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Exact solutions for the longitudinal flow of a generalized Maxwell fluid in a circular cylinder

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This paper deals with the longitudinal flow of a generalized Maxwell fluid in an infinite circular cylinder, due to the longitudinal variable time-dependent shear stress that is prescribed on the boundary of the cylinder. The fractional calculus approach in the constitutive relationship model of a Maxwell fluid is introduced. The velocity field and the resulting shear stress are obtained by means of the Laplace and finite Hankel transforms and satisfy all the imposed initial and boundary conditions. The solutions corresponding to ordinary Maxwell fluids as well as those for Newtonian fluids are obtained as limiting cases of our general solutions. Finally, the influence of the fractional coefficient on the velocity and shear stress of the fluid is analyzed by graphical illustrations.

Key words: generalized Maxwell fluid, velocity field, shear stress, exact solutions.

1. Introduction

Generally speaking, rheological properties of materials are specified by their so-called constitutive equations. The simple constitutive equation for a fluid is a Newtonian one, and the classical Navier–Stokes theory is based on this equation. The mechanical behavior of many fluids is well-described by this theory. However, there are many rheological complex fluids such as polymer solutions, blood, heavy oils and many emulsions, which are inadequately described by a Newtonian constitutive equation that does not show any relaxation and retardation phenomena. For this reason, various fluid concepts have been proposed and studied by different authors.

The fluids that cannot be modeled by Navier–Stokes equations are called non-Newtonian fluids. The term of non-Newtonian is used to classify all fluids in which shear stress is not directly proportional to the shear rate. Among the many constitutive assumptions that were employed to study the non-Newtonian fluid behavior, rate-type fluids [1] as well as differential-type fluids [2] have gained the acceptance of both the theoreticians and experimentalists. The first viscoelastic rate-type model is due to Maxwell [3] and this model had some success in describing the response of some polymeric liquids.

In the last decade, many authors have made use of rheological equations with fractional derivatives [4, 5] to describe the properties of polymers. In general, the constitutive equations with fractional derivative are obtained from the known non-Newtonian models by replacing the ordinary time derivatives by derivatives of fractional order, e.g. [6]. Other very important models regarding non-Newtonian fluids with a fractional derivatve were investigated by TONG *et al.* [7, 8], AKHTAR *et al.* [9], C. FETECAU *et al.* [10], CORINA FETECAU *et al.* [11, 12] and VIERU *et al.* [13].

In this paper, we study the unsteady longitudinal flow of a generalized Maxwell fluid with a fractional derivative model within a circular cylinder of radius R. Generally, in one dimension, the constitutive equation of fractional Maxwell fluid can be expressed as [14, 15]

$$\tau(t) + \lambda^{\beta} D_t^{\beta} \tau(t) = \mu \frac{d\varepsilon(t)}{dt}, \tag{1.1}$$

where $\tau(t)$ is the shear stress, $\varepsilon(t)$ is the shear strain, λ is the relaxation time, μ is the dynamic viscosity and β is the fractional parameter such that $0 \le \beta \le 1$. Also D_t^{β} is the Riemann–Liouville fractional differential operator defined as [4, 5]:

$$D_t^{\beta}[f(t)] = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(u)}{(t-u)^{\beta}} du, \qquad 0 < \beta < 1, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function. This model can be reduced to the ordinary Maxwell model when $\beta \to 1$, because in this case $D_t^1 = df(t)/dt$. Furthermore, this model can be reduced to the classical Newtonian model for $\beta \to 1$ and $\lambda \to 0$.

The flow of the fluid is generated by the shear stress which is prescribed on the surface of the cylinder in the form

$$\tau(R,t) = \frac{f}{\lambda} \int_0^{\infty} (t-s)^a G_{\beta,0,1}(-1/\lambda, s) ds, \qquad t > 0,$$

where f is a constant, $a \ge 0$ and $G_{a,b,c}(\cdot,\cdot)$ is the generalized G-function [16]. The velocity fields and the resulting shear stresses, obtained by means of the Laplace and finite Hankel transforms, are presented in terms of generalized G-functions. The solutions corresponding to ordinary Maxwell fluids or Newtonian fluids are obtained as special cases of our solutions. Finally, for comparison, the profiles of the velocity $v(r,t)$, for Newtonian, Maxwell and generalized Maxwell fluids, for different values of the fractional coefficient β and for a constant shear stress on the boundary, are plotted as functions of cylindrical coordinate r.

2. Governing equations

Let us consider an incompressible fractional Maxwell fluid at rest in an infinite circular cylinder of radius R. At time $t = 0^+$, the cylinder is pulled by a time-dependent shear stress along its axis. Obviously, the motion is axial symmetric, so we choose the cylindrical coordinates (r, θ, z) and the components of the velocity are $v_r = 0$, $v_\theta = 0$, $v_z = v(r, t)$. Under the above assumptions, the constitutive equation of fractional Maxwell fluid is [17, 18]

$$\tau(r,t) + \lambda^\beta D_t^\beta \tau(r,t) = \mu \frac{\partial v(r,t)}{\partial r}, \tag{2.1}$$

where $\tau(r,t) = \tau_{rz}(r,t)$ is the shear stress.

In absence of the pressure gradient in the axial direction and neglecting the body forces, the balance of linear momentum leads to the partial differential equation [19]

$$\rho \frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau(r,t), \tag{2.2}$$

where ρ is the constant density of the fluid.

By eliminating $\tau(r,t)$ between Eqs. (2.1) and (2.2) we acquire the following motion equation of fractional Maxwell fluid:

$$(1 + \lambda^\beta D_t^\beta)\frac{\partial v(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial^2 r} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t), \qquad r \in (0, R),\ t > 0, \tag{2.3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The appropriate initial and boundary conditions are

$$v(r,0) = \frac{\partial v(r,0)}{\partial t} = 0, \qquad \tau(r,0) = 0, \qquad r \in [0, R), \tag{2.4}$$

$$(1 + \lambda^\beta D_t^\beta)\tau(R,t) = \mu \frac{\partial v(r,t)}{\partial r}\bigg|_{r=R} = f t^a, \qquad t > 0, \tag{2.5}$$

where f is a constant and $a \geq 0$.

3. Analytical solution of the model

The velocity field and the associated shear stress corresponding to the aforemention problem will be determined by means of the Laplace and finite Hankel transforms. Applying the Laplace transform to Eqs. (2.3) and $(2.5)_2$, using $(2.4)_{1,2}$ and formulae

$$L\{D_t^\beta f(t)\} = q^\beta L\{f(t)\}, \qquad L\{t^a\} = \frac{\Gamma(a+1)}{q^{a+1}}, \qquad a > -1, \tag{3.1}$$

we obtain the following problem with boundary condition (for simplicity, we take $\lambda^\beta = \lambda$ hereinafter):

$$(q + \lambda q^{\beta+1})\bar{v}(r,q) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q), \tag{3.2}$$

$$\left.\frac{\partial \bar{v}(r,q)}{\partial r}\right|_{r=R} = \frac{f}{\mu}\frac{\Gamma(a+1)}{q^{a+1}}, \tag{3.3}$$

where $\bar{v}(r,q) = \int_0^\infty v(r,t)e^{-qt}dt$ is the Laplace transform of function $v(r,t)$ and q is the transform parameter.

In the following, let us denote by [20]

$$\bar{v}_H(r_n,q) = \int_0^R r\bar{v}(r,q)J_0(rr_n)dr, \tag{3.4}$$

the finite Hankel transform of $\bar{v}(r,q)$, where r_n, $n = 1,2,3,\ldots$ are the positive roots of the transcendental equation $J_1(Rr) = 0$. In the above relations, $J_\nu(\cdot)$ is the first-kind, ν-order Bessel function.

By using the following formulae [21, 22]

$$\begin{aligned} &\frac{d}{dr}J_0[u(r)] = -J_1[u(r)]u'(r), \\ &\frac{d}{dr}J_1[u(r)] = \left[J_0[u(r)] - \frac{1}{u(r)}J_1[u(r)]\right]u'(r), \end{aligned} \tag{3.5}$$

we obtain that

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q)J_0(rr_n)dr = \frac{RJ_0(Rr_n)}{\mu}\frac{\partial \bar{v}(R,q)}{\partial r} - r_n^2\bar{v}_H(r_n,q). \tag{3.6}$$

Applying the Hankel transform to Eq. (3.2) and taking into account Eqs. (3.3) and (3.6), we find that

$$\bar{v}_H(r_n,q) = RfJ_0(Rr_n)\Gamma(a+1)\frac{1}{\rho q^{a+1}(q + \lambda q^{\beta+1} + \nu r_n^2)}. \tag{3.7}$$

Now, for a more suitable presentation of the final results, we rewrite Eq. (3.7) in the following equivalent form:

$$\bar{v}_H(r_n,q) = \frac{RfJ_0(Rr_n)}{\mu r_n^2}\frac{\Gamma(a+1)}{q^{a+1}} - \frac{RfJ_0(Rr_n)}{\mu r_n^2}\frac{\Gamma(a+1)}{q^a}\frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2}. \tag{3.8}$$

The inverse Hankel transform of the function $\bar{v}_H(r_n,q)$ is [20]

$$(3.9)\quad \bar{v}(r,q)=\frac{fr^2}{2\mu R}\frac{\Gamma(a+1)}{q^{a+1}}-\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\frac{\Gamma(a+1)}{q^a}\frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2}.$$

In order to determine the inverse Laplace transform of function $\bar{v}(r,q)$, we introduce the following notations:

$$(3.10)\qquad F_1(q)=\frac{\Gamma(a+1)}{q^{a+1}},\qquad F_2(q)=\frac{\Gamma(a+1)}{q^a},$$

$$(3.11)\qquad F_3(r_n,q)=\frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2}=\frac{q^{\beta-1}+\frac{1}{\lambda}q^{-1}}{\left(q^\beta+\frac{1}{\lambda}\right)+\frac{\nu r_n^2}{\lambda}q^{-1}}$$
$$=\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{\beta-k-1}+\frac{1}{\lambda}q^{-k-1}}{\left(q^\beta+\frac{1}{\lambda}\right)^{k+1}}.$$

Using $(3.1)_2$ and the known result [16, Eq. (97)]

$$(3.12)\quad L^{-1}\left\{\frac{q^c}{(q^b-p)^d}\right\}=G_{b,c,d}(p,t),\quad \mathrm{Re}(bd-c)>0,\quad \mathrm{Re}(q)>0,\quad |p|>|q^a|,$$

where

$$(3.13)\qquad G_{b,c,d}(p,t)=\sum_{j=0}^{\infty}\frac{\Gamma(d+j)p^j}{\Gamma(d)\Gamma(j+1)}\frac{t^{(d+j)b-c-1}}{\Gamma[(d+j)b-c]}$$

is the generalized G-function, we find that the inverse Laplace transforms of functions F_i, $i=1,2,3$, are [23]:

$$(3.14)\qquad \begin{aligned} f_1(t)&=L^{-1}\{F_1(q)\}=t^a,\quad a\geq 0,\\ f_2(t)&=L^{-1}\{F_2(q)\}=at^{a-1},\quad a>0,\end{aligned}$$

$$(3.15)\quad f_3(r_n,t)=L^{-1}\{F_3(r_n,q)\}$$
$$=\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left[G_{\beta,\beta-k-1,k+1}\left(-\frac{1}{\lambda},t\right)+\frac{1}{\lambda}G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda},t\right)\right].$$

Applying the inverse Laplace transform to Eq. (3.9), using (3.12), (3.14), (3.15) and the convolution theorem, we find the velocity field $v(r,t)$ under the following forms.

If $a > 0$, then

$$v(r,t) = \frac{fr^2}{2\mu R}t^a - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}f_2(t) * f_3(r_n,t), \tag{3.16}$$

and if $a = 0$, then

$$v(r,t) = \frac{fr^2}{2\mu R} - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}f_3(r_n,t), \tag{3.17}$$

where $h(t) * g(t) = \int_0^t h(t-\tau)g(\tau)d\tau = \int_0^t h(\tau)g(t-\tau)d\tau$ represents the convolution of functions h and g.

3.1. Calculation of the shear stress

Applying the Laplace Transform to Eq. (2.1) and using the initial condition $(2.4)_3$, we find that

$$(1+\lambda q^\beta)\bar{\tau}(r,q) = \mu\frac{\partial \bar{v}(r,q)}{\partial r}. \tag{3.18}$$

Differentiating Eq. (3.9) with respect to r and using the identity $(3.5)_1$, we find that

$$\begin{aligned}\bar{\tau}(r,q) = \frac{fr}{R}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{1+\lambda q^\beta} \\ + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\lambda q^{\beta+1}+\nu r_n^2}.\end{aligned} \tag{3.19}$$

To determine the inverse Laplace transform of function $\bar{\tau}(r,q)$, we introduce the following notations:

$$F_4(q) = \frac{1}{1+\lambda q^\beta} = \frac{1}{\lambda}\frac{1}{q^\beta + \frac{1}{\lambda}}, \tag{3.20}$$

$$\begin{aligned}F_5(r_n,q) &= \frac{1}{q+\lambda q^{\beta+1}+\nu r_n^2} = \frac{1}{\lambda q\left(q^\beta + \frac{1}{\lambda} + \frac{\nu r_n^2}{\lambda}q^{-1}\right)} \\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \frac{q^{-k-1}}{\left(q^\beta + \frac{1}{\lambda}\right)^{k+1}}.\end{aligned} \tag{3.21}$$

The inverse Laplace transform of the above functions, by using (3.12), are [23]

$$f_4(t) = L^{-1}\{F_4(q)\} = \frac{1}{\lambda} G_{\beta,0,1}\Big(-\frac{1}{\lambda}, t\Big), \tag{3.22}$$

$$f_5(r_n, t) = L^{-1}\{F_5(r_n, q)\} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \Big(\frac{-\nu r_n^2}{\lambda}\Big)^k G_{\beta,-k-1,k+1}\Big(-\frac{1}{\lambda}, t\Big). \tag{3.23}$$

Applying again the inverse Laplace transform to Eq. (3.19), using $(3.14)_1$, (3.22), (3.23) and the convolution theorem, we find the shear stress $\tau(r,t)$ in the following form.

If $a > 0$, then

$$\tau(r,t) = \frac{fr}{R} f_1(t) * f_4(t) + \frac{2f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} f_2(t) * f_5(r_n, t), \tag{3.24}$$

and if $a = 0$, then

$$\tau(r,t) = \frac{fr}{\lambda R} \int_0^t G_{\beta,0,1}\Big(-\frac{1}{\lambda}, t\Big) + \frac{2f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} f_5(r_n, t). \tag{3.25}$$

By using (3.13) we obtain

$$\int_0^t G_{b,c,d}(p,\tau)\,d\tau = G_{b,c-1,d}(p,t)$$

and Eq. (3.25) can be written in the form

$$\tau(r,t) = \frac{fr}{\lambda R} G_{\beta,-1,1}\Big(-\frac{1}{\lambda}, t\Big) + \frac{2f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} f_5(r_n, t). \tag{3.26}$$

4. Limiting cases

4.1. Flow of Maxwell fluid due to longitudinal constant shear stress ($\beta = 1$, $a = 0$)

For $\beta \to 1$ our model is reduced to the ordinary Maxwell fluid and for $a = 0$, the shear stress on the boundary of the cylinder is constant, equal to f.

Introducing $\beta \to 1$ into Eqs. (3.17) and (3.26), we obtain the velocity field

$$\begin{aligned} v(r,t) = \frac{fr^2}{2\mu R} - \frac{2f}{\mu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \\ \times \sum_{k=0}^{\infty} \Big(\frac{-\nu r_n^2}{\lambda}\Big)^k \Big[G_{1,-k,k+1}\Big(-\frac{1}{\lambda}, t\Big) + \frac{1}{\lambda} G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda}, t\Big)\Big], \end{aligned} \tag{4.1}$$

and the associated shear stress

$$\tau(r,t) = \frac{rf}{\lambda R} G_{1,-1,1}\Big(-\frac{1}{\lambda}, t\Big) + \frac{2f}{\lambda R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \sum_{k=0}^{\infty} \Big(\frac{-\nu r_n^2}{\lambda}\Big)^k G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda}, t\Big), \tag{4.2}$$

corresponding to the ordinary Maxwell fluid, performing the same motion.

By using (3.13) we find that

$$G_{1,-1,1}\Big(-\frac{1}{\lambda}, t\Big) = \lambda\Big[1 - \exp\Big(-\frac{t}{\lambda}\Big)\Big], \tag{4.3}$$

and the expression (4.2) can be written in the simplified form

$$\tau(r,t) = \frac{fr}{R}\Big[1 - \exp\Big(-\frac{t}{\lambda}\Big)\Big] + \frac{2f}{\lambda R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \sum_{k=0}^{\infty} \Big(\frac{-\nu r_n^2}{\lambda}\Big)^k G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda}, t\Big). \tag{4.4}$$

4.2. Flow of a Newtonian fluid due to torsional constant shear stress ($\beta = 1$, $a = 0$, $\lambda \to 0$)

Assuming $\lambda \to 0$ in Eqs. (4.1) and (4.4), the known solutions

$$v(r,t) = \frac{fr^2}{2\mu R} - \frac{2f}{\mu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \exp(-\nu r_n^2 t), \tag{4.5}$$

and

$$\tau(r,t) = \frac{fr}{R} + \frac{2f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \exp(-\nu r_n^2 t), \tag{4.6}$$

corresponding to the Newtonian fluid, are recovered [24, Eqs. (34) and (44)].

4.3. Flow of a Maxwell fluid due to the torsional time-variable shear stress ($\beta = 1$, $a > 0$)

Assuming $\beta \to 1$ in Eqs. (3.16) and (3.24) we obtain the velocity field

$$v(r,t) = \frac{fr^2}{2\mu R} t^a - \frac{2af}{\mu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \sum_{k=0}^{\infty} \Big(-\frac{\nu r_n^2}{\lambda}\Big)^k \times \int_0^t (t-\tau)^{a-1} \Big[G_{1,-k,k+1}\Big(-\frac{1}{\lambda}, \tau\Big) + \frac{1}{\lambda} G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda}, \tau\Big)\Big] d\tau, \tag{4.7}$$

and the associated shear stress

$$\tau(r,t) = \frac{fr}{\lambda R}\int_0^t \tau^a \exp\Big[-\frac{1}{\lambda}(t-\tau)\Big]d\tau + \frac{2af}{\lambda R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)} \times \sum_{k=0}^{\infty}\Big(-\frac{\nu r_n^2}{\lambda}\Big)^k \int_0^t (t-\tau)^{a-1}G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda},\tau\Big)d\tau, \tag{4.8}$$

corresponding to the Maxwell fluid, performing the same motion.

For $a = 1$, Eqs. (4.7) and (4.8) can be written in the following forms:

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\sum_{k=0}^{\infty}\Big(-\frac{\nu r_n^2}{\lambda}\Big)^k \times \int_0^t \Big[G_{1,-k,k+1}\Big(-\frac{1}{\lambda},\tau\Big) + \frac{1}{\lambda}G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda},\tau\Big)\Big]d\tau, \tag{4.9}$$

and

$$\tau(r,t) = \frac{fr}{\lambda R}\Big[t - \lambda\Big(1-\exp\Big(\frac{-t}{\lambda}\Big)\Big)\Big] + \frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)} \times \sum_{k=0}^{\infty}\Big(\frac{-\nu r_n^2}{\lambda}\Big)^k \int_0^t G_{1,-k-1,k+1}\Big(-\frac{1}{\lambda},\tau\Big)d\tau. \tag{4.10}$$

4.4. Flow of a Newtonian fluid due to torsional time-variable shear stress ($\beta = 1$, $a > 0$, $\lambda \to 0$)

Assuming $\lambda \to 0$ in Eqs. (4.7) and (4.8) or (4.9) and (4.10), similar solutions [24, Eqs. (32) and (42)]

$$v(r,t) = \frac{fr^2}{2\mu R}t^a - \frac{2af}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\int_0^t (t-\tau)^{a-1}\exp(-\nu r_n^2\tau)d\tau, \tag{4.11}$$

and

$$\tau(r,t) = \frac{fr}{R}t^a + \frac{2af}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\int_0^t (t-\tau)^{a-1}\exp(-\nu r_n^2\tau)d\tau, \tag{4.12}$$

corresponding to the Newtonian fluid are recovered.

For $a = 1$ in Eqs. (4.11) and (4.12), the simple solutions

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4J_0(Rr_n)}[1-\exp(-\nu r_n^2 t)], \tag{4.13}$$

and

$$\tau(r,t) = \frac{fr}{R}t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}[1-\exp(-\nu r_n^2 t)], \tag{4.14}$$

for a Newtonian fluid are recovered in [10, 11, 24, 25].

5. Conclusions

In this paper, the velocity fields and the resulting shear stresses corresponding to the axial flow of generalized Maxwell fluids through a circular cylinder due to a longitudinal shear stress are determined. The solutions determined by means of the Laplace and finite Hankel transforms are presented in integral and series forms in terms of generalized G-functions, and satisfy all the imposed initial and boundary conditions.

For $\beta = 1$, the model of the fluid with fractional derivatives is reduced to the classical Maxwell fluid and for $\beta = 1$ and $\lambda \to 0$, our generalized model is reduced to the Newtonian fluid. Finally, several relevant physical aspects of the obtained solutions have been shown by means of graphical illustrations. The diagrams of the velocity fields corresponding to Newtonian (continuous thick line), Maxwell (black circle line) and generalized Maxwell (circle and triangle line) fluids are plotted in Fig. 1 for $a = 0$. From this figure we see that in the

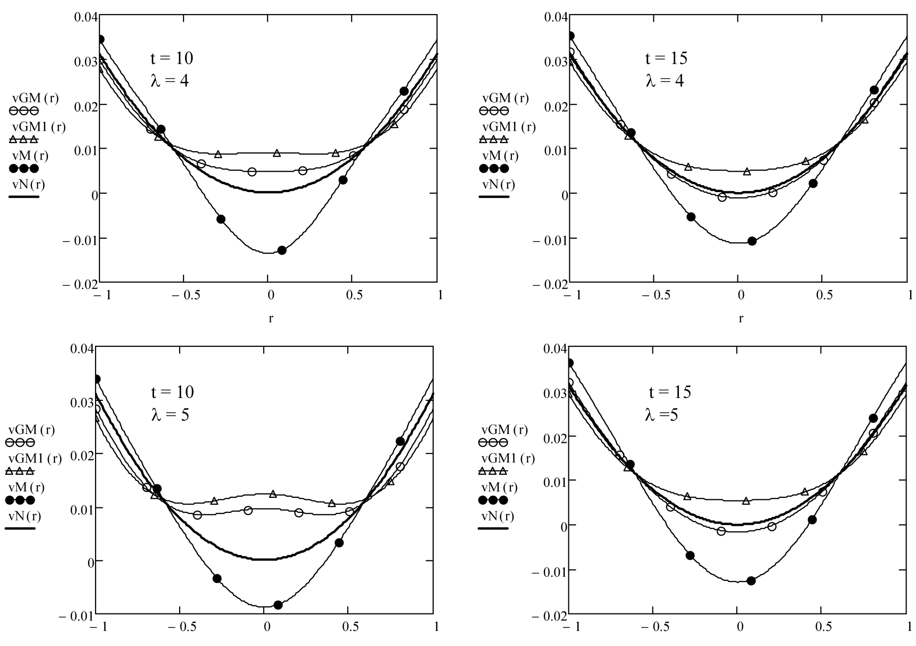

Fig. 1. Profiles of velocity field$v(r,t)$ for $\nu = 0.0357$, $\mu = 32$, $R = 1$, $f = 2$ and for different values of the fractional coefficient β; —— Newtonian fluids, ●●● Maxwell fluids ($\beta = 1$), ○○○ generalized Maxwell fluid ($\beta = 0.5$), △△△ generalized Maxwell fluid ($\beta = 0.2$).

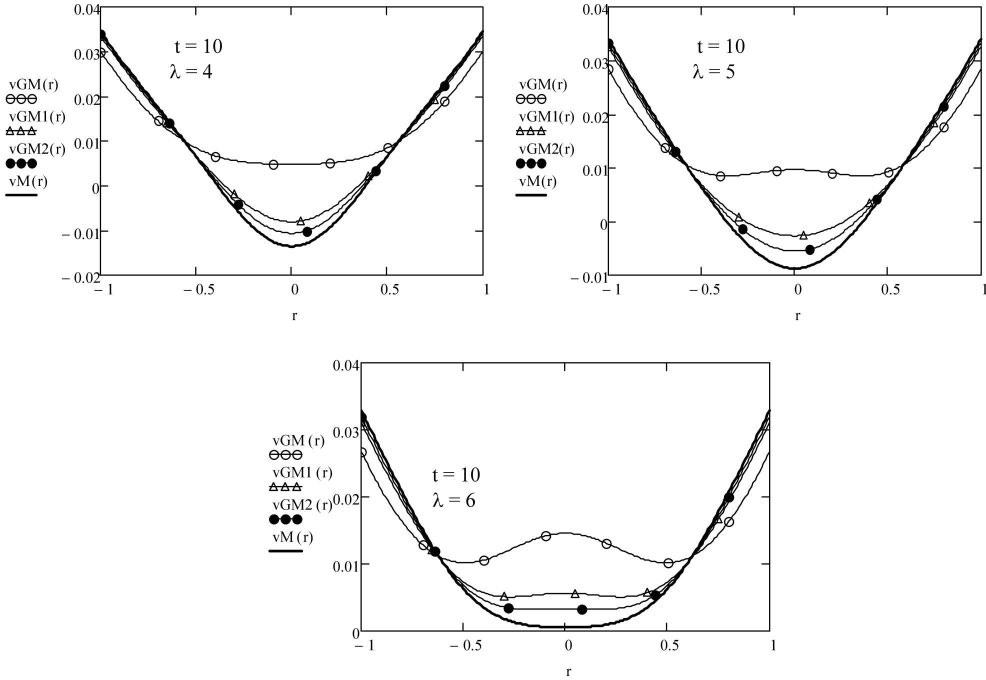

FIG. 2. Profiles of velocity field$v(r,t)$ of generalized Maxwell fluids for different values of the fractional coefficient β and for $\nu = 0.0357$, $\mu = 32$, $R = 1$, $f = 2$.

central area of the channel, the Maxwell fluid has lower velocity and the velocity of the generalized Maxwell fluid increases if the fractional coefficient decreases. In the closed wall area, the velocity of the generalized Maxwell fluid decreases if the fractional coefficient decreases. For high values of the time t, the differences between velocity fields of Maxwell, generalized Maxwell and Newtonian fluids, disappear. The diagrams of the velocity field corresponding to the generalized Maxwell fluid are plotted in Fig. 2 for $a = 0$ and for different values of the fractional coefficient β. We see that for $\beta \to 1$, the diagrams of the velocity field tend to the diagram corresponding to the Maxwell fluid. The units of the parameters in Figs. 1 and 2 are from SI units and the roots r_n have been approximated by $r_n \simeq (4n+1)\pi/4R$ [21].

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