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Profile reconstruction of a continuously-stratified layer from reflection data on acoustic waves

G. CAVIGLIA[1)], A. MORRO[2)]

[1)] *Department of Mathematics, University of Genova*
Via Dodecaneso 35, I-16146 Genova, Italy

[2)] *DIBE, University of Genova*
Via Opera Pia 11a, 16145 Genova, Italy
e-mail: angelo.morro@unige.it

The paper investigates the reflection-transmission process of acoustic waves, generated by an inhomogeneous fluid layer of finite thickness, which is sandwiched between two semi-infinite homogeneous half-spaces. First a direct problem is solved by determining the reflection and transmission coefficients along with the wave solution in the layer, produced by a known incident wave. Owing to the planar stratification of the layer, the unknown acoustic pressure is looked at as a generalized plane wave. Upon the Fourier transformation, the second-order wave equation is written as a first-order system of equations for the dependence on the depth of the pressure and the partial derivative. The corresponding Volterra integral equation gives the pressure in the layer as a series of repeated integrals of powers of the pertinent depth-dependent matrix of the system. The reflection and transmission coefficients of the layer are then determined for any incidence angle. Next an inverse problem is investigated. The derivatives of the reflection coefficient, with respect to the frequency, are shown to provide the thickness of the layer, the speed beyond the layer and the moments, of any order, of the refractive index.

Key words: stratified layer, acoustic waves, Volterra integral equation, profile reconstruction

1. Introduction

This paper investigates the reflection-transmission process of acoustic waves, generated by an inhomogeneous fluid layer of finite thickness, L, which is sandwiched between two semi-infinite homogeneous half-spaces. The model is intended to be representative, e.g. of an inhomogeneous marine sediment overlying a uniform substrate or of common seismological settings (see, e.g., [1]). Seismological exploration may well adopt more involved models such as that in [2] for electrokinetics in a porous medium. For definiteness, though, here we restrict to acoustic waves and assume the inhomogeneity to be planar in that the material parameters depend only on the depth coordinate, say $z \in [0, L]$. The functions describing the material parameters, here the mass density and sound speed, are only required to be bounded everywhere.

In the direct problem, a known incident wave comes from, e.g., $z < 0$ and we have to determine the waves produced in the layer and in the half-spaces. Exact solutions for planar stratified layers are limited to particular cases (see [3, 4]). Hence, the approximate methods are often applied. The most common approach to find the wave solutions is to approximate the layer by a finely layered medium, where the material parameters are piecewise constant, on the view that when the homogeneous sublayers are infinitesimally thin, the solution would approach that of the continuously-stratified layer (see e.g. [5, 6] and references therein). If the layer consists of, or is approximated by a stalk of homogeneous sublayers, then we can determine the solution directly in the time domain [7] in the form of an appropriate series. If, instead, the inhomogeneity is generic, then the recourse to integral equation methods is widely used (see, e.g., [8, 9]).

In addition to the approximation of a stalk of homogeneous sublayers, the literature shows a variety of approximate mathematical methods. Such is the case of difference equations [4] and of the WKBJ, eikonal and Born approximations [10]. Often the approach involves exact methods and leads to the solution in the form of a series of integrals of Green's operators such as the forward scattering series [1]. To our mind, handier forms of exact solutions are highly desirable. In this regard we point out that the use of Volterra integral equations [11] may be more convenient than that of Fredholm-type equations [1].

The purpose of this paper is twofold. First, to solve the direct problem by determining the wave solution produced by a known incident wave. Secondly, to determine an approximation of the refractive index N in the layer, in terms of the reflection data.

As a starting point, owing to the planar stratification of the layer we look for the unknown acoustic pressure p as a generalized plane wave, which means that p has the plane wave character relative to the time t and the transverse coordinates x, y. Upon the Fourier transformation, we can then write the second-order wave equation as a first-order system of equations for the dependence of p and $\partial_z p$ on z. The corresponding integral equation allows us to find explicitly the pressure p in the layer as a series of integrals of any order of powers of the pertinent z-dependent matrix of the system. However, the solution is parameterized by the initial values of p and $\partial_z p$ and this looks as a drawback because, in a reflection-transmission problem, the initial values are unknown. Nevertheless we show that the application of the standard jump (or matching) conditions at the interfaces $z = 0, L$ of the layer allows us to find the solution generated by the incident wave. In particular we find the reflection and transmission coefficients of the layer, parameterized by the frequency of the incident wave for any incidence angle.

Next an inverse problem is investigated. Given the reflection coefficient R, we want to determine the thickness L of the layer, the refractive index $N(z)$, $z \in (0, L)$ and the speed c_1 of the transmitted wave ($z > L$). We observe that R is a function of the frequency $\omega \in \mathbb{R}$ in the form of the ratio of two power series. The function $R(\omega)$, through the derivatives at $\omega = 0$, allows us to find L, c_1 and the moments of N of any order. It is of interest that, once we know the moments, we can express the function N as a series of the Legendre polynomials.

2. Acoustic wave equation

Let $\Omega \subseteq \mathbb{R}^3$ be the space region occupied by the body under consideration. We denote by $\mathbf{x} \in \Omega$ the vector position of a point. Acoustic waves in inhomogeneous inviscid fluids are governed by the differential equation [12]

$$\frac{1}{c^2}\partial_t^2 p - \Delta p = 0 \tag{2.1}$$

for the unknown pressure p on the space-time domain $\Omega \times \mathbb{R}$. We let the body be continuously stratified so that c^2 and the mass density ρ depend on one Cartesian coordinate only, say z.

It is worth remarking that the wave equation for p is often written in the form

$$\frac{1}{c^2}\partial_t^2 p - \rho\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = 0,$$

which traces back to Bergmann (see, e.g., [13]). In our opinion, Eq. (2.1) is the proper equation for p (see [11]).

The positive quantity c^2 is the derivative of the pressure with respect to the mass density ρ at constant entropy. We assume that p is related to ρ by a function such that

$$c^2(\rho) = \frac{dp}{d\rho}(\rho) \tag{2.2}$$

is invertible. If, for instance, p is related to ρ by Laplace's hypothesis $p = \kappa\rho^\gamma$, where γ is the specific-heat ratio, then

$$c^2(\rho) = \gamma\kappa\rho^{\gamma-1}$$

is invertible.

A function f, on $\Omega \times \mathbb{R}$, is a plane wave propagating in the direction of the constant vector $\mathbf{q}$, with speed $1/|\mathbf{q}|$, if

$$f(\mathbf{x}, t) = \mathcal{F}(t - \mathbf{q} \cdot \mathbf{x})$$

for some function $\mathcal{F}$, on $\mathbb{R}$, which describes the profile of the wave. In general inhomogeneous media, plane-wave solutions to (2.1) do not hold. In stratified fluids we consider generalised plane waves in the form

$$g(\mathbf{x}, t) = \mathcal{G}(z, t - \mathbf{m} \cdot \mathbf{x}),$$

where $\mathbf{m}$ is perpendicular to the z-axis, for some function $\mathcal{G}$ on $\mathbb{R}^2$ (see [14]). By means of appropriate Cartesian axes we assume $\mathbf{m}$ to be directed along the x-axis and write

$$g(\mathbf{x}, t) = \mathcal{G}(z, t - \xi x), \qquad \xi \in \mathbb{R}.$$

To simplify the connection with the literature on the subject, we let the Fourier transform $\tilde{g}$, with respect to time t, be given by

$$\tilde{g}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} g(\mathbf{x}, t) \exp(i\omega t) dt.$$

Hence we have

$$\tilde{g}(\mathbf{x}, \omega) = \exp(i\omega\xi x)\tilde{\mathcal{G}}(z, \omega). \tag{2.3}$$

Apply now the Fourier transform to (2.1) to obtain

$$\frac{\omega^2}{c^2(z)}\tilde{p}(\mathbf{x}, \omega) + \Delta\tilde{p}(\mathbf{x}, \omega) = 0. \tag{2.4}$$

By (2.3) we can write

$$\tilde{p}(\mathbf{x}, \omega) = \exp(i\omega\xi x)P(z, \omega).$$

Substitution in (2.4) gives

$$\partial_z^2 P + \omega^2 n(z; \xi)P = 0, \tag{2.5}$$

where n is a function of z, parameterized by ξ, given by

$$n(z; \xi) = N(z) - \xi^2, \qquad N(z) := \frac{1}{c^2(z)}. \tag{2.6}$$

With a small error we regard N as the refractive index, though we should consider $c_0^2 N$ as the refractive index, c_0 being a reference speed. It is understood that $n > 0$.

Letting

$$\mathbf{w} = \begin{bmatrix} P \\ \partial_z P \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ -\omega^2 n & 0 \end{bmatrix},$$

we can write (2.5) in the form of a first-order system,

$$\partial_z \mathbf{w} = \mathbf{M}\mathbf{w}. \tag{2.7}$$

If N is constant, then

$$\exp(i(\omega\xi x + kz)) \tag{2.8}$$

is a solution to (2.5) if $k = \omega\sqrt{n}$. Hence

$$(\omega\xi)^2 + k^2 = \omega^2 N$$

and (2.8) is a plane wave solution with $\omega\xi$ being the x-component and $\omega\sqrt{n}$ the z-component of the wave number vector $\boldsymbol{\kappa}$ such that

$$\boldsymbol{\kappa}^2 = \omega^2 N.$$

If θ is the angle of $\boldsymbol{\kappa}$ with the z-axis, then

$$\omega\xi = \kappa \sin\theta, \qquad k = \kappa\cos\theta. \tag{2.9}$$

The constancy of ξ means that

$$\kappa \sin\theta = \text{constant}, \tag{2.10}$$

independent of z, which may be viewed as the content of Snell's law.

2.1. Integral equation

It is understood that the function P is parameterized by ω. Denote by $P(0)$, $P'(0)$ the values of P, $\partial_z P$ at $z = 0$. The obvious integration of (2.7) provides

$$\mathbf{w}(z) = \mathbf{w}(0) + \int_0^z \mathbf{M}(\eta)\mathbf{w}(\eta)d\eta, \tag{2.11}$$

where $\mathbf{w}(0) = [P(0), P'(0)]^T$. In component form we can write (2.11) as

$$w_1(z) = w_1(0) + \int_0^z w_2(\eta)d\eta, \qquad w_2(z) = w_2(0) - \omega^2 \int_0^z n(\eta)w_1(\eta)d\eta. \tag{2.12}$$

Also (2.11) can be viewed as a Volterra integral equation for $\mathbf{w}$ with initial value $\mathbf{w}(0)$. We now apply (2.11) to obtain $P(z)$ in terms of $P(0), P'(0)$ and of the parameter ω.

Preliminarily we prove the following statement.

Proposition 1. *If n is bounded in $[0,L]$ then w is bounded in $[0,L]$.*

The proof is immediate. Let $z\in[0,L]$. Define

$$\psi(z)=|w_1(z)|+|w_2(z)|.$$

By (2.12) we obtain

$$\psi(z)\le\psi(0)+\int_0^z h(\eta)\psi(\eta)d\eta,$$

where

$$h(\eta)=1+\omega^2|n(\eta)|.$$

By the Gronwall inequality we obtain

$$\psi(z)\le\psi(0)|+\int_0^z\left[h(\eta)\exp\left(\int_\eta^z h(u)du\right)\right]d\eta.$$

The boundedness of n implies the boundedness of ψ, and hence of P and P', in $[0,L]$. □

As a consequence of Proposition 1 it follows that

$$\Psi=\sup_{z\in(0,L)}\psi(z)<\infty.$$

3. Representation of the solution

Substitution of $\mathbf{w}(\eta)$ in (2.11), $\eta=s_1$, provides

$$\mathbf{w}(z)=\mathbf{w}(0)+\int_0^z\mathbf{M}(s_1)ds_1\,\mathbf{w}(0)+\int_0^z ds_1\mathbf{M}(s_1)\int_0^{s_1}\mathbf{M}(s_2)\mathbf{w}(s_2)ds_2.$$

Subsequent substitutions give

$$\mathbf{w}(z)=\left[\mathbf{1}+\sum_{h=1}^m\int_0^z\int_0^{s_1}\dots\int_0^{s_{h-1}}\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_h)ds_1ds_2\dots ds_h\right]\mathbf{w}(0)+\hat{\mathbf{w}}(z),$$

where $\mathbf{1}$ is the identity and $\hat{\mathbf{w}}$ is the remainder given by

$$\hat{\mathbf{w}}(z)=\int_0^z\int_0^{s_1}\dots\int_0^{s_m}\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_{m+1})\mathbf{w}(s_{m+1})ds_1ds_2\dots ds_{m+1}.$$

Proposition 2. *The boundedness of n implies that $\hat{\mathbf{w}} \to 0$ as $m \to \infty$.*

Since n is bounded then M must be bounded too. Let

$$|M_{jk}(z)| < C, \qquad z \in (0, L), \quad j, k = 1, 2.$$

The entries of $\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_{m+1})$ are the sum of 2^m products of $m+1$ entries of $\mathbf{M}$. Hence

$$|[\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_{m+1})\mathbf{w}(s_{m+1})]_j| \le 2^m C^{m+1}\Psi, \qquad j = 1, 2.$$

Accordingly,

$$|\hat{\mathbf{w}}_j(z)| \le 2^m C^{m+1}\Psi \int_0^z \int_0^{s_1} \dots \int_0^{s_m} ds_1 ds_2 \dots ds_{m+1}.$$

Because

$$\int_0^z \int_0^{s_1} \dots \int_0^{s_{m+1}} ds_1 ds_2 \dots ds_{m+1} = \frac{z^{(m+1)}}{(m+1)!} \le \frac{L^{(m+1)}}{(m+1)!},$$

we obtain

$$|\hat{w}_j(z)| \le \frac{(2CL)^{m+1}\Psi}{2(m+1)!}, \qquad j = 1, 2.$$

The limit as $m \to \infty$ provides the conclusion. □

It is of interest that the remainder $\hat{\mathbf{w}}$ approaches zero for any values of C and L and hence without any requirements on the bound C and on the thickness L.

By Proposition 2 we can write the solution $\mathbf{w}(z)$ as a series of integrals of increasing dimensions. Precisely, letting

$$\boldsymbol{\mathcal{M}}(z) = \mathbf{1} + \sum_{h=1}^{\infty} \int_0^z \int_0^{s_1} \dots \int_0^{s_{h-1}} \mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_h) ds_1 ds_2 \dots ds_h \tag{3.1}$$

we can write

$$\mathbf{w}(z) = \boldsymbol{\mathcal{M}}(z)\mathbf{w}(0). \tag{3.2}$$

By (3.2) we can say that $\mathbf{w}(z)$ is a linear superposition of the entries of $\boldsymbol{\mathcal{M}}(z)$. Indeed,

$$P(z) = w_1(0)\mathcal{M}_{11}(z) + w_2(0)\mathcal{M}_{12}(z), \tag{3.3}$$

and

$$P'(z) = w_1(0)\mathcal{M}_{21}(z) + w_2(0)\mathcal{M}_{22}(z). \tag{3.4}$$

It is then of interest to investigate the form of the entries $\mathcal{M}_{jk}$, $j, k = 1, 2$.

3.1. On the form of the entries of $\mathcal{M}$

The product of an even number of matrices $\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_{2p})$ is a diagonal matrix whereas the product of an odd number of matrices $\mathbf{M}(s_1)\mathbf{M}(s_2)\dots\mathbf{M}(s_{2p+1})$ has non-zero entries only in the secondary diagonal. Indeed, we find the following result.

Proposition 3. *Let* $\mathbf{A} : [0, L] \to \mathbb{R}^{2\times 2}$ *be given by*

$$\mathbf{A}(s) = \begin{bmatrix} 0 & f(s) \\ g(s) & 0 \end{bmatrix},$$

f, g being given functions on $[0, L]$. Hence

$$\mathbf{A}(s_1)\mathbf{A}(s_2)\dots\mathbf{A}(s_{2p}) = \begin{bmatrix} f(s_1)g(s_2)\dots f(s_{2p-1})g(s_{2p}) & 0 \\ 0 & g(s_1)f(s_2)\dots g(s_{2p-1})f(s_{2p}) \end{bmatrix},$$

$$\mathbf{A}(s_1)\mathbf{A}(s_2)\dots\mathbf{A}(s_{2p+1}) = \begin{bmatrix} 0 & f(s_1)g(s_2)\dots g(s_{2p})f(s_{2p+1}) \\ g(s_1)f(s_2)\dots f(s_{2p})g(s_{2p+1}) & 0 \end{bmatrix}.$$

The application of Proposition 3 to $\mathbf{M}$, where $f(s) = 1, g(s) = -\omega^2 n(s)$, allows us to find the form of $\boldsymbol{\mathcal{M}}(z)$. It follows that

$$\mathcal{M}_{11}(z) = 1 + \sum_{p=1}^{\infty} (-1)^p \omega^{2p} \times \int_0^z \int_0^{s_1} \dots \int_0^{s_{2p-1}} n(s_2)n(s_4)\dots n(s_{2p})ds_1 ds_2 \dots ds_{2p}, \tag{3.5}$$

$$\mathcal{M}_{12}(z) = z + \sum_{p=1}^{\infty} (-1)^p \omega^{2p} \times \int_0^z \int_0^{s_1} \dots \int_0^{s_{2p}} n(s_2)n(s_4)\dots n(s_{2p})ds_1 ds_2 \dots ds_{2p+1}, \tag{3.6}$$

$$\mathcal{M}_{21}(z) = \sum_{p=0}^{\infty} (-1)^{p+1} \omega^{2(p+1)} \times \int_0^z \int_0^{s_1} \dots \int_0^{s_{2p}} n(s_1)n(s_3)\dots n(s_{2p+1})ds_1 ds_2 \dots ds_{2p+1}, \tag{3.7}$$

(3.8) $$\mathcal{M}_{22}(z) = 1 + \sum_{p=1}^{\infty} (-1)^p \omega^{2p} \times \int_0^z \int_0^{s_1} \dots \int_0^{s_{2p-1}} n(s_1)n(s_3)\dots n(s_{2p-1}) ds_1 ds_2 \dots ds_{2p}.$$

By (2.7) and (3.2) we have

$$w_1'(z) = \mathcal{M}_{11}'(z)w_1(0) + \mathcal{M}_{12}'(z)w_2(0) = \mathcal{M}_{21}(z)w_1(0) + \mathcal{M}_{22}w_2(0).$$

The arbitrariness of $w_1(0), w_2(0)$ requires that

(3.9) $$\mathcal{M}_{11}'(z) = \mathcal{M}_{21}(z), \qquad \mathcal{M}_{12}'(z) = \mathcal{M}_{22}.$$

Moreover, $\mathcal{M}_{11}$ and $\mathcal{M}_{12}$ satisfy the initial conditions

(3.10) $$\mathcal{M}_{11}(0) = 1, \qquad \mathcal{M}_{11}'(0) = 0, \qquad \mathcal{M}_{12}(0) = 0, \qquad \mathcal{M}_{12}'(0) = 1.$$

As it must be, direct differentiation of (3.5) and (3.6) shows that Eq. (3.9) holds.

4. The reflection-transmission problem

The Fourier components P are solutions to (2.5). Let

$$c(z) = c_0, \qquad n(z) = n_0 = N_0 - \xi^2, \qquad z < 0;$$
$$c(z) = c_1, \qquad n(z) = n_1 = N_1 - \xi^2, \qquad z > L,$$

where

$$N_0 = \frac{1}{c_0^2}, \qquad N_1 = \frac{1}{c_1^2}.$$

It is understood that $n_0, n_1 > 0$. As $z < 0$ we have

$$P(\omega, z) = A_\pm \exp(\pm ik_0 z), \qquad k_0 = \omega\sqrt{n_0},$$

A being complex-valued, and the like for $z > L$. The $+(-)$ sign denotes propagation in the forward (backward) z-direction. For, going back to the time domain we have

$$\exp[i(k_0 z - \omega t)] = \exp[i\omega(\sqrt{n_0}z - t)],$$

that is a wave propagating with speed $1/\sqrt{n_0}$ in the z-direction.

As $z \in (0, L)$, by (3.3) we have

$$P(z) = af_1(z) + bf_2(z),$$

where $a = P(0), b = P'(0)$, and

$$f_1(z) = \mathcal{M}_{11}(z), \qquad f_2(z) = \mathcal{M}_{12}(z). \tag{4.1}$$

This means that the pair f_1, f_2 is a basis for the solution P, the coefficients a, b being the initial values of P, P'. Remarkably, the basis f_1, f_2 is known explicitly in terms of a series of integrals of n, parameterized by ω. Also, since f_1, f_2 are solutions to (2.5), the Wronskian $W(z) = f_1(z)f_2'(z) - f_2(z)f_1'(z)$ is constant and hence is equal to the initial value, $W(0) = 1$,

$$f_1(z)f_2'(z) - f_2(z)f_1'(z) = 1, \qquad z \in (0, L). \tag{4.2}$$

Look at the reflection-transmission process generated by a plane wave $P_{\rm inc}$ which impinges obliquely on the layer from the $z < 0$ half-space. For formal convenience we let $A_{\rm inc} = 1$ so that

$$P_{\rm inc}(z) = \exp(ik_0 z), \qquad z < 0.$$

The solution P produced by $P_{\rm inc}$ is written in the form

$$P(z) = \begin{cases} \exp(ik_0 z) + R\exp(-ik_0 z), & z < 0, \\ af_1(z) + bf_2(z), & z \in (0, L), \\ T\exp(ik_1(z - L)), & z > L, \end{cases} \tag{4.3}$$

where $k_1 = \omega n_1$ and $R, a, b, T \in \mathbb{C}$. As a consequence,

$$P'(z) = \begin{cases} ik_0[\exp(ik_0 z) - R\exp(-ik_0 z)], & z < 0, \\ af_1'(z) + bf_2'(z), & z \in (0, L), \\ ik_1 T\exp(ik_1(z - L)), & z > L. \end{cases} \tag{4.4}$$

It is natural to regard R as the reflection coefficient and T as the transmission coefficient. Also, k_0 is the z-component of the wave-number vector of the incident wave, k_1 is the analogous z-component of the transmitted wave.

The mass density ρ is allowed to suffer jump discontinuities at $z = 0, L$ and hence the jump conditions are taken in the standard form

$$[\![P]\!] = 0, \qquad \left[\!\!\left[\frac{1}{\rho}P'\right]\!\!\right] = 0, \qquad z = 0, L, \tag{4.5}$$

where $[\![\quad]\!]$ denotes the jump at the pertinent value of z,

$$[\![P]\!](z) = P(z_+) - P(z_-).$$

Also let

$$J_0 = \frac{\rho(0_+)}{\rho(0_-)}, \qquad J_1 = \frac{\rho(L_-)}{\rho(L_+)}.$$

The continuity of P'/ρ gives

$$J_0 P'(0_-) = P'(0_+), \qquad P'(L_-) = J_1 P'(L_+).$$

Let $\hat{f}_m, \hat{f}'_m$, $m = 1, 2$, be the values of f_m, f'_m at $z = L$ and

$$F_m = \hat{f}_m + i\frac{\hat{f}'_m}{k_1 J_1}, \qquad m = 1, 2. \tag{4.6}$$

Also, letting

$$\gamma = \frac{k_0 J_0}{k_1 J_1} = r\frac{J_0}{J_1},$$

by (4.2) we have

$$ik_0 J_0 (F_1 \hat{f}_2 - F_2 \hat{f}_1) = \gamma.$$

Hence, by applying the jump conditions (4.5) to (4.3) and (4.4) we eventually obtain

$$R = -\frac{F_1 + ik_0 J_0 F_2}{F_1 - ik_0 J_0 F_2}, \qquad T = \frac{2\gamma}{F_1 - ik_0 J_0 F_2}, \tag{4.7}$$

$$a = -\frac{2ik_0 J_0 F_2}{F_1 - ik_0 J_0 F_2}, \qquad b = \frac{2ik_0 J_0 F_1}{F_1 - ik_0 J_0 F_2}. \tag{4.8}$$

The ratios J_0, J_1 are the given data. Once F_1 and F_2 are determined, Eq. (4.7) provides the reflection and transmission coefficients of the layer. By (4.3), substitution of a and b from (4.8) gives the wave solution in the layer.

The parameter γ is independent of ω. Indeed, in terms of the incidence and transmission angles θ_0, θ_1 and of the densities $\rho(0_\pm), \rho(L_\pm)$, γ takes the form

$$\gamma = \frac{c_1 \cos\theta_0 \rho(0_+)\rho(L_+)}{c_0 \cos\theta_1 \rho(0_-)\rho(L_-)}. \tag{4.9}$$

Because, by Snell's law (2.10),

$$\frac{1}{c_0}\sin\theta_0 = \frac{1}{c_1}\sin\theta_1,$$

we can determine the dependence of γ on the incidence angle θ_0 to obtain

$$\gamma(\theta_0) = \frac{c_1 \cos^2\theta_0 \; \rho(0_+)\rho(L_+)}{\sqrt{c_0^2 - c_1^2 \sin^2\theta_0} \; \rho(0_-)\rho(L_-)}. \tag{4.10}$$

5. Reflection and transmission coefficients

By means of (4.7) we now investigate the dependence of R and T on the frequency ω. In view of (3.9), substitution of (4.1) in (4.6) gives

$$F_1 = \mathcal{M}_{11}(L) + i\frac{\mathcal{M}_{21}(L)}{k_1 J_1}, \qquad F_2 = \mathcal{M}_{12}(L) + i\frac{\mathcal{M}_{22}(L)}{k_1 J_1}.$$

Hence by some rearrangements we find that

$$F_1 + ik_0 J_0 F_2 = \sum_{h=0}^{\infty} \left[a_{2h}^{+}\omega^{2h} + ia_{2h+1}^{+}\omega^{2h+1}\right],$$

$$F_1 - ik_0 J_0 F_2 = \sum_{h=0}^{\infty} \left[a_{2h}^{-}\omega^{2h} + ia_{2h+1}^{-}\omega^{2h+1}\right],$$

where

$$a_0^{\pm} = 1 \mp \gamma, \qquad a_1^{\pm} = \pm\sqrt{n_0}J_0 L - \frac{1}{\sqrt{n_1}J_1}\int_0^L n(s_1)ds_1, \tag{5.1}$$

$$a_{2h}^{\pm} = \tag{5.2}$$

$$(-1)^h \int_0^L \int_0^{s_1} \dots \int_0^{s_{2h-1}} [n(s_2)n(s_4)\dots n(s_{2h}) \mp \gamma n(s_1)n(s_3)\dots n(s_{2h-1})]ds_1\dots ds_{2h},$$

$$a_{2h+1}^{\pm} = \frac{1}{\sqrt{n_1}J_1}(-1)^{h+1} \tag{5.3}$$

$$\times \int_0^L \int_0^{s_1} \dots \int_0^{s_{2h}} [n(s_1)n(s_3)\dots n(s_{2h+1}) \mp \nu n(s_2)n(s_4)\dots n(s_{2h})]ds_1\dots ds_{2h+1},$$

where $h \in \mathbb{N}$ and

$$\nu = \sqrt{n_0 n_1}J_0 J_1.$$

It is convenient to let

$$a_k^{\pm} = \hat{a}_k \pm \tilde{a}_k, \qquad k \in \mathbb{N}, \tag{5.4}$$

where $\hat{a}_k, \tilde{a}_k$ are unaffected by the change of sign. Also, let

$$p_{2h} = a_{2h}^{+}, \qquad p_{2h+1} = ia_{2h+1}^{+}, \qquad m_{2h} = a_{2h}^{-}, \qquad m_{2h+1} = ia_{2h+1}^{-}, \tag{5.5}$$

the functions $a^{\pm}_{2h}, a^{\pm}_{2h+1}$ being real-valued. Hence, by (4.7), we can write R and T as functions of ω in the series form:

$$R(\omega) = -\frac{\sum_{k=0}^{\infty} p_k \omega^k}{\sum_{k=0}^{\infty} m_k \omega^k}, \tag{5.6}$$

$$T(\omega) = \frac{2\gamma}{\sum_{k=0}^{\infty} m_k \omega^k}. \tag{5.7}$$

Equations (5.6) and (5.7) provide the reflection and transmission coefficients for any layer profile $n(z)$ and incidence angle θ_0. By the same way, Eqs. (4.8) provide a and b as series of powers of ω.

As a check of consistency we look at the particular case when n is constant within the layer, $n(z) = n$, $z \in (0, L)$. We find that

$$\mathcal{M}_{11}(L) = 1 + \sum_{h=1}^{\infty} (-1)^h \omega^{2h} n^h \frac{L^{2h}}{(2h)!} = \cos(\sqrt{n}\omega L),$$

$$\omega \mathcal{M}_{12}(L) = \omega L + \sum_{h=1}^{\infty} (-1)^h \omega^{2h+1} n^h \frac{L^{2h+1}}{(2h+1)!} = \frac{1}{\sqrt{n}} \sin(\sqrt{n}\omega L),$$

$$\frac{1}{\omega}\mathcal{M}_{21}(L) = \sum_{h=0}^{\infty} (-1)^{h+1} \omega^{2h+1} n^{p+1} \frac{L^{2h+1}}{(2h+1)!} = -\sqrt{n}\sin(\sqrt{n}\omega L),$$

$$\mathcal{M}_{22}(L) = 1 + \sum_{h=1}^{\infty} (-1)^h \omega^{2h} n^h \frac{L^{2h}}{(2h)!} = \cos(\sqrt{n}\omega L).$$

Hence we have

$$R = -\frac{(1-\gamma)\cos(\sqrt{n}\omega L) + i[\sqrt{n}/\sqrt{n_1}J_1](-1 + J_0 J_1 \sqrt{n_0 n_1}/n)\sin(\sqrt{n}\omega L)}{(1+\gamma)\cos(\sqrt{n}\omega L) - i[\sqrt{n}/\sqrt{n_1}J_1](1 + J_0 J_1 \sqrt{n_0 n_1}/n)\sin(\sqrt{n}\omega L)}$$

and

$$T = \frac{2\gamma}{(1+\gamma)\cos(\sqrt{n}\omega L) - i[\sqrt{n}/\sqrt{n_1}J_1](1 + J_0 J_1 \sqrt{n_0 n_1}/n)\sin(\sqrt{n}\omega L)}.$$

If the parameters of the layer coincide with those of the half-space $z > L$, then we let $J_1 = 1$, $n = n_1$, $L = 0$ and find that

$$R = \frac{\gamma - 1}{\gamma + 1}, \qquad T = \frac{2\gamma}{\gamma + 1}.$$

By (4.9) we have

$$\gamma = \frac{\rho_1 c_1 \cos\theta_0}{\rho_0 c_0 \cos\theta_1}.$$

Hence it follows that

$$R = \frac{\rho_1 c_1 \cos\theta_0 - \rho_0 c_0 \cos\theta_1}{\rho_1 c_1 \cos\theta_0 + \rho_0 c_0 \cos\theta_1}, \qquad T = \frac{2\rho_1 c_1 \cos\theta_0}{\rho_1 c_1 \cos\theta_0 + \rho_0 c_0 \cos\theta_1},$$

see, e.g., [15].

6. Inverse problem

We now consider the inverse problem that consists in evaluation of the material properties, say $\rho(0_+), \rho(L_-), \rho(L_+), c_1$, and $N(z), z \in (0, L)$, by means of the reflection coefficient $R(\omega)$. Preliminarily we look for the determination of the coefficients p_k, m_k (and hence $a_k^{\pm}$), $k = 0, 1, 2, \ldots$, in (5.6) by means of $R(\omega)$, which is now regarded as a known function. Indeed, we show that interesting results follow by merely exploiting $R(\omega)$ in a neighbourhood of $\omega = 0$.

As we will see in a moment, the inverse problem is made easier by starting from (5.6), in the form

$$R(\omega) = -\frac{p_0 + p_1\omega + \ldots + p_n\omega^n + O(\omega^{n+1})}{m_0 + m_1\omega + \ldots + m_n\omega^n + O(\omega^{n+1})}, \qquad m_0 \neq 0, \tag{6.1}$$

and looking for $p_0, p_1, \ldots, p_n$ and $m_0, m_1, \ldots, m_n$ in terms of $R(\omega)$.

Letting $R(\omega)$ be n times differentiable at $\omega = 0$ we can write R by means of the corresponding Taylor's polynomial of degree n. Indeed, for formal convenience let

$$r_k = -\frac{R^{(k)}(0)}{k!}, \qquad k = 0, 1, \ldots, n.$$

Hence (6.1) becomes

$$[r_0 + r_1\omega + \ldots + r_n\omega^n + O(\omega^{n+1})][m_0 + m_1\omega + \ldots + m_n\omega^n + O(\omega^{n+1})]$$
$$= [p_0 + p_1\omega + \ldots + p_n\omega^n + O(\omega^{n+1})].$$

Equating the coefficients of equal powers of ω we obtain the recursive relations

$$p_0 - r_0 m_0 = 0, \qquad p_k - r_0 m_k = \sum_{h=0}^{k-1} r_{k-h} m_h, \qquad k = 1, 2, \ldots, n. \tag{6.2}$$

In terms of $\{\hat{a}_k\}$ and $\{\tilde{a}_k\}$ we can write (6.2) as

$$(1 - r_0)\hat{a}_0 + (1 + r_0)\tilde{a}_0 = 0, \tag{6.3}$$

$$(1 - r_0)\hat{a}_{2j+1} + (1 + r_0)\tilde{a}_{2j+1} = -i\sum_{h=0}^{2j} r_{2j+1-h} m_h, \tag{6.4}$$

$$(1 - r_0)\hat{a}_{2j+2} + (1 + r_0)\tilde{a}_{2j+2} = \sum_{h=0}^{2j+1} r_{2j+2-h} m_h, \tag{6.5}$$

for the pertinent values of $j \in \mathbb{N}$. We might then say that $\hat{a}_0$, $\tilde{a}_0$, $\{\hat{a}_{2j+1}, \tilde{a}_{2j+1}\}$, $\{\hat{a}_{2j}, \tilde{a}_{2j}\}$ are the unknowns to be determined. Indeed, we might think that (6.3) determines $\hat{a}_0$, $\tilde{a}_0$ and hence Eqs. (6.4)–(6.5) provide recursively $\hat{a}_k, \tilde{a}_k$ in terms of $a_0^-, a_1^-, \ldots, a_{k-1}^-$. Yet, seemimgly we have a system of $n+1$ equations in $2(n+1)$ unknowns. Moreover, the set $\{a_k^\pm\}$ is parameterized by θ_0 as well as $r_0, r_1, \ldots, r_n$. Hence a further analysis is necessary to fix the effective unknowns and the corresponding structure of the system.

The unknown material function $N(z)$, $z \in (0, L)$, occurs in the quantities $a_k^\pm$ through integrals of

$$n(z) = N(z) - \xi^2, \qquad \xi^2 = N_0 \sin^2 \theta_0. \tag{6.6}$$

For instance,

$$\tilde{a}_3 = \frac{1}{\sqrt{n_1}\, J_1} \int_0^L \int_0^{s_1} \int_0^{s_2} n(s_1) n(s_3) ds_1 ds_2 ds_3.$$

Substitution of n from (6.6) shows that ultimately, the effective unknowns become appropriate integrals of the unknown function N. Based upon these qualitative remarks, we now evaluate step by step the structure of Eqs. (6.3)–(6.5) and determine the pertinent unknowns.

6.1. Material properties from the reflection data

The following steps show how Eqs. (6.3)–(6.5) allow us to determine c_1, J_1, γ, L and the moments

$$\mu_k = \int_0^L z^k N(z) dz \tag{6.7}$$

of order $k = 0, 1, \ldots$.

Step 1. By (5.1) we have

$$\hat{a}_0 = 1, \qquad \tilde{a}_0 = -\gamma.$$

Substitution in (6.3) gives

$$\gamma = \frac{1 - r_0}{1 + r_0}. \tag{6.8}$$

As a consequence, both a_0^+ and a_0^- are determined,

$$a_0^+ = \frac{2r_0}{1 + r_0}, \qquad a_0^- = \frac{2}{1 + r_0}.$$

Since r_0 is parameterized by θ_0, so are also γ, a_0^+, a_0^-.

Henceforth we need to evaluate separately the effects of different incidence angles. For normal incidence we have

$$\xi = 0, \qquad n = N.$$

By (4.10) we have

$$\frac{c_0^2 \cos^2\theta_0}{c_0^2 - c_1^2 \sin^2\theta_0} = \frac{\gamma^2(\theta_0)}{\gamma^2(0)}.$$

Hence, for any non-zero value of θ_0 we can determine c_1 as

$$c_1 = c_0 \frac{\sqrt{[\gamma(\theta_0)/\gamma(0)]^2 - \cos^2\theta_0}}{\sin\theta_0 \gamma(\theta_0)/\gamma(0)}, \tag{6.9}$$

where $\gamma(0), \gamma(\theta_0)$ are given by (6.8). As a consequence, we determine J_1 in the form

$$J_1 = \frac{c_0 J_0}{c_1 \gamma(0)} \tag{6.10}$$

and say that $N_0 = 1/c_0^2$ and $N_1 = 1/c_1^2$ are known.

Step 2. By (6.4), $j = 0$, we have

$$(1 - r_0)\hat{a}_1 + (1 + r_0)\tilde{a}_1 = -i a_0^- r_1.$$

Now, by (5.1) and (6.6) we have

$$(1 + r_0)\sqrt{n_0}\, J_0 L - (1 - r_0)\frac{1}{\sqrt{n_1}\, J_1}(\mu_0 - \xi^2 L) = -i a_0^- r_1, \tag{6.11}$$

μ_0 being the integral (or the zero-order moment) of N. The validity of (6.11) at normal incidence gives

$$[1 + r_0(0)]\sqrt{N_0}\, J_0 L - [1 - r_0(0)]\frac{1}{\sqrt{N_1}\, J_1}\mu_0 = -i a_0^-(0) r_1(0).$$

Hence we have

$$\mu_0 = \frac{\sqrt{N_1} J_1}{1 - r_0(0)}\{[1 + r_0(0)]\sqrt{N_0} J_0 L + i a_0^-(0) r_1(0)\}.$$

The validity of (6.11) for oblique incidence, at angle θ_0, provides then

$$L\left\{ [1+r_0(\theta_0)] J_0 \sqrt{N_0 - \xi^2} + \frac{[1 - r_0(\theta_0)]\xi^2}{J_1\sqrt{N_1 - \xi^2}} - J_0 \frac{[1 - r_0(\theta_0)][1 + r_0(0)]}{1 - r_0(0)}\sqrt{\frac{N_0 N_1}{N_1 - \xi^2}} \right\}$$
$$= -i a_0^-(\theta_0) r_1(\theta_0) + i a_0^-(0) r_1(0) \frac{1 - r_0(\theta_0)}{1 - r_0(0)}\sqrt{\frac{N_1}{N_1 - \xi^2}},$$

whence we determine L. The knowledge of L and μ_0 in turn determines $a_1^{\pm}(\theta_0)$, for any θ_0.

Step 3. By (6.5), $j = 0$, we have

$$(1 - r_0)\hat{a}_2 + (1 + r_0)\tilde{a}_2 = ia_1^- r_1 + a_0^- r_2, \tag{6.12}$$

where the right-hand side and r_0 are known. Now, by (5.2) and (5.4) we have

$$\begin{aligned}\hat{a}_2 \pm \tilde{a}_2 &= -\int_0^L \int_0^{s_1} n(s_2) ds_1 ds_2 \mp \gamma \int_0^L \int_0^{s_1} n(s_1) ds_1 ds_2 \\ &= \int_0^L \int_0^{s_1} [-N(s_2) \mp \gamma N(s_1)] ds_1 ds_2 + \frac{1}{2}\xi^2 L^2 (1 \pm \gamma).\end{aligned}$$

Moreover, an integration by parts gives

$$\int_0^L \int_0^{s_1} N(s_2) ds_1 ds_2 = L\mu_0 - \mu_1.$$

Hence, because

$$\int_0^L \int_0^{s_1} N(s_1) ds_1 ds_2 = \mu_1,$$

we find that

$$\hat{a}_2 \pm \tilde{a}_2 = -L\mu_0 + (1 \mp \gamma)\mu_1, + \frac{1}{2}\xi^2 L^2 (1 \pm \gamma). \tag{6.13}$$

Because, at this step, μ_0 is known it follows that $\hat{a}_2, \tilde{a}_2$ contain the single unknown μ_1. Substitution in (6.12) gives

$$\begin{aligned}\mu_1 = {} & \frac{1}{1 - \gamma - r_0(1 + \gamma)} \\ & \times \left\{ ia_1^- r_1 + a_0^- r_2 + (1 - r_0) L\mu_0 - \frac{1}{2}\xi^2 L^2 [1 + \gamma - r_0(1 - \gamma)] \right\}.\end{aligned} \tag{6.14}$$

Evaluation of the right-hand side for e.g. normal incidence, provides the value of μ_1. Applying (6.13) for two values of the incidence angle, 0 and θ_0, and hence with $\gamma(0)$, $r_0(0)$, $\xi^2 = 0$ and $\gamma(\theta_0)$, $r_0(\theta_0)$, $\xi^2 = N_0 \sin^2 \theta_0$, we find the values $a_2^{\pm}(\theta_0)$, for any θ_0.

Step 4. By (6.4), $j=1$, we have

$$(1-r_0)\hat{a}_3 + (1+r_0)\tilde{a}_3 = -ia_2^- r_1 + a_1^- r_2 - ia_0^- r_3. \tag{6.15}$$

By (5.3) and (5.4) can write

$$\hat{a}_3 \pm \tilde{a}_3 = \frac{1}{J_1\sqrt{N_1-\xi^2}} \int_0^L \int_0^{s_1} \int_0^{s_2} [N(s_1)-\xi^2][N(s_3)-\xi^2] ds_1 ds_2 ds_3 \mp J_0\sqrt{N_0-\xi^2} \int_0^L \int_0^{s_1} [N(s_2)-\xi^2] s_2 ds_1 ds_2.$$

Letting

$$\mathcal{N}_{13} = \int_0^L \int_0^{s_1} \int_0^{s_2} N(s_1)N(s_3) ds_1 ds_2 ds_3, \qquad \mathcal{N}_2 = \int_0^L \int_0^{s_1} N(s_2) s_2 ds_1 ds_2$$

we obtain

$$\hat{a}_3 \pm \tilde{a}_3 = \frac{1}{J_1\sqrt{N_1-\xi^2}} \left\{ \mathcal{N}_{13} - \xi^2 \left[\frac{1}{2}\mu_2 + \int_0^L ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} N(s_3) ds_3 \right] + \xi^4 \frac{L^3}{6} \right\} \mp J_0\sqrt{N_0-\xi^2} \left[\mathcal{N}_2 - \xi^2 \frac{L^3}{6} \right].$$

Now, an integration by parts shows that

$$\int_0^{s_1} N(s_2) s_2 ds_2 = s_1 \int_0^{s_1} N(s_2) ds_2 - \int_0^{s_1} ds_2 \int_0^{s_2} N(s_3) ds_3$$

and hence

$$\mathcal{N}_2 = \frac{1}{2}\mu_0 L^2 - \frac{1}{2} M_2 - \int_0^L ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} N(s_3) ds_3.$$

As a consequence, we can write

$$\hat{a}_3 \pm \tilde{a}_3 = \frac{1}{J_1\sqrt{N_1-\xi^2}} \left[\mathcal{N}_{13} + \xi^2 \mathcal{N}_2 - \frac{\xi^2 L^2}{2}\mu_0 + \xi^4 \frac{L^3}{6} \right] \mp J_0\sqrt{N_0-\xi^2} \left[\mathcal{N}_2 - \xi^2 \frac{L^3}{6} \right].$$

Since μ_0 is by now a known quantity, it follows that for any value of ξ, $a_3^{\pm}$ involves two unknowns, $\mathcal{N}_{13}$ and $\mathcal{N}_2$.

Returning to (6.15) we can write the equation for two values of the incidence angle, 0 and θ_0, to obtain

$$\begin{aligned}\frac{1-r_0(0)}{J_1\sqrt{N_1}}\mathcal{N}_{13} - [1+r_0(0)]J_0\sqrt{N_0}\,\mathcal{N}_2 \\ = -ia_2^-(0)r_1(0) + a_1^-(0)r_2(0) - ia_0^-(0)r_3(0),\end{aligned} \tag{6.16}$$

$$\begin{aligned}&\frac{1-r_0(\theta_0)}{J_1\sqrt{N_1-\xi^2}}\mathcal{N}_{13} + \left\{\frac{[1-r_0(\theta_0)]\xi^2}{J_1\sqrt{N_1-\xi^2}} - [1+r_0(\theta_0)]J_0\sqrt{N_0-\xi^2}\right\}\mathcal{N}_2 \\ &= \frac{1-r_0(\theta_0)}{J_1\sqrt{N_1-\xi^2}}\left[\frac{\xi^2L^2}{2}\mu_0 - \frac{\xi^4L^3}{6}\right] - [1+r_0(\theta_0)]J_0\sqrt{N_0-\xi^2}\frac{\xi^2L^3}{6} \\ &\quad - ia_2^-(\theta_0)r_1(\theta_0) + a_1^-(\theta_0)r_2(\theta_0) - ia_0^-(\theta_0)r_3(\theta_0).\end{aligned} \tag{6.17}$$

Hence we conclude that $\mathcal{N}_{13}$ and $\mathcal{N}_2$ are determined.

In addition we observe that an integration by parts gives

$$\mathcal{N}_2 = L\mu_1 - \mu_2.$$

Since μ_1 is known from (6.14), then at this step we determine μ_2 as

$$\mu_2 = L\mu_1 - \mathcal{N}_2.$$

Step 5. By (6.5), $j=1$, we have

$$(1-r_0)\hat{a}_4 + (1+r_0)\tilde{a}_4 = ia_3^-r_1 + a_2^-r_2 + ia_1^-r_3 + a_0^-r_4. \tag{6.18}$$

Also by (5.2) we have

$$\begin{aligned}&\hat{a}_4 \pm \tilde{a}_4 \\ &= \int_0^L\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} \left\{[N(s_2)-\xi^2][N(s_4)-\xi^2] \mp \gamma[N(s_1)-\xi^2][N(s_3)-\xi^2]\right\} ds_1\ldots ds_4\end{aligned}$$

Let

$$\mathcal{Q}_{24} = \int_0^L\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} N(s_2)N(s_4)ds_1\ldots ds_4$$

and

$$\mathcal{Q}_{13} = \int_0^L\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} N(s_1)N(s_3)ds_1\ldots ds_4$$

so that

$$\hat{a}_4 \pm \tilde{a}_4 = \mathcal{Q}_{24} \mp \gamma \mathcal{Q}_{13} - \xi^2 \int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} [N(s_2) + N(s_4)] ds_1 \dots ds_4$$
$$\pm \xi^2 \gamma \int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} [N(s_1) + N(s_3)] ds_1 \dots ds_4 + \frac{\xi^4 L^4}{4!}(1 \pm \gamma).$$

Direct integrations and integrations by parts allow us to show that

$$\int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} N(s_1) ds_1 \dots ds_4 = \frac{1}{6}\mu_3,$$
$$\int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} N(s_2) ds_1 \dots ds_4 = \frac{1}{2}(L\mu_2 - \mu_3),$$
$$\int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} N(s_3) ds_1 \dots ds_4 = \frac{1}{2}(L^2\mu_1 - 2L\mu_2 + \mu_3),$$
$$\int_0^L \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} N(s_4) ds_1 \dots ds_4 = \frac{1}{6}(L^3\mu_0 - \mu_3 + 3L\mu_2 - 3L^2\mu_1).$$

Susbstitution in (6.18) provides an equation with the unknowns $\mathcal{Q}_{13}$, $\mathcal{Q}_{24}$ and μ_3 while $r_0, r_1, \dots, r_4$ and μ_0, μ_1, μ_2 are known. Applying the equation for three incidence angles provides a system of three equations in the three unknowns.

By iterating the procedure to the next steps we can determine the moments of higher order $\mu_4, \mu_5, \dots$ along with nonlinear terms such as $\mathcal{N}_{13}$, $\mathcal{Q}_{13}$, $\mathcal{Q}_{24}$. Quite naturally, the equations become more and more involved at the next steps.

While

$$\rho_0, c_0, \rho(0_+), c(0_+) \tag{6.19}$$

are known from the outset, upon steps 1 to 5 we conclude that

$$c_1, J_1, \rho(L_-), c(L_-), L, \tag{6.20}$$

and

$$\mu_0,\ \mu_1,\ \mu_2,\ \mu_3 \tag{6.21}$$

are determined from the reflection data. Really, we determine γ from (6.8) and hence c_1 and J_1 from (6.9), (6.10).

If the constitutive equation (2.2) is regarded as known at $z = L_-, L_+$ then we can determine also $\rho(L_+)$ and $c^2(L_-)$ by

$$\rho(L_+) = \hat{\rho}(c_1^2), \qquad c^2(L_-) = \hat{c}^2(\rho(L_-)).$$

7. Profile reconstruction in terms of the moments

Consider the Legendre polynomials (see, e.g. [16])

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n = 0, 1, \ldots.$$

They constitute an orthogonal system

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}.$$

Hence we let

$$p_n(x) = \sqrt{n+1/2}\; P_n(x)$$

so that $\{p_n\}$ is an orthonormal set of polynomials. For later convenience we assume that

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2-1),$$
$$P_3(x) = \frac{1}{2}x(5x^2-3), \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Look at the n-th moment μ_n of N on $[0, L]$. The change of variable

$$z \mapsto x, \qquad x = -1 + \frac{2z}{L}, \qquad z = \frac{L}{2}(x+1),$$

so that $[0, L] \to [-1, 1]$, allows us to write μ_n in the form

$$\mu_n = \left(\frac{L}{2}\right)^{n+1} \int_{-1}^{1} \hat{N}(x)(x+1)^n dx, \tag{7.1}$$

where

$$\hat{N}(x) = N(z(x)).$$

The moments $\{\hat{\mu}_n\}$ of $\hat{N}$,

$$\hat{\mu}_n = \int_{-1}^{1} \hat{N}(x) x^n dx, \tag{7.2}$$

can be related to the moments $\{\mu_n\}$ of N. The change of variable $x \to z$ in (7.2) provides

$$\hat{\mu}_n = \frac{2}{L} \int_0^L N(z) \left(\frac{2}{L} z - 1 \right)^n dz.$$

Hence we find that

$$\hat{\mu}_n = \frac{2}{L} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\frac{2}{L} \right)^k \mu_k. \tag{7.3}$$

For any function $f \in L^2[-1,1]$ we can write

$$f(x) = \sum_{n=0}^{\infty} c_n p_n(x), \qquad c_n = \langle f, p_n \rangle := \int_{-1}^{1} f(x) p_n(x) dx.$$

Hence we have

$$f(x) = \sum_{n=0}^{\infty} \alpha_n P_n(x), \qquad \alpha_n = \left(n + \frac{1}{2} \right) \langle f, P_n \rangle. \tag{7.4}$$

Denote by $a_{n0}, a_{n1}, \ldots, a_{nn}$ the coefficients occurring in the polynomial P_n,

$$P_n(x) = \sum_{k=0}^{n} a_{nk} x^k.$$

The inner product $\langle \hat{N}, P_n \rangle$ can then be written in terms of the moments $\{\hat{\mu}_k\}$ as

$$\langle \hat{N}, P_n \rangle = \int_{-1}^{1} \hat{N}(x) \sum_{k=0}^{n} a_{nk} x^k dx = \sum_{k=0}^{n} a_{nk} \hat{\mu}_k.$$

By means of (7.4) we have

$$\hat{N}(x) = \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{2} \right) \sum_{k=0}^{n} a_{nk} \hat{\mu}_k \right] P_n(x). \tag{7.5}$$

To obtain $N(z)$, $z \in [0, L]$, we have only to express x in terms of z. By means of (7.3) we find that

$$N(z) = \frac{1}{L}\sum_{n=0}^{\infty}\left[(2n+1)\sum_{k=0}^{n} a_{nk}\sum_{h=0}^{k}\binom{k}{h}(-1)^{k-h}\left(\frac{2}{L}\right)^{h}\mu_h\right]P_n((2z-L)/L). \tag{7.6}$$

The relation (7.6) provides the function (profile) $N(z)$, $z \in [0, L]$, as a series of the Legendre polynomials $\{P_n\}$ with coefficients determined by the moments $\{\mu_n\}$.

7.1. Approximate representation of N

In practical applications we know only a few moments $\{\mu_n\}$, say $\mu_0, \mu_1, \ldots, \mu_j$. In such cases we can only approximate N by replacing the series (7.6) with the finite sum

$$N_j(z) = \frac{1}{L}\sum_{n=0}^{j}\left[(2n+1)\sum_{k=0}^{n} a_{nk}\sum_{h=0}^{k}\binom{k}{h}(-1)^{k-h}\left(\frac{2}{L}\right)^{h}\mu_h\right]P_n((2z-L)/L). \tag{7.7}$$

For definiteness we let $j = 4$ and show how N_4 approximates N by selecting

$$N(z) = \lambda + \epsilon \sin \pi z/L, \qquad z \in [0, L]. \tag{7.8}$$

To this end we evaluate the pertinent moments $\mu_0, \mu_1, \ldots, \mu_4$ as it would be given by a real experiment, and then determine N_4. It is convenient to denote by I_n the n-th moment of $\sin(\pi z/L)$,

$$I_n = \int_0^L z^n \sin(\pi z/L)dz,$$

so that

$$\mu_n = \lambda\frac{L^{n+1}}{n+1} + \epsilon I_n.$$

Two integrations by parts provide a recurrence relation for the sequence $\{I_n\}$,

$$I_n = \frac{L^{n+1}}{\pi} - n(n-1)\frac{L^2}{\pi^2}I_{n-2}. \tag{7.9}$$

A direct calculation for $n = 0, 1$ and use of (7.9) for $n = 2, 3, 4$ yield

$$\mu_0 = \lambda L + \epsilon\frac{2L}{\pi}, \qquad \mu_1 = \lambda\frac{L^2}{2} + \epsilon\frac{L^2}{\pi},$$

$$\mu_2 = \lambda\frac{L^3}{3} + \epsilon\frac{L^3}{\pi^3}(\pi^2 - 4), \qquad \mu_3 = \lambda\frac{L^4}{4} + \epsilon\frac{L^4}{\pi^3}(\pi^2 - 6),$$

$$\mu_4 = \lambda\frac{L^5}{5} + \epsilon\frac{L^5}{\pi^5}(\pi^4 - 12\pi^2 + 48).$$

A direct application of (7.7) gives

$$N_0(z) = \lambda + \epsilon \frac{2}{\pi}, \qquad N_1(z) = N_0,$$

$$N_2(z) = N_0 + \epsilon \frac{5}{\pi^3}(\pi^2 - 12)\left[3\left(\frac{2z-L}{L}\right)^2 - 1\right], \qquad N_3(z) = N_2(z),$$

$$N_4(z) = N_2(z) + \epsilon \frac{9(\pi^4 - 180\pi^2 + 1680)}{4\,\pi^5}\left[35\left(\frac{2z}{L} - 1\right)^4 - 30\left(\frac{2z}{L} - 1\right)^2 + 3\right].$$

As an example, we consider the function (7.8) in the case $\lambda = 2$, $\epsilon = 1$, $L = 10$. Figure 1 shows the function (7.8) along with the approximations N_0, $N_2(z)$, $N_4(z)$, while $N_1 = N_0$, $N_3 = N_2$. The errors with N_2, N_4, namely

$$\max_{z\in(0,L)} |N(z) - N_2(z)| \qquad \text{and} \qquad \max_{z\in(0,L)} |N(z) - N_4(z)|,$$

turns out to be smaller than 0.05 and 0.0012, thus confirming the good fit of N with N_2 and the excellent one with N_4.

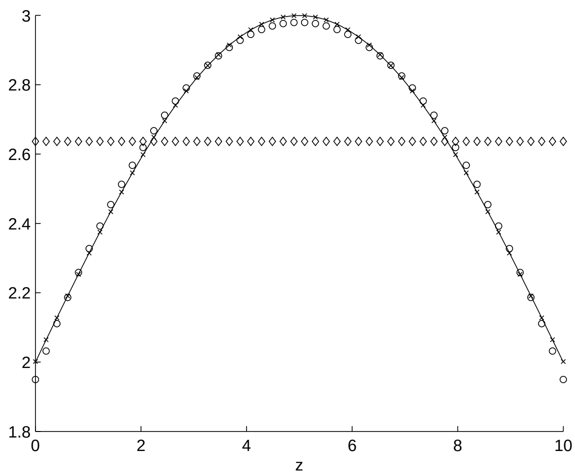

Fig. 1. The function $N(z) = 2 + \sin(\pi z/10)$ (solid line) and the approximations N_0 (diamonds), N_2 (circles), N_4 (crosses).

8. Conclusions

The paper deals with the reflection-transmission problem for acoustic waves, associated with an obliquely-incident plane wave which impinges on a continuously-stratified layer. The direct problem is solved and the reflection and transmission coefficients, R and T, are determined for any value of the incidence angle θ_0. No approximation is made such as regarding the layer as a stalk of homogeneous sublayers or confining to ray theory. By means of the natural integral formulation for a first-order system, $R(\omega)$ and $T(\omega)$ are determined and found to be given by suitable series of powers of the angular frequency ω. The results are of interest also because the integral formulation is naturally established for an initial-value problem, while the known data are confined to the incident plane wave.

Next the inverse problem is faced which consists in determination of the geometrical and material properties of the layer from the reflection data. Also because $R(\omega)$, as well as $T(\omega)$, is parameterized by the incidence angle θ_0, it follows that the moments of N and the thickness L, are determined in terms of derivatives of $R(\omega)$ of various orders, at $\omega = 0$, for a small number of values of θ_0. As an application, it is shown that a few moments are sufficient to provide an excellent fit of the refractive index by means of a superposition of the Legendre polynomials.

Sometimes the analogous inverse problem is developed in relation to the transmission coefficient T, instead of R. The problem with T would be much easier however all terms $p_1, p_2, \ldots$ disappear. For practical purposes it is more realistic to deal with reflection data and that is why we have chosen to face the inverse problem from the reflection data.

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