

The effect of surface elasticity on a Mode-III interface crack

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WE STUDY THE CONTRIBUTION of surface elasticity to the anti-plane deformations of a linearly elastic bi-material with Mode-III interface crack. The surface elasticity is incorporated using a version of the continuum-based surface/interface model of Gurtin and Murdoch. We obtain a complete semi-analytic solution valid throughout the solid (including the crack tips) via a Cauchy singular, integro-differential equation of the first kind. Our solution demonstrates that the surface elasticity on the crack face leads to finite stresses at the crack tips and stress discontinuities across the material interface.

Key words: surface elasticity, Mode-III interface crack, bonded dissimilar isotropic half-planes, anti-plane deformations, complete semi-analytic solution, cauchy singular integro-differential equations.

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1. Introduction

THE ANALYSIS OF THE PROBLEM of a crack at the interface between dissimilar elastic materials is critical for the understanding of failure modes and in the general stress analysis of advanced composite materials (for example, laminar and fiber-reinforced composites) where, for example, a high possibility of material debonding and cracking or sliding of the interface exists. Consequently, this problem has been the subject of research and discussion in the classical literature on elasticity theory.

Recently, it has been shown that a more accurate and comprehensive analysis of the deformation of an elastic solid with one or more surfaces, can be achieved by incorporating a description of the separate surface mechanics near each surface of the solid. In the case of a solid containing a crack, a comprehensive model includes surface effects corresponding to the two surfaces (faces) of the crack. In the context of continuum-based analytical models, the surface model proposed in [1, 2] has been used extensively in a number of studies including several problems dealing with fracture mechanics (see, for example, [3, 4] and the bibliographies contained therein). In this paper, we further extend this idea and consider anti-plane deformations of a linearly elastic solid consisting of two

(perfectly) bonded *dissimilar* isotropic elastic materials (here represented by two bonded half-spaces), in the presence of a crack along the material interface. Most significantly, the crack faces are assumed to have elastic properties different from those of each of the bulk materials.

Using complex variable methods, we reduce the corresponding problem to a system of coupled Cauchy singular integro-differential equations [5] which is solved numerically using an adapted collocation technique [6]. This leads to a complete semi-analytic solution, valid throughout the entire domain of interest (including at the crack tips). Finally, we show that, among various other interesting phenomena, the stress component (σ_{xz}) demonstrates a discontinuity across the bi-material interface, which is in contrast to the classical results from linear elastic fracture mechanics [7, 8].

Throughout the paper, we make use of a number of well-established symbols and conventions. Thus, unless otherwise stated, Greek and Latin subscripts take the values 1, 3 and 1, 2, 3, respectively, summation over repeated subscripts is understood, (x, z) and (x, y, z) are generic points in the (x, z) -plane and \mathbb{R}^3 , respectively and δ_{ij} are the Kronecker delta. Finally, we note that the generic points (x, z) and (x, y, z) may also be labelled (x_1, x_3) and (x_1, x_2, x_3) , when reference is made to $\{\underline{e}_i\}_{i=1}^3$, the standard basis for \mathbb{R}^3 .

2. Anti-plane interface crack problem with surface stress: governing equations

It is well-known that in absence of the body forces, the equilibrium and constitutive equations describing the deformation of a linearly elastic, homogeneous and isotropic (bulk) solid are given by:

$$(2.1) \quad \sigma_{ij,j} = 0, \quad \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij},$$

$$(2.2) \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where λ and μ are the Lamé constants of the material, u_i is the i^{th} component of the displacement vector \mathbf{u} in \mathbb{R}^3 , $(\cdot)_{,j}$ denotes differentiation with respect to x_j , and σ_{ij} , ε_{ij} are the components of the stress and strain tensors in the bulk material, respectively.

2.1. Surface equation

Although Eqs. (2.1)–(2.2) remain true in the bulk material, equilibrium on the (crack) surface is now described by the equations (see [1, 2] and [9] for detailed derivations):

$$(2.3)_1 \quad \sigma_{\alpha\beta,\beta}^s \underline{e}_\alpha + [\sigma_{\alpha j} n_j \underline{e}_\alpha] = 0, \text{ (tangential-direction),}$$

$$(2.3)_2 \quad k_{\alpha\beta}\sigma_{\alpha\beta}^s = [\sigma_{ij}n_in_j], \text{ (normal direction),}$$

and

$$(2.3)_3 \quad \sigma_{\alpha\beta}^s = \sigma_0\delta_{\alpha\beta} + 2(\mu^s - \sigma_0)\varepsilon_{\alpha\beta}^s + (\lambda^s + \sigma_0)\varepsilon_{\gamma\gamma}^s\delta_{\alpha\beta} + \sigma_0\nabla_s u.$$

Here, the suffix s denotes the corresponding quantity on the crack face as a result of the surface elasticity, $[*] = (*)^{\text{in}} - (*)^{\text{out}}$ denotes the jump of the quantity “ $*$ ” across the surface (here “in” and “out” refer, respectively, to the inside and outside of the body) and σ_0 is the surface tension. The curvatures $k_{\alpha\beta}$ are defined in such a way that they are positive if the center of curvature is within the “ $-$ ” side (here, the “ $-$ ” and “ $+$ ” sides denote the lower ($y < 0$) and upper ($y > 0$) half-planes, as depicted in Fig. 1). Finally, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal vector of the surface pointing from the “ $-$ ” side to the “ $+$ ” side. We mention here that Eqs. (2.3) lack the additional term (namely the surface gradient term) present in the original Gurtin–Murdoch model. This term is intentionally neglected here since it does not contribute to the resulting equilibrium condition on the (crack) surface, i.e. in the analysis of anti-plane deformations only the out-of-plane displacement component $w(x, y)$ is considered.

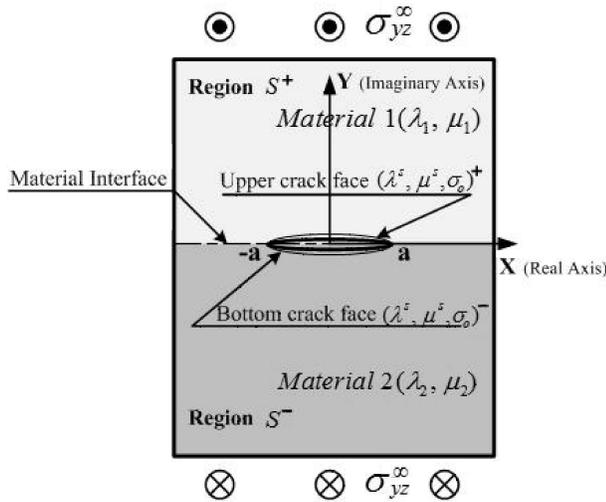


FIG. 1. Scheme of the problem.

REMARK 1. It is well-known that the relation between the surface stresses (σ^s) and surface energy (Γ) can be determined by

$$\sigma_{\alpha\beta}^s = \sigma_0\delta_{\alpha\beta} + \frac{\partial\Gamma}{\partial\varepsilon_{\alpha\beta}^s}.$$

In the Gurtin–Murdoch surface elasticity model, it is suggested that the surface energy (Γ) takes the form of a quadratic function of the surface strain invariants:

$$\Gamma = \frac{1}{2}(\lambda^s + \sigma_0)(\varepsilon_{\alpha\alpha}^s)^2 + (\mu^s - \sigma_0)(\varepsilon_{\alpha\beta}^s \varepsilon_{\alpha\beta}^s) + \sigma_0 \frac{1}{2} |\nabla_s u|^2,$$

from which the expression for the surface stresses can be obtained. It should be noted here that the interface stress-strain law depends on several factors, including the physical assumptions of the surface/interface elasticity and the geometrical changes of the surface with initial stress. This means that the stress-strain law may assume different forms, depending on the particular mathematical/physical assumptions adopted. Currently, there is no clear physical evidence in favor of any specific surface model.

2.2. Complex-variable formulation

In the anti-plane shear of an isotropic elastic medium, we assume that the displacement vector \mathbf{u} with components now denoted by (u, v, w) , satisfies the condition

$$(2.4) \quad w = w(x, y), \quad u = v = 0, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

In view of Eqs. (2.1)–(2.2), we have:

$$(2.5) \quad \begin{aligned} \sigma_{xz} &= 2\mu\varepsilon_{xz} = \mu \frac{\partial w}{\partial x}, & \sigma_{yz} &= 2\mu\varepsilon_{yz} = \mu \frac{\partial w}{\partial y}, \\ \sigma_{xy} &= \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0. \end{aligned}$$

Since $w(x, y)$ is a harmonic function, we denote by $\psi(x, y)$ its conjugate harmonic function. Introducing the complex variable $z = x + iy$, we can now write

$$(2.6) \quad w = \text{Re}[\Omega(z)], \quad \Omega(z) = w(x, y) + i\psi(x, y),$$

where $\Omega(z)$ is an analytic function of z in the domain under consideration (in our case, $S^+ \cup S^- = S$ exterior to the crack, as depicted in Fig. 1). From Eq. (2.6) we then have that

$$(2.7) \quad \frac{d\Omega}{dz}(z) = \Omega'(z) = \frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} = \frac{1}{\mu}(\sigma_{xz} - i\sigma_{yz})$$

and

$$(2.8) \quad \sigma_{yz} = \frac{\mu i}{2}[\Omega'(z) - \overline{\Omega'(z)}], \quad \sigma_{xz} = \frac{\mu}{2}[\Omega'(z) + \overline{\Omega'(z)}].$$

In addition, noting that (in our case) the normal to the crack face is aligned with the \underline{e}_2 or y -direction, the equilibrium conditions on the (crack) surface is now obtained from Eqs. (2.3)₁–(2.3)₂ as:

$$(2.9) \quad \sigma_{xz,x}^s + [\sigma_{yz}] = 0.$$

2.3. A traction-free Mode-III crack problem with surface stress

We consider anti-plane deformations of two bonded dissimilar half-planes, incorporating a single crack $[-a \leq x \leq a]$, ($y = 0$) on its interface and subjected to uniform remote shear stress $\sigma_{yz} = \sigma_{yz}^\infty$ (see Fig. 1). Let the upper half-plane ($y > 0$, occupied by *material "1"*) and the lower half-plane ($y < 0$, occupied by *material "2"*) be designated the “-” and “+” sides of the crack, respectively. The elastic properties of *material "1"* and *material "2"* are, in general, different. We further consider the situation when the interface under consideration is perfectly bonded, across which the traction (σ_{yz}) and displacement (w) are continuous (note that σ_{xz} is not necessarily continuous across the interface). Then the displacements and stresses for the upper and lower half-plane can be expressed as:

$$(2.10) \quad \begin{aligned} w^+ &= \frac{1}{2}[\Omega_1(z) + \overline{\Omega_1(z)}], & \text{for upper half } y > 0, S^+, \\ w^- &= \frac{1}{2}[\Omega_2(z) + \overline{\Omega_2(z)}], & \text{for lower half } y < 0, S^-, \end{aligned}$$

$$(2.11) \quad \begin{aligned} \sigma_{yz}^+ &= \frac{\mu_1 i}{2}[\Omega_1'(z) - \overline{\Omega_1'(z)}], & \text{for upper half } y > 0, S^+, \\ \sigma_{xz}^+ &= \frac{\mu_1}{2}[\Omega_1'(z) + \overline{\Omega_1'(z)}], \\ \sigma_{yz}^- &= \frac{\mu_2 i}{2}[\Omega_2'(z) - \overline{\Omega_2'(z)}], & \text{for lower half } y < 0, S^-, \\ \sigma_{xz}^- &= \frac{\mu_2}{2}[\Omega_2'(z) + \overline{\Omega_2'(z)}], \end{aligned}$$

where subscripts “1” and “2” represent the quantities from the upper half-plane (S^+) plane and lower half-plane (S^-), respectively.

From Eq. (2.9), the boundary conditions on the (crack) surface can be written as

$$(2.12) \quad \begin{aligned} \frac{\partial \sigma_{xz}^s}{\partial x} + (\sigma_{yz})^+ - (\sigma_{yz})^- &= 0, & \text{on the upper face,} \\ \frac{\partial \sigma_{xz}^s}{\partial x} + (\sigma_{yz})^+ - (\sigma_{yz})^- &= 0, & \text{on the lower face.} \end{aligned}$$

where, in the case of the present crack problem, the terms $(\sigma_{yz})^-$ in (2.12)₁ and $(\sigma_{yz})^+$ in (2.12)₂ are zero. In view of Eqs. (2.1), (2.2) and (2.3)₃ and the assumption of a coherent interface ($\varepsilon_{\alpha\beta}^s = \varepsilon_{\alpha\beta}$), the surface stresses can be expressed explicitly in terms of body (bulk) stresses as

$$(2.13) \quad \sigma_{xz}^s = 2(\mu^s - \sigma_0)\varepsilon_{xz} = \frac{\mu^s - \sigma_0}{\mu}\sigma_{xz}.$$

Therefore, the surface conditions on either side of the crack face $[-a < x < a]$, ($y = \pm 0$) can then be formulated as follows:

$$(2.14) \quad \begin{aligned} (\sigma_{yz})^+ &= -\frac{\partial \sigma_{xz}^s}{\partial x} = -(\mu^s - \sigma_0)^+ \frac{\partial^2 w^+}{\partial x^2}, & \text{on the upper face,} \\ (\sigma_{yz})^- &= +\frac{\partial \sigma_{xz}^s}{\partial x} = +(\mu^s - \sigma_0)^- \frac{\partial^2 w^-}{\partial x^2}, & \text{on the lower face,} \end{aligned}$$

where $(\mu^s - \sigma_0)^+ \neq (\mu^s - \sigma_0)^-$, in general. Adding and subtracting Eqs. (2.14) we obtain

$$(2.15) \quad \begin{aligned} (\sigma_{yz})^+ + (\sigma_{yz})^- &= -(\mu^s - \sigma_0)^+ \left(\frac{\partial^2 w^+}{\partial x^2} \right) + (\mu^s - \sigma_0)^- \left(\frac{\partial^2 w^-}{\partial x^2} \right), \\ (\sigma_{yz})^+ - (\sigma_{yz})^- &= -(\mu^s - \sigma_0)^+ \left(\frac{\partial^2 w^+}{\partial x^2} \right) - (\mu^s - \sigma_0)^- \left(\frac{\partial^2 w^-}{\partial x^2} \right). \end{aligned}$$

where, from Eqs. (2.10),

$$(2.16) \quad \frac{\partial^2 w^+}{\partial x^2} = \frac{1}{2}[\Omega_1''(z)^+ + \overline{\Omega_1''(z)^+}], \quad \frac{\partial^2 w^-}{\partial x^2} = \frac{1}{2}[\Omega_2''(z)^- + \overline{\Omega_2''(z)^-}].$$

The aforementioned assumptions imply that the displacements and stresses are continuous across the bi-material interface away from the crack ($y = 0$, $|x| > a$). Therefore, we derive from Eqs. (2.10)–(2.11) that

$$\begin{aligned} \mu_1[\Omega_1'(z)^+ - \overline{\Omega_1'(z)^+}] &= \mu_2[\Omega_2'(z)^- - \overline{\Omega_2'(z)^-}], \\ \Omega_1(z)^+ + \overline{\Omega_1(z)^+} &= \Omega_2(z)^- + \overline{\Omega_2(z)^-}, \quad y = 0, \quad x > |a|. \end{aligned}$$

By applying the relations $\overline{\Omega_1'(z)^+} = \overline{\Omega_1'(z)^-}$ on $y = \pm 0$, we have

$$(2.17) \quad \begin{aligned} \mu_1 \Omega_1'(z)^+ + \mu_2 \overline{\Omega_2'(z)^+} &= \mu_2 \Omega_2'(z)^- + \mu_1 \overline{\Omega_1'(z)^-}, \\ \Omega_1(z)^+ - \overline{\Omega_2(z)^+} &= \Omega_2(z)^- - \overline{\Omega_1(z)^-}. \end{aligned}$$

Now, in view of Eqs. (2.17), we define analytic functions $\theta(z)$ and $\psi(z)$ in the whole plane ($S^+ \cup S^- = S$) cut along $L = -a \leq x \leq a$, $y = 0$ as:

$$(2.18) \quad \begin{aligned} \mu_1 \Omega_1'(z) + \mu_2 \overline{\Omega_2'(z)} &= \mu_2 \Omega_2'(z) + \mu_1 \overline{\Omega_1'(z)} \equiv \theta(z), \\ \Omega_1'(z) - \overline{\Omega_2'(z)} &= \Omega_2'(z) - \overline{\Omega_1'(z)} \equiv \psi(z). \end{aligned}$$

(Again, with the surface energy $\theta(z) \neq 0$, as $w^+ \neq w^-$ on $-a < x < a$, $y = \pm 0$). Therefore, Eq. (2.18)₁ can be rewritten for the upper and lower half-planes as:

$$(2.19) \quad \begin{aligned} \overline{\Omega_2'(z)} &\equiv -\frac{\mu_1}{\mu_2} \Omega_1'(z) + \frac{1}{\mu_2} \theta(z), & \text{for upper half-plane,} \\ \overline{\Omega_1'(z)} &\equiv -\frac{\mu_2}{\mu_1} \Omega_2'(z) + \frac{1}{\mu_1} \theta(z), & \text{for lower half-plane.} \end{aligned}$$

From Eq. (2.19), Eq. (2.18)₂ becomes

$$(2.20) \quad \begin{aligned} \Omega'_1(z) &= \frac{\mu_2\psi(z)}{\mu_1 + \mu_2} + \frac{1}{\mu_1 + \mu_2}\theta(z), & \text{for upper half-plane,} \\ \Omega'_2(z) &= \frac{\mu_1\psi(z)}{\mu_1 + \mu_2} + \frac{1}{\mu_1 + \mu_2}\theta(z), & \text{for lower half-plane.} \end{aligned}$$

Then, by applying Eqs. (2.19)–(2.20), Eq. (2.16) can now be re-formulated in terms of $\psi(z)$ and $\theta(z)$:

$$(2.21) \quad \begin{aligned} \frac{\partial^2 w^+}{\partial x^2} &= \frac{1}{2} \left[\frac{\mu_2}{\mu_1 + \mu_2} (\psi'(z)^+ - \psi'(z)^-) + \frac{1}{\mu_1 + \mu_2} (\theta'(z)^+ + \theta'(z)^-) \right], \\ \frac{\partial^2 w^-}{\partial x^2} &= \frac{1}{2} \left[\frac{\mu_1}{\mu_1 + \mu_2} (\psi'(z)^- - \psi'(z)^+) + \frac{1}{\mu_1 + \mu_2} (\theta'(z)^+ + \theta'(z)^-) \right], \end{aligned}$$

By substituting Eqs. (2.21) back into Eqs. (2.15), we have

$$(2.22) \quad \begin{aligned} (\sigma_{yz})^+ + (\sigma_{yz})^- &= -\frac{A^s}{\mu_1 + \mu_2} \left(\frac{\psi'(z)^+ - \psi'(z)^-}{2} \right) - \frac{B^s}{\mu_1 + \mu_2} \left(\frac{\theta'(z)^+ + \theta'(z)^-}{2} \right), \\ (\sigma_{yz})^+ - (\sigma_{yz})^- &= -\frac{C^s}{\mu_1 + \mu_2} \left(\frac{\psi'(z)^+ - \psi'(z)^-}{2} \right) - \frac{D^s}{\mu_1 + \mu_2} \left(\frac{\theta'(z)^+ + \theta'(z)^-}{2} \right), \end{aligned}$$

where

$$\begin{aligned} A^s &\equiv \mu_2(\mu^s - \sigma_0)^+ + \mu_1(\mu^s - \sigma_0)^-, & B^s &\equiv (\mu^s - \sigma_0)^+ - (\mu^s - \sigma_0)^-, \\ C^s &\equiv \mu_2(\mu^s - \sigma_0)^+ - \mu_1(\mu^s - \sigma_0)^-, & D^s &\equiv (\mu^s - \sigma_0)^+ + (\mu^s - \sigma_0)^-. \end{aligned}$$

In addition, now the left-hand side of Eqs. (2.22) can be expressed via Eqs. (2.11) and (2.19) as:

$$\begin{aligned} (\sigma_{yz})^+ + (\sigma_{yz})^- &= \frac{i}{2} [2\mu_1\Omega'_1(z)^+ + 2\mu_2\Omega'_2(z)^- - (\theta(z)^+ + \theta(z)^-)], \\ (\sigma_{yz})^+ - (\sigma_{yz})^- &= \frac{i}{2} [\theta(z)^+ - \theta(z)^-]. \end{aligned}$$

Therefore, we obtain from Eqs. (2.20) that

$$(2.23) \quad \begin{aligned} (\sigma_{yz})^+ + (\sigma_{yz})^- &= \frac{i}{2} \left[\frac{2\mu_1\mu_2}{\mu_1 + \mu_2} (\psi(z)^+ + \psi(z)^-) + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} (\theta(z)^+ - \theta(z)^-) \right], \\ (\sigma_{yz})^+ - (\sigma_{yz})^- &= \frac{i}{2} [\theta(z)^+ - \theta(z)^-]. \end{aligned}$$

Consequently, from Eqs. (2.23), Eqs. (2.22) take the following forms:

$$\begin{aligned}
& i \left[\frac{2\mu_1\mu_2}{\mu_1 + \mu_2} (\psi(z)^+ + \psi(z)^-) + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} (\theta(z)^+ - \theta(z)^-) \right] \\
& = -\frac{A^s}{\mu_1 + \mu_2} (\psi'(z)^+ - \psi'(z)^-) - \frac{B^s}{\mu_1 + \mu_2} (\theta'(z)^+ + \theta'(z)^-), \\
(2.24) \quad & i[\theta(z)^+ - \theta(z)^-] \\
& = -\frac{C^s}{\mu_1 + \mu_2} (\psi'(z)^+ - \psi'(z)^-) - \frac{D^s}{\mu_1 + \mu_2} (\theta'(z)^+ + \theta'(z)^-).
\end{aligned}$$

Next, if we write the unknowns $\psi(z)$ and $\theta(z)$ as Cauchy integrals [5], noting the endpoint conditions which characterize the requirement that the stresses should be bounded at the crack tips, we have that

$$\begin{aligned}
(2.25) \quad \psi(z) &= \frac{1}{2i\pi} \int_{-a}^{+a} \frac{f(t)}{t-z} dt + \frac{\mu_1 + \mu_2}{i\mu_1\mu_2} [\sigma_{yz}^\infty], \\
\psi'(z) &= \frac{1}{2\pi i} \int_{-a}^{+a} \frac{f(t) dt}{(t-z)^2} = - \left[\frac{f(t)}{t-z} \right]_{-a}^a + \frac{1}{2\pi i} \int_{-a}^{+a} \frac{f'(t) dt}{t-z} = \frac{1}{2\pi i} \int_{-a}^{+a} \frac{f'(t) dt}{t-z},
\end{aligned}$$

where

$$\begin{aligned}
f(t_0) &= \psi(z)^+ - \psi(z)^-, \quad f(a) = f(-a) = 0, \\
\psi(z)^+ &= \frac{1}{2} f(t_0) + \frac{1}{2i\pi} \int_{-a}^{+a} \frac{f(t)}{t-t_0} dt + \frac{\mu_1 + \mu_2}{i\mu_1\mu_2} [\sigma_{yz}^\infty], \\
\psi(z)^- &= -\frac{1}{2} f(t_0) + \frac{1}{2i\pi} \int_{-a}^{+a} \frac{f(t)}{t-t_0} dt + \frac{\mu_1 + \mu_2}{i\mu_1\mu_2} [\sigma_{yz}^\infty].
\end{aligned}$$

In addition, in view of Eqs. (2.24)₂ and (2.25)₂, $(\theta(z)^+ - \theta(z)^-)$ must be purely imaginary. Therefore, we express the unknown $\theta(z)$ as:

$$\begin{aligned}
(2.26) \quad \theta(z) &= \frac{1}{2i\pi} \int_{-a}^{+a} \frac{i\alpha(t)}{t-z} dt, \\
\theta'(z) &= \frac{1}{2\pi i} \int_{-a}^{+a} \frac{i\alpha(t) dt}{(t-z)^2} = - \left[\frac{i\alpha(t)}{t-z} \right]_{-a}^a + \frac{1}{2\pi i} \int_{-a}^{+a} \frac{i\alpha'(t) dt}{t-z} \\
&= \frac{1}{2\pi i} \int_{-a}^{+a} \frac{i\alpha'(t) dt}{t-z},
\end{aligned}$$

where

$$i\alpha(t_0) = \theta(z)^+ - \theta(z)^-, \quad \alpha(a) = \alpha(-a) = 0,$$

$$\theta(z)^+ = \frac{1}{2}i\alpha(t_0) + \frac{1}{2i\pi} \int_{-a}^{+a} \frac{i\alpha(t)}{t-t_0} dt,$$

$$\theta(z)^- = -\frac{1}{2}i\alpha(t_0) + \frac{1}{2i\pi} \int_{-a}^{+a} \frac{i\alpha(t)}{t-t_0} dt.$$

Finally, from Eqs. (2.24–2.26), we obtain the following Cauchy singular integro-differential equations for the unknowns $f(t)$ and $\alpha(t)$, $t \in (-a, a)$:

$$(2.27) \quad \frac{2\mu_1\mu_2}{\pi} \int_{-a}^{+a} \frac{f(t)}{t-t_0} dt + 4(\mu_1 + \mu_2)[\sigma_{yz}^\infty] - (\mu_1 - \mu_2)\alpha(t_0) = -A^s f'(t_0) - \frac{B^s}{\pi} \int_{-a}^{+a} \frac{\alpha'(t)}{t-t_0} dt,$$

$$\alpha(t_0) = \frac{C^s}{\mu_1 + \mu_2} f'(t_0) + \frac{D^s}{\pi(\mu_1 + \mu_2)} \int_{-a}^{+a} \frac{\alpha'(t)}{t-t_0} dt.$$

3. Solution of singular integro-differential equations by a collocation method

The equations appearing in Eqs. (2.27) are coupled first-order Cauchy singular integro-differential equations. Although similar types of equations have been well studied, classical methods of their solution are not directly applicable here without additional mathematical intervention. In this section, we employ the T^{-1} operator from [8] and [10] and a collocation method [6] to analyze the problems mentioned above.

By replacing $\alpha(t_0)$ in Eq. (2.27)₁ and $f'(t_0)$ in Eq. (2.27)₂ by their counterparts, we derive the following new system of equations:

$$(3.1) \quad \int_{-a}^{+a} \frac{-\frac{2C^s\mu_1\mu_2}{A^s} f(t) + (D^s - \frac{C^s B^s}{A^s}) \alpha'(t)}{t-t_0} dt = \pi \left(\mu_1 + \mu_2 - \frac{C^s(\mu_1 - \mu_2)}{A^s} \right) \alpha(t_0) + \frac{4\pi C^s(\mu_1 + \mu_2)}{A^s} [\sigma_{yz}^\infty].$$

$$\int_{-a}^{+a} \frac{-2\mu_1\mu_2 f(t) + \left(\frac{D^s(\mu_1 - \mu_2)}{\mu_1 + \mu_2} - B^s \right) \alpha'(t)}{t-t_0} dt = \pi \left(A^s - \frac{C^s(\mu_1 - \mu_2)}{\mu_1 + \mu_2} \right) f'(t_0) + 4\pi(\mu_1 + \mu_2)[\sigma_{yz}^\infty].$$

Assume $t/a = x$ in Eqs. (3.1) and obtain

$$\begin{aligned}
 & \int_{-1}^{+1} \frac{-\frac{2C^s\mu_1\mu_2}{A^s}f(ax) + \left(D^s - \frac{C^sB^s}{A^s}\right)\frac{d\alpha(ax)}{d(ax)}}{a(x-x_0)}(a)dx \\
 &= \pi \left(\mu_1 + \mu_2 - \frac{C^s(\mu_1 - \mu_2)}{A^s} \right) \alpha(ax_0) + \frac{4\pi C^s(\mu_1 + \mu_2)}{A^s} [\sigma_{yz}^\infty], \\
 (3.2) \quad & \int_{-1}^{+1} \frac{-2\mu_1\mu_2f(ax) + \left(\frac{D^s(\mu_1-\mu_2)}{\mu_1+\mu_2} - B^s\right)\frac{d\alpha(ax)}{d(ax)}}{a(x-x_0)}(a)dx \\
 &= \pi \left(A^s - \frac{C^s(\mu_1 - \mu_2)}{\mu_1 + \mu_2} \right) \frac{df(ax_0)}{d(ax_0)} + 4\pi(\mu_1 + \mu_2)[\sigma_{yz}^\infty].
 \end{aligned}$$

Rewriting $x \rightarrow t$, $x_0 \rightarrow t_0$ and further defining $f(at) = u(t)$, $\alpha(at) = \eta(t)$, from Eqs. (2.21), we obtain

$$\begin{aligned}
 & \int_{-1}^{+1} \frac{-\frac{2C^s\mu_1\mu_2}{A^s}u(t) + \left(\frac{D^s}{a} - \frac{C^sB^s}{aA^s}\right)\eta'(t)}{t-t_0}dt \\
 &= \pi \left(\mu_1 + \mu_2 - \frac{C^s(\mu_1 - \mu_2)}{A^s} \right) \eta(t_0) + \frac{4\pi C^s(\mu_1 + \mu_2)}{A^s} [\sigma_{yz}^\infty], \\
 (3.3) \quad & \int_{-1}^{+1} \frac{-2\mu_1\mu_2u(t) + \left(\frac{D^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)} - \frac{B^s}{a}\right)\eta'(t)}{t-t_0}dt \\
 &= \pi \left(\frac{A^s}{a} - \frac{C^s(\mu_1 - \mu_2)}{a(\mu_1 + \mu_2)} \right) u'(t_0) + 4\pi(\mu_1 + \mu_2)[\sigma_{yz}^\infty],
 \end{aligned}$$

where $1 < t_0 < 1$, $u(1) = u(-1) = \eta(1) = \eta(-1) = 0$.

We now utilize the first inverse operator T_{1st}^{-1} defined in the following manner (see [8] or [10]):

$$\begin{aligned}
 (3.4) \quad T_{1st}^{-1}\psi(t) &= \frac{\sqrt{1-t_0^2}}{\pi} \int_{-1}^1 \psi(t)dt \\
 &\quad - \frac{\sqrt{1-t_0^2}}{\pi^2} \int_{-1}^1 \frac{\psi(t)}{(t-t_0)\sqrt{1-t^2}}dt, \quad t_0 \in (-1, 1), \\
 T(T^{-1}\psi) &= \psi.
 \end{aligned}$$

It follows then from Eq. (3.3)₁ that

$$\begin{aligned}
 (3.5) \quad & -\frac{2C^s\mu_1\mu_2}{A^s}u(t_0) + \left(\frac{D^s}{a} - \frac{C^sB^s}{aA^s}\right)\eta'(t_0) \\
 & = \frac{\sqrt{1-t_0^2}}{\pi} \int_{-1}^1 \left[-\frac{2C^s\mu_1\mu_2}{A^s}u(t) + \left(\frac{D^s}{a} - \frac{C^sB^s}{aA^s}\right)\eta'(t)\right] dt \\
 & \quad - \frac{\sqrt{1-t_0^2}}{\pi} \int_{-1}^1 \frac{\left(\mu_1 + \mu_2 - \frac{C^s(\mu_1-\mu_2)}{A^s}\right)\eta(t) + \frac{4C^s(\mu_1+\mu_2)}{A^s}[\sigma_{yz}^\infty]}{(t-t_0)\sqrt{1-t^2}} dt.
 \end{aligned}$$

Similarly, by applying the second inverse operator T_{2nd}^{-1} as defined by the relation in [6]

$$\begin{aligned}
 (3.6) \quad T_{2nd}^{-1}\psi(t) & = \frac{1}{\pi\sqrt{1-t_0^2}} \int_{-1}^1 \psi(t) dt \\
 & \quad - \frac{1}{\pi^2\sqrt{1-t_0^2}} \int_{-1}^1 \frac{\sqrt{1-t^2}\psi(t)}{t-t_0} dt, \quad t_0 \in (-1, 1),
 \end{aligned}$$

$$T(T^{-1}\psi) = \psi,$$

we obtain from Eq. (3.3)₂ that [3]

$$\begin{aligned}
 (3.7) \quad & \sqrt{1-t_0^2} \left[-2\mu_1\mu_2u(t_0) + \left(\frac{D^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)} - \frac{B^s}{a}\right)\eta'(t_0)\right] \\
 & = \frac{1}{\pi} \int_{-1}^1 \left[-2\mu_1\mu_2u(t) + \left(\frac{D^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)} - \frac{B^s}{a}\right)\eta'(t)\right] dt \\
 & \quad - \frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-t^2}}{t-t_0} \left[\left(\frac{A^s}{a} - \frac{C^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)}\right)u'(t) + 4(\mu_1+\mu_2)[\sigma_{yz}^\infty]\right] dt.
 \end{aligned}$$

If we assume that the functions u and η have an (approximate) expansion of the form

$$(3.8) \quad u(t_0) = \sum_{m=0}^N a_m T_m(t_0), \quad \eta(t_0) = \sum_{m=0}^N b_m T_m(t_0), \quad m = 0, 1, 2, \dots,$$

where $T_m(t_0)$ represents the m -th Chebychev polynomial of the first kind, the Eqs. (3.6)–(3.7) can then be transformed into the following system of equations (see details in [3, 11] and the associated properties of Chebychev polynomials therein):

$$\begin{aligned}
(3.9) \quad & \sum_{m=0}^N \left[a_m T_m(t_0) \left\{ 2\mu_1\mu_2\sqrt{1-t_0^2} + m \left(\frac{A^s}{a} - \frac{C^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)} \right) \right\} \right. \\
& \quad \left. - \frac{2\mu_1\mu_2}{\pi} a_m \left(\frac{1+(-1)^m}{1-m^2} \right) + b_m \left(\frac{D^s(\mu_1-\mu_2)}{a(\mu_1+\mu_2)} - \frac{B^s}{a} \right) \right. \\
& \quad \left. \times \left\{ \frac{(1-(-1)^m)}{\pi} - \sqrt{1-t_0^2} m U_{m-1}(t_0) \right\} \right] \\
& \quad = -4t_0(\mu_1+\mu_2)[\sigma_{yz}^\infty],
\end{aligned}$$

where

$$\begin{aligned}
(3.10) \quad & b_m = a_m \left(-\frac{2C^s\mu_1\mu_2}{A^s} \right) \left[\frac{\sqrt{1-t_0^2}}{\pi} \left(\frac{1+(-1)^m}{1-m^2} \right) - T_m(t_0) \right] \\
& \times \left[U_{m-1}(t_0) \left\{ \left(\frac{D^s}{a} - \frac{C^s B^s}{a A^s} \right) m + \sqrt{1-t_0^2} \left(\mu_1 + \mu_2 - \frac{C^s(\mu_1-\mu_2)}{A^s} \right) \right\} \right. \\
& \quad \left. - \frac{(1-(-1)^m)\sqrt{1-t_0^2}}{\pi} \left(\frac{D^s}{a} - \frac{C^s B^s}{a A^s} \right) \right]^{-1},
\end{aligned}$$

and the end conditions $u(1) = u(-1) = \eta(1) = \eta(-1) = 0$ as:

$$\begin{aligned}
(3.11) \quad & \sum_{m=0}^N a_m (-1)^m = 0, \quad \sum_{m=0}^N a_m = 0, \\
& \sum_{m=0}^N b_m (-1)^m = 0, \quad \sum_{m=0}^N b_m = 0, \quad m = 0, 1, 2, \dots
\end{aligned}$$

We now select the set of collocation points as given by $t_0 = t_{0i} = -\cos(i\pi/N)$ for $i = 1, 2, \dots, N-1$. In addition, by evaluating Chebychev polynomials of the first kind $T_m(t_{0i})$ and the second kind $U_{m-1}(t_{0i})$ with respect to each collocation point, we obtain that

$$\begin{aligned}
(3.12) \quad & T_m \left(-\cos \left(\frac{i\pi}{N} \right) \right) = -\cos \left(\frac{im\pi}{N} \right), \\
& U_{m-1} \left(-\cos \left(\frac{i\pi}{N} \right) \right) = \frac{\sin \left(\frac{mi\pi}{N} \right)}{\sin \left(\frac{i\pi}{N} \right)}.
\end{aligned}$$

Consequently, in view of Eq. (3.12), Eqs. (3.10)–(3.11) further reduce to the following system of linear equations:

$$\begin{aligned}
 (3.13) \quad & \sum_{m=0}^N \left[-a_m \cos\left(\frac{im\pi}{N}\right) \left\{ 2\mu_1\mu_2 \sqrt{1 - \left(\cos\left(\frac{i\pi}{N}\right)\right)^2} + m \left(\frac{A^s}{a} - \frac{C^s(\mu_1 - \mu_2)}{a(\mu_1 + \mu_2)} \right) \right\} \right. \\
 & \quad \left. - \frac{2\mu_1\mu_2}{\pi} a_m \left(\frac{1 + (-1)^m}{1 - m^2} \right) \right. \\
 & \quad \left. + b_m \left(\frac{D^s(\mu_1 - \mu_2)}{a(\mu_1 + \mu_2)} - \frac{B^s}{a} \right) \left\{ \frac{(1 - (-1)^m)}{\pi} - \sqrt{1 - \left(\cos\left(\frac{i\pi}{N}\right)\right)^2} m \left(\frac{\sin\left(\frac{mi\pi}{N}\right)}{\sin\left(\frac{i\pi}{N}\right)} \right) \right\} \right] \\
 & = 4 \cos\left(\frac{i\pi}{N}\right) (\mu_1 + \mu_2) [\sigma_{yz}^\infty],
 \end{aligned}$$

where

$$\begin{aligned}
 (3.14) \quad & b_m = \mathbf{K} a_m, \\
 & \mathbf{K} = a_m \left(-\frac{2C^s\mu_1\mu_2}{A^s} \right) \left[\frac{\sqrt{1 - \left(\cos\left(\frac{i\pi}{N}\right)\right)^2}}{\pi} \left(\frac{1 + (-1)^m}{1 - m^2} \right) + \cos\left(\frac{im\pi}{N}\right) \right] \\
 & \quad \times \left[\frac{\sin\left(\frac{mi\pi}{N}\right)}{\sin\left(\frac{i\pi}{N}\right)} \left\{ \left(\frac{D^s}{a} - \frac{C^s B^s}{a A^s} \right) m + \sqrt{1 - \left(\cos\left(\frac{i\pi}{N}\right)\right)^2} \left(\mu_1 + \mu_2 - \frac{C^s(\mu_1 - \mu_2)}{A^s} \right) \right\} \right. \\
 & \quad \left. - \frac{(1 - (-1)^m) \sqrt{1 - \left(\cos\left(\frac{i\pi}{N}\right)\right)^2}}{\pi} \left(\frac{D^s}{a} - \frac{C^s B^s}{a A^s} \right) \right]^{-1},
 \end{aligned}$$

for $1 \leq i \leq N - 1$. In addition, from the end conditions Eq. (38), we have that

$$\begin{aligned}
 (3.15) \quad & \sum_{m=0}^N a_m = \sum_{m=0}^N b_m = 0, \quad \text{for } i = 0, \\
 & \sum_{m=0}^N a_m (-1)^m = \sum_{m=0}^N b_m (-1)^m = 0, \quad \text{for } i = N.
 \end{aligned}$$

4. Results and discussion

In this section, the numerical solution of Eqs. (3.13)–(3.15) is performed for a range of surface parameters. The listed values are estimated properties of “GaN” obtained from the work of SHARMA and GANTI in [12]. A series of Gallium nitride (GaN) material distinguishes itself by high heat capacity and mechanical stability and therefore, it is used in the manufacture of semi-conductors.

$$(4.1) \quad S_e = \frac{\mu^s - \sigma_0}{a(\mu_1 + \mu_2)} : 8.65 \times 10^{-5} < S_e < 0.0865, \quad 10 \text{ nm} < a < 10 \text{ }\mu\text{m},$$

$$\mu^s = 161.73 \text{ (J/m}^2\text{)}, \quad \sigma_0 = 1.3 \text{ (J/m}^2\text{)}, \quad \mu = 168 \text{ (Gpa)}.$$

Throughout the analysis, we have considered the situation where the material properties of the upper half-plane are assumed to be ten times greater than those of lower half-plane (i.e. $\mu_1 = 168$ (Gpa), $\mu_2 = 16.8$ (Gpa)), whereas the surface material properties on the upper and lower crack faces are set to be equal (i.e. $(\mu^s - \sigma_0)^+ = (\mu^s - \sigma_0)^-$). This is only because we currently have very limited sources of surface material properties available [12]. However, the method presented here is sufficiently general since it incorporates the case in which the surface material properties from the upper and lower crack faces are different ($(\mu^s - \sigma_0)^+ \neq (\mu^s - \sigma_0)^-$, see Eqs. (2.22) and (3.13)–(3.15)) and a wide range of surface parameters in the physical domain.

4.1. Comparison with known classical results

We first examine how the solution obtained here, in the presence of surface effects, differs from the solution of the classical interface anti-plane crack problem. The corresponding analytical solution of the latter problem can be found in [5] and [7]:

$$\psi(z) = \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{-i\sigma_{yz}^\infty z}{\sqrt{z^2 - a^2}}.$$

Evaluating $\psi(z)$ at $(-a < t < a)$, we have that

$$(4.2) \quad \begin{aligned} \psi(z)^+ &= \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{-i\sigma_{yz}^\infty t}{\sqrt{-(a^2 - t^2)}} \\ &= \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{-\sigma_{yz}^\infty t}{\sqrt{a^2 - t^2}}, \quad \text{on the upper face,} \\ \psi(z)^- &= \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{i\sigma_{yz}^\infty t}{\sqrt{-(a^2 - t^2)}} \\ &= \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{\sigma_{yz}^\infty t}{\sqrt{a^2 - t^2}}, \quad \text{on the lower face.} \end{aligned}$$

Then the corresponding difference between the upper and lower faces can be defined from Eqs. (2.25) by

$$(4.3) \quad \psi(z)^+ - \psi(z)^- = f(t) = \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \frac{-2\sigma_{yz}^\infty t}{\sqrt{a^2 - t^2}}, \quad -a < t < a.$$

Also, in the classical case, $\theta(z)$ is found to be zero. Thus, we have that

$$(4.4) \quad \theta(z)^+ - \theta(z)^- = \alpha(t) = 0.$$

Returning to our problem, the values of $f(t)$ can be estimated using Eqs. (3.13)–(3.15) and are plotted in Fig. 2, where the parameter S_e is varied by changing the dimension of the crack (i.e. $20 \text{ nm} < 2a < 20 \text{ }\mu\text{m}$).

Figure 2 clearly indicates that our solution reduces to that of the classical case as the surface effect becomes negligible. We have also found that, in contrast to the classical results (see Eq. (4.4)), $\alpha(t)$ has indeed non-zero values (see Fig. 3) resulting in a noticeable contribution to the stress field, especially on the real axis. This will be the subject of the following section.

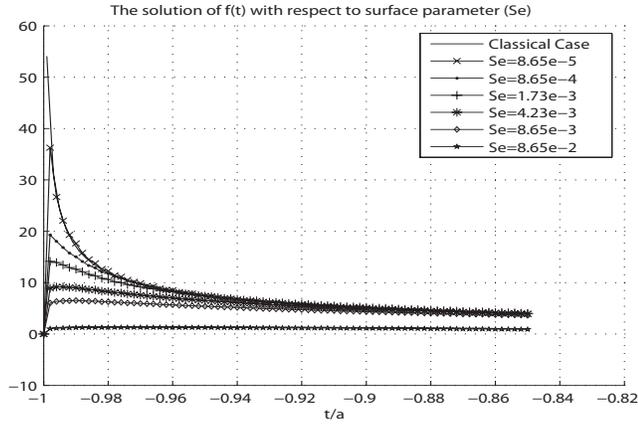


FIG. 2. The solution of $f(t)$ with respect to surface parameter (S_e), when $\frac{\sigma_{yz}^\infty}{\mu_1 + \mu_2} = 0.1$.

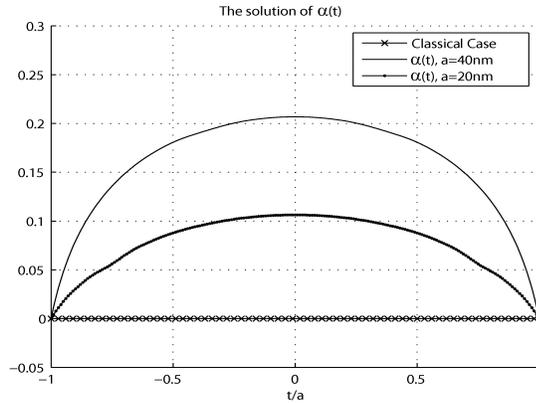


FIG. 3. The solution of $\alpha(t)$, when $\frac{\sigma_{yz}^\infty}{\mu_1 + \mu_2} = 0.1$.

4.2. Analysis of the stress distribution under the influence of surface effects

From Eqs. (2.11) and (2.20), stresses (σ_{yz}) on the upper and lower half-planes can be determined by

$$\begin{aligned}\sigma_{yz}^+ &= \frac{\mu_1 i}{2(\mu_1 + \mu_2)} \left[\mu_2 \psi(z) + \theta(z) - \mu_2 \overline{\psi(z)} - \overline{\theta(z)} \right], \\ \sigma_{xz}^+ &= \frac{\mu_1}{2(\mu_1 + \mu_2)} \left[\mu_2 \psi(z) + \theta(z) + \mu_2 \overline{\psi(z)} + \overline{\theta(z)} \right], \\ &\text{for upper half-plane } y > 0, S^+, \\ \sigma_{yz}^- &= \frac{\mu_2 i}{2(\mu_1 + \mu_2)} \left[\mu_1 \psi(z) + \theta(z) - \mu_1 \overline{\psi(z)} - \overline{\theta(z)} \right], \\ \sigma_{xz}^- &= \frac{\mu_2}{2(\mu_1 + \mu_2)} \left[\mu_1 \psi(z) + \theta(z) + \mu_1 \overline{\psi(z)} + \overline{\theta(z)} \right], \\ &\text{for lower half-plane } y < 0, S^-, \end{aligned}$$

where the complex potentials $\psi(t)$ and $\theta(t)$ can be obtained via Eqs. (2.25)₁ and (2.26)₁ with known solutions of $f(t)$ and $\alpha(t)$. These results are presented in Figs. 4–5 with clear indication of rapid convergence of the method (in approximately 30 iterations).

We have found that, in contrast to the classical case (where surface effects are completely neglected), the stresses at the crack tips remain finite and do indeed converge to the solutions from classical linear elastic fracture mechanics as the surface effects become insignificant (see Fig. 6). More importantly, the results in Figs. 6 and 7 illustrate the fact that σ_{yz} is continuous across the bi-material interface, whereas σ_{xz} jumps across the interface. The result is in sharp contrast to the classical fracture mechanics solution and due mainly to the non-zero contribution of the complex potential $\theta(z)$ in presence of the surface effects describing more accurate situation of stress discontinuity across the interface.

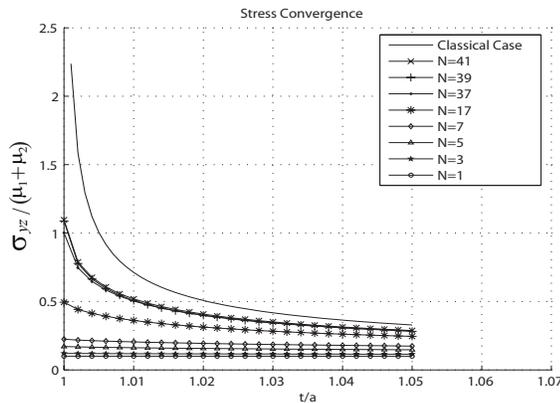


FIG. 4. Stress convergence for σ_{yz} , when $\frac{\sigma_{yz}^\infty}{\mu_1 + \mu_2} = 0.1$.

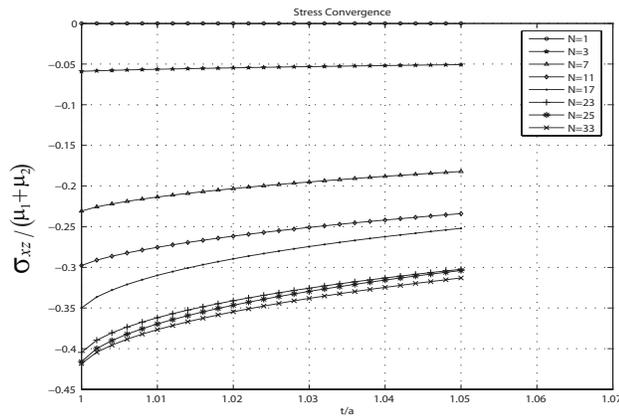


FIG. 5. Stress convergence for σ_{xz} , when $\frac{\sigma_{yz}^\infty}{\mu_1 + \mu_2} = 0.1$.

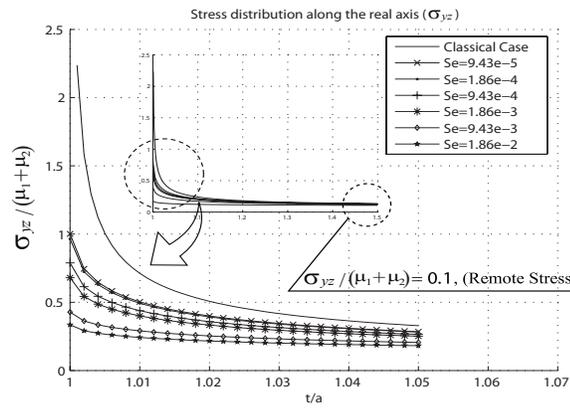


FIG. 6. Stress distribution (σ_{yz}) with respect to surface parameter, when $\frac{\sigma_{yz}^\infty}{\mu_1 + \mu_2} = 0.1$

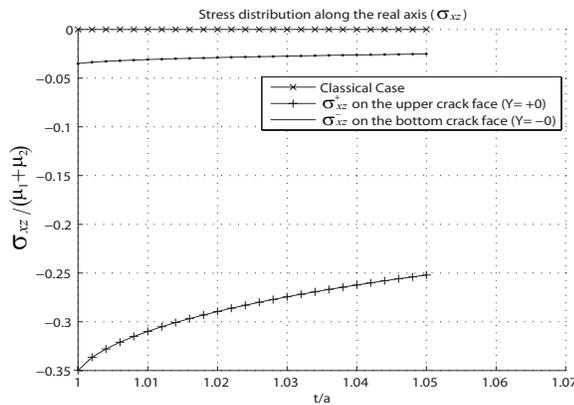


FIG. 7. Stress (σ_{xz}) jump across the bi-material interface.

REMARK 2. In the theory of linear elastic fracture mechanics, σ_{xz} is continuous across the bi-material interface, since its values are zero on both sides of the interface:

$$\begin{aligned}\sigma_{xz}^+ &= \frac{\mu_1}{2(\mu_1 + \mu_2)} \left[\mu_2 \psi(z) + \overline{\mu_2 \psi(z)} \right] \\ &= \sigma_{xz}^- = \frac{\mu_2}{2(\mu_1 + \mu_2)} \left[\mu_1 \psi(z) + \overline{\mu_1 \psi(z)} \right] = 0, \\ \therefore \psi(z) &= \text{Im}, \quad \text{on } y = \pm 0, \quad x > |a|.\end{aligned}$$

However, the interface condition under consideration indicates that traction (σ_{yz}) and displacements (w) are continuous across the interface, yet σ_{xz} are not necessarily continuous. Perhaps, the continuity in stress ($\sigma_{xz}^+ = \sigma_{xz}^-$) indicates the symmetrical nature of the problem in which the solution of Mode-III interface crack problem can be obtained by superposing solutions of two half-plane problems with distinct material properties on either side of the interface. In the case when the surface elasticity is present, the symmetry breaks down due to the effect of surface mechanics and therefore, the above-mentioned statement can no longer be satisfied. In fact, stress (σ_{xz}) on both sides of the interface can be estimated as:

$$\begin{aligned}\sigma_{xz}^+ &= \frac{\mu_1}{2(\mu_1 + \mu_2)} \left[\theta(z) + \overline{\theta(z)} \right], \quad \text{on } y = 0^+, \quad x > |a|, \\ \sigma_{xz}^- &= \frac{\mu_2}{2(\mu_1 + \mu_2)} \left[\theta(z) + \overline{\theta(z)} \right], \quad \text{on } y = 0^-, \quad x > |a|.\end{aligned}$$

This clearly indicates that the estimated stresses differ according to the material properties (μ_1, μ_2) of the upper and lower half-planes.

Finally, we see from Fig. 6 that stress distribution along the real axis increases when the surface effect becomes negligible and converges to the value 0.1 (which is the magnitude of the applied remote stress use in the computations) as we move away from the crack tips. Further, since the surface parameter S_e is controlled by variations in the crack length, our results also indicate that the corresponding stresses are strongly dependent on crack size [12, 13].

5. Conclusions

In this paper, we have incorporated the effects of surface elasticity into a classical Mode-III interface crack problem arising in the anti-plane shear deformations of a linearly elastic solid. The surface mechanics are employed using a version of the continuum-based surface/interface model of Gurtin and Murdoch. Complex variable methods are used to obtain a system of coupled

Cauchy singular integro-differential equations of the first-order which is solved numerically using an adapted collocation method. We have obtained a complete semi-analytic solution (not simply a crack-tip solution) which demonstrates several interesting phenomena, when the bi-material solid incorporates a traction-free crack on its interface and is subjected to uniform, remote loading. In particular, we note that the stresses at the (sharp) crack tip remain finite but more importantly, the stress (σ_{xz}) jumps across the bi-material interface, in contrast to the classical result from linear elastic fracture mechanics. We also mention that the solutions obtained here reduce to those obtained for the homogeneous material case (see [3]) by setting $\mu_1 = \mu_2$ and $(\mu^s - \sigma_0)^+ = (\mu^s - \sigma_0)^-$.

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References

1. M.E. GURTIN, A.I. MURDOCH, *A continuum theory of elastic material surfaces*, Arch. Ration. Mech. Anal., **57**, 4, 291–323, 1975.
2. M.E. GURTIN, J. WEISSMULLER, F. LARCHE, *A general theory of curved deformable interface in solids at equilibrium*, Philos. Mag. A, **78**, 5, 1093–1109, 1998.
3. C.I. KIM, P. SCHIAVONE, C.Q. RU, *The effects of surface elasticity on an elastic solid with mode-III crack: complete solution*, ASME J. Appl. Mech., **77**, 2, 021011 (1–7), 2009.
4. C.I. KIM, P. SCHIAVONE, C.Q. RU, *Analysis of plane-strain crack problems (mode-I and mode-II) in the presence of surface elasticity*, J. Elasticity, doi: 10.1007/s10659-010-9287-0, 2010.
5. N.I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff, Groningen, The Netherlands, 1953.
6. A. CHAKRABARTI, HAMSAPRIYE, *Numerical solution of a singular integro-differential equation*, ZAMM. Z. Angew. Math. Mech., **79**, 4, 233–241, 1999.
7. A.H. ENGLAND, *Complex Variable Methods in Elasticity*, John Wiley & Sons Ltd., London 1971.
8. F.D. GAKHOV, *Boundary value problems*, Pergamon Press, Oxford 1963.
9. C-Q. RU, *Simple geometrical explanation of Gurtin–Murdoch model of surface elasticity with clarification of its related versions*, Science China, **53**, 3, 536–544, 2010.
10. A. CHAKRABARTI, A.J. GEORGE, *Solution of a singular integral equation involving two intervals arising in the theory of water waves*, Appl. Math. Lett., **7**, 5, 43–47, 1994.
11. A.C. KAYA, F. ERDOGAN, *On the solution of integral equations with a generalized Cauchy kernel*, Quart. Appl. Math., **XLV**, 3, 455–469, 1987.

12. P. SHARMA, S. GANTI, *Size-dependent Eshelby's tensor for embedded nano-inclusions incorporating surface/interface energies*, JASME J. Appl. Mech., **71**, 5, 663–671, 2004.
13. L. TIAN, R.K.N.D. RAJAPAKSE, *Analytical solution of size-dependent elastic field of a nano-scale circular inhomogeneity*, JASME J. Appl. Mech., **74**, 3, 568–574, 2007.

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