **EW2D** (2.3).
2. Conversely, the Euler potential system **EW2D** (2.3) is a potential system for the (transformed) Lagrange system **L2D** (2.4). In particular, the potential system arises from the conservation law (3.5) satisfied by two of its dependent variables y_1, y_2.

REMARK. In the one-dimensional context, one can set $w_0 = 0 = w_2$ so that $w := w_1$ is the sole potential variable. With the identification $2 \leftrightarrow x$; $0 \leftrightarrow t$, system (3.4) yields the system

$$\begin{cases} w_x = 1/\rho, \\ w_t = -w_x.x_t. \end{cases}$$

Consequently, system **EW2D** (2.3) directly projects to the 1D system **EW1D** (1.2).

3.2. Construction of the potential functions: example

The general strategy for expressing the partial derivatives of the potential functions in terms of the Lagrangian coordinates consists of two steps:

1. Find a specific solution of this problem denoted by $\mathbf{w}_p$, for instance by imposing some gauge constraint on the potentials.
2. Consider another arbitrary solution for the same gauge constraint, denoted by $\mathbf{w}_g$. The right-hand side of the gauge system GC is a given vector, called $\mathbf{Y}$, such that $\operatorname{curl}\mathbf{w}_p = \mathbf{Y}$ and $\operatorname{curl}\mathbf{w}_g = \mathbf{Y}$. Taking the difference of these two

equalities gives $\mathrm{curl}(\mathbf{w}_p - \mathbf{w}_g) = \mathbf{0}$, from which it follows, using the Helmholtz decomposition theorem, that the difference $\mathbf{w}_p - \mathbf{w}_g$ is the gradient of an arbitrary scalar-valued function of the Eulerian coordinates, say $L(t, x_1, x_2)$. This results in the general solution of (3.4) expressed as

$$\mathbf{w}_g = \mathbf{w}_p(y_{1,0}, y_{2,0}, y_{1,1}, y_{2,2}, y_{2,1}, y_{1,2}) + \mathrm{grad}_{(t,y_1,y_2)} L(t, x_1, x_2).$$

Mass conservation in the Eulerian configuration can be satisfied by a representation involving only two potential functions w_1, w_2, acting as independent degrees of freedom. A specific solution is first found after setting $w_0 = 0$. A general solution of the potential functions satisfying these constraints is constructed as follows after introducing the quantities

$$\begin{aligned} A &:= -y_{1,0}y_{2,2}/(y_{1,1}y_{2,2} - y_{2,1}y_{1,2}),\\ B &:= y_{2,0}y_{1,2}/(y_{1,1}y_{2,2} - y_{2,1}y_{1,2}),\\ C &:= y_{1,0}y_{2,1}/(y_{1,1}y_{2,2} - y_{2,1}y_{1,2}),\\ D &:= -y_{2,0}y_{1,1}/(y_{1,1}y_{2,2} - y_{2,1}y_{1,2}). \end{aligned} \tag{3.6}$$

Integration of the second and third equations of (3.4) gives

$$w_1 = -\int dtj(C + D) + \varphi_1(x_1, x_2); \quad w_2 = \int_t dtj(A + B) + \varphi_2(x_1, x_2) \tag{3.7}$$

with $j = 1/J$ and in which the otherwise arbitrary functions $\varphi_1(x_1, x_2)$, $\varphi_2(x_1, x_2)$ have to satisfy the constraint resulting from the first equation in (3.4), i.e.,

$$\varphi_{2,1}(x_1, x_2) - \varphi_{1,2}(x_1, x_2) = y_{1,1}y_{2,2} - y_{1,2}y_{2,1}.$$

Any pair of functions $\varphi_1(x_1, x_2)$, $\varphi_2(x_1, x_2)$ satisfying the previous equation provides a solution for the potentials. For example, one may select the specific solution

$$\begin{aligned} &\varphi_1(x_1, x_2) = 0 \Rightarrow w_1 = \int_t (-y_{1,0}y_{2,1} + y_{2,0}y_{1,1})ds,\\ &w_2 = \int_t (y_{2,0}y_{1,2} - y_{1,0}y_{2,2})ds + \int_{x_1} (y_{1,1}y_{2,2} - y_{1,2}y_{2,1})dx_1. \end{aligned} \tag{3.8}$$

Note that one has the freedom to add to the components w_1, w_2, the gradient of an arbitrary function of the Eulerian coordinates.

The general expression for the gauge constrained potentials involving an arbitrary function $L(t, x_1, x_2)$, is then given by

$$(3.9)\quad \begin{cases} w_0 = \dfrac{\partial L(t,x_1,x_2)}{\partial t}, \\ w_1 = \displaystyle\int_t (-y_{1,0}y_{2,1} + y_{2,0}y_{1,1})ds + \dfrac{\partial L(t,x_1,x_2)}{\partial x_1}, \\ w_2 = \displaystyle\int_t (y_{2,0}y_{1,2} - y_{1,0}y_{2,2})ds + \int_{x_1} (y_{1,1}y_{2,2} - y_{1,2}y_{2,1})dx_1 + \dfrac{\partial L(t,x_1,x_2)}{\partial x_2}. \end{cases}$$

This can be further transformed, expressing the right-hand side of (3.9) fully in terms of Lagrangian coordinates. For this purpose, one differentiates $L(t, x_1, x_2)$, accounting for the mapping between the Lagrangian and Eulerian coordinates. Since the gradient transforms as a contravariant vector under the change of basis, system (3.9) becomes

$$(3.10)\quad \begin{cases} w_0 = \dfrac{\partial G(s,y_1,y_2)}{\partial s}, \\ w_1 = \displaystyle\int_s (-y_{1,0}y_{2,1} + y_{2,0}y_{1,1})ds, \\ \qquad + \dfrac{\partial G(s,y_1,y_2)}{\partial y_1}\dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial G(s,y_1,y_2)}{\partial y_2}\dfrac{\partial y_2}{\partial x_1}, \\ w_2 = \displaystyle\int_s (y_{2,0}y_{1,2} - y_{1,0}y_{2,2})ds + \int_{x_1} (y_{1,1}y_{2,2} - y_{1,2}y_{2,1})dx_1 \\ \qquad + \dfrac{\partial G(s,y_1,y_2)}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial G(s,y_1,y_2)}{\partial y_2}\dfrac{\partial y_2}{\partial x_2}, \end{cases}$$

with $G(s, y_1, y_2)$ being an arbitrary function of the Lagrangian coordinates. The multiplicative factors of the partial derivatives of $G(s, y_1, y_2)$ yield an inverse Jacobian matrix. This specific solution for the potential function provides a one-to-one mapping between the potential functions w_1, w_2 and the Lagrangian coordinates y_1, y_2.

In order to exemplify the construction of the potential functions from the partial derivatives of the Lagrangian functions $y_1(x_1, x_2, t)$, $y_2(x_1, x_2, t)$, relying on system (3.4), a second model problem is considered. Here the initial square domain in the (y_1, y_2)-plane with unit edge lengths has the following imposed displacements on its boundary:

$$(3.11)\quad \begin{cases} y_1 = 0 : \mathbf{u} = \mathbf{0}, \\ y_2 = 0 : \mathbf{u} = \mathbf{0}, \\ y_1 = 1,\ y_2 \in [0,1] : \mathbf{u} = k(t)y_2\mathbf{e}_1, \\ y_2 = 1,\ y_1 \in [0,1] : \mathbf{u} = \mathbf{0}, \end{cases}$$

with $k(t)$ being a given function of time, and $\mathbf{e}_1$ a unit basis vector along the y_1 axis. Consider a Mooney–Rivlin constitutive behavior given by the strain energy density function in (2.5). It is straightforward to obtain the solution of system **L2D** (2.4) satisfying the initial conditions (3.11) as the following mapping relating the Lagrangian and Eulerian coordinates:

$$\begin{cases} x_1 = y_1 + k(t)y_2, \\ x_2 = y_2. \end{cases} \tag{3.12}$$

This mapping results in the transformation gradients

$$\mathbf{F} = \mathbf{F}(t) = \begin{pmatrix} 1 & k(t) \\ 0 & 1 \end{pmatrix} \Rightarrow \mathbf{C} = \mathbf{F}^t.\mathbf{F} = \begin{pmatrix} 1 & k(t) \\ k(t) & 1 + k^2(t) \end{pmatrix}. \tag{3.13}$$

The Jacobian is evaluated as $J = 1$, corresponding to an incompressible simple shear motion. The Lagrangian stress is then obtained as

$$\mathbf{T} := 2(a + 3b) \begin{pmatrix} 1 & k(t) \\ k(t) & 1 + \left(\dfrac{2b}{a + 3b}\right) k^2(t) \end{pmatrix}. \tag{3.14}$$

The inverse transformation gradient is given by

$$\mathbf{F}^{-1}(t) = \begin{pmatrix} 1 & -k(t) \\ 0 & 1 \end{pmatrix} \Rightarrow \frac{\partial y_1}{\partial x_1} = 1;\ \frac{\partial y_1}{\partial x_2} = -k(t);\ \frac{\partial y_2}{\partial x_1} = 0;\ \frac{\partial y_2}{\partial x_2} = 1. \tag{3.15}$$

The construction of the potentials w_1, w_2 through (3.10) yields the inverse velocity with components $y_{1,0}$, $y_{2,0}$, obtained from the following lemma.

LEMMA 1. *The Lagrangian (inverse) velocity*

$$\mathbf{V}(\mathbf{y}, t) := \frac{\partial \varphi^{-1}}{\partial t}_{|x} = \mathbf{V}(\varphi^{-1}(\mathbf{x}, t), t),$$

obtained from the inverse mapping $\varphi^{-1}(\mathbf{x}, t)$, *with components* $(\varphi^{-1})_1 = V_1 = y_{1,t}$; $(\varphi^{-1})_2 = V_2 = y_{2,t}$ *and the Eulerian velocity* $\mathbf{v}(\mathbf{x}, t)$ *are related by the kinematic constraint*

$$\mathbf{v} + \mathbf{F}.\mathbf{V} = \mathbf{0}.$$

Proof. Apply the differentiation rule

$$dx^i = \frac{\partial \varphi^i}{\partial t}_{|y} dt + \frac{\partial \varphi^i}{\partial y^k} \left[\frac{\partial (\varphi^{-1})^k}{\partial t}_{|x} dt + \frac{\partial (\varphi^{-1})^k}{\partial x^j}_{|t} dx^j \right]$$

and use the relation

$$\frac{\partial}{\partial t}_{|y} = \frac{\partial}{\partial t}_{|x} + \mathbf{v}.\nabla_x,$$

which is the tensorial generalization of the 1D derivative rule $\partial_s = \partial_t + v.\partial_x$, from the mapping $s = s(x,t)$. Hence, we get

$$dx^i = \frac{\partial \varphi^i}{\partial t}_{|x} dt + (\mathbf{v}.\nabla\varphi)dt + \frac{\partial \varphi^i}{\partial y^k}\left[\frac{\partial(\varphi^{-1})^k}{\partial t}_{|x} dt + \frac{\partial(\varphi^{-1})^k}{\partial x^j}_{|t} dx^j\right]$$
$$= v^i dt + F^i_{.K}[V^K dt + (F^{-1})^{.j}_K dx^j].$$

Cancellation of the factor of the differential dt gives $v^i + F^i_{.K}V^K = 0$, i.e., $\mathbf{v} + \mathbf{F}.\mathbf{V} = \mathbf{0}$ in tensorial format (the cancellation of the factor of the differential dx^j has not been used).

Using (3.12) and (3.15), the inverse velocity is calculated as

$$\mathbf{V} = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = -\mathbf{F}^{-1}.\mathbf{v} = -\begin{pmatrix} 1 & -k(t) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$
$$= \begin{pmatrix} k(t)x_{2,t} - x_{1,t} \\ -x_{2,t} \end{pmatrix} = \begin{pmatrix} k(t)v_2 - v_1 \\ -v_2 \end{pmatrix} \equiv \begin{pmatrix} -k'(t)y_2 \\ 0 \end{pmatrix}.$$

Hence, the potentials are finally expressed in terms of Lagrangian coordinates by

$$(3.16)\quad \begin{cases} w_0 = 0, \\ w_1 = \dfrac{\partial G(s,y_1,y_2)}{\partial y_1}, \\ w_2 = \displaystyle\int\limits_s -k'(u)y_2 du + \int\limits_{x_1} dx_1 - k(s)\frac{\partial G(s,y_1,y_2)}{\partial y_1} + \frac{\partial G(s,y_1,y_2)}{\partial y_2} \\ \quad = -k(s)y_2 + x_1 - k(s)\dfrac{\partial G(s,y_1,y_2)}{\partial y_1} + \dfrac{\partial G(s,y_1,y_2)}{\partial y_2}. \end{cases}$$

□

4. An extended tree of nonlocally related systems arising for a constant spatial body force

Suppose that the spatial body force $\mathbf{f} = (f_1, f_2)$ is constant. Then one obtains a second conservation law for the Euler system **E2D** (2.1):

$$(4.1)\quad (\rho f_1 v_2 - \rho f_2 v_1)_{,0} + (\rho f_1 v_1 v_2 - \rho f_2 v_1^2 + f_2\sigma_{11} - f_1\sigma_{12})_{,1}$$
$$+ (\rho f_1 v_2^2 - \rho f_2 v_1 v_2 + f_2\sigma_{12} - f_1\sigma_{22})_{,2} = 0.$$

Consequently, in terms of the curl of the vector potential function $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$, one obtains another Euler potential system given by

$$\text{(4.2)} \qquad \mathbf{E}\alpha 2\mathrm{D}\{x_1, x_2, t; v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, \rho, \alpha_0, \alpha_1, \alpha_2\}: \\ \begin{cases} \rho_t + \nabla_x \cdot (\rho \mathbf{v}) = 0, \\ \alpha_{2,1} - \alpha_{1,2} = \rho(f_1 v_2 - f_2 v_1), \\ \alpha_{0,2} - \alpha_{2,0} = v_1(\alpha_{2,1} - \alpha_{1,2}) + f_2\sigma_{11} - f_1\sigma_{12}, \\ \alpha_{1,0} - \alpha_{0,1} = v_2(\alpha_{2,1} - \alpha_{1,2}) + f_2\sigma_{12} - f_1\sigma_{22}, \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{B}). \end{cases}$$

Furthermore, by combining the potential systems **EW2D** (2.2) and **E$\boldsymbol{\alpha}$2D** (4.2), one obtains the additional nonlocally related couplet system **E$\boldsymbol{\alpha}$W2D**. The resulting tree of nonlocally related systems is presented in Fig. 1. A tree that includes the Lagrange, Euler and Euler potential systems is included within this tree. The importance of such trees is discussed in Sec. 1.

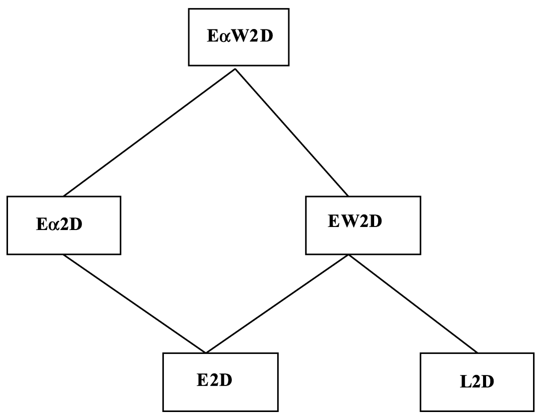

FIG. 1. Tree of nonlocally related systems when the spatial body force is constant. Note that the tree would consist of the three nonlocally related systems **E2D**, **L2D** and **EW2D** when the spatial body force is not constant.

5. Summary and future directions

Nonlocally related systems of equations of two-dimensional dynamical nonlinear elasticity, including the Euler and Lagrange systems, have been constructed in the present contribution, using conservation of mass to represent the physical density and the Eulerian velocity components as the curl of a vector of three potential functions for the Euler system. The corresponding Euler potential system of differential equations has the same set of solutions as the Lagrange and Euler systems. After a point transformation that interchanges the two Lagrangian material (independent) variables with the two Lagrangian (dependent) spatial position variables to obtain the transformed Lagrange system, one sees that the Euler potential system is also a potential system for the transformed Lagrange system. Hence the Euler potential and Lagrange systems are nonlocally related to the Euler system of equations.

Additional conservation laws may arise in two spatial dimensions for specific constitutive functions, as has been shown in the one-dimensional case in [7]. For each conservation law, three potential variables can be introduced, allowing a rewriting of a posed BVP in terms of a BVP for a potential system. As shown in Sec. 4, an additional conservation law arises in the case of a constant spatial body force.

In nonlinear elasticity, the finding of closed-form solutions for new BVPs and additional conservation laws is of great importance, especially for compressible materials. Very few closed-form solutions have been obtained in the literature (contrary to incompressible elasticity), due to the absence of the kinematic incompressibility constraint [14].

The search for extended trees of nonlocally related systems for 2D and 3D dynamical nonlinear elasticity and for useful gauge conditions connecting potential variables to yield nonlocal symmetries and useful invariant solutions for the Lagrange and Euler systems, requires further investigation.

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