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Bounds on the effective isotropic moduli of thin elastic composite plates

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The main aim of this paper is to estimate the effective moduli of an isotropic elastic composite, analyzed within the framework of the Kirchhoff-Love theory of thin plates in bending. Results of calculations provide explicit functional correlations between the homogenized properties of a composite plate made of two isotropic materials, thus yielding more restrictive bounds on pairs of effective moduli than the classical (uncoupled) Hashin–Shtrikman–Walpole ones. Applying the static-geometric analogy of Lurie and Goldenveizer, enables rewriting of these new bounds in the two-dimensional elasticity (plane stress) setting, thus revealing a link to the formulae previously found by Gibiansky and Cherkaev. Consequently, simple cross-property estimates are proposed for the plate subject to the simultaneous bending and in-plane loads.

Key words: microstructures, inhomogeneous material, plates, optimization, translation method.

1. Introduction

The history of estimating the averaged constitutive moduli of multiphase materials goes back at least as far as the introduction of the harmonic and arithmetic (Voigt–Reuss) estimates and it is worth pointing out that, despite their simplicity, they proved to be optimal (i.e. attainable on certain microstructural layouts of constituent materials) for some macroscopically anisotropic mixtures. However, in general these bounds can be improved, hence new ideas leading to more accurate estimates have emerged, see Milton [11] for the treatment of this topic in exhaustive detail.

In this paper we tackle the problem of calculating the range of the homogenized effective moduli (k_h, μ_h) of a Kirchhoff–Love isotropic elastic composite plate made of two materials, whose Hooke's tensors are also isotropic. To this end we estimate the energy accumulated in a particle of a two-material mixture subject to three linearly independent plate moments or curvature tensor components with fixed macroscopic values. Our research is thus related to the

"G_m-closure" problem for a set of basic materials U (G_mU for short), which consists in finding the totality of effective constitutive tensors for all possible microstructural layouts of constituents, mixed in fixed proportions expressed by $m \in (0,\ 1)$. In CHERKAEV [1, Ch. 10], see also MILTON [11, Ch. 30.3], it is shown that G_mU is estimated from outside by minimizing sums of energies and dual energies, but the question of its precise calculation is still open in broad elasticity setting.

In the present work we investigate the Kirchhoff-Love theory of thin plates in bending for an outside estimate of a particular subset of G_mU, consisting of its isotropic members. It is well known that for "well-ordered" basic materials, the smallest rectangle containing this subset can be determined through the Hashin–Shtrikman variational principle in two-dimensional elasticity as well as plate theory settings, see HASHIN and SHTRIKMAN [6], LI [8]. The counterpart of Hashin–Shtrikman estimates in case of "badly-ordered" materials was found by WALPOLE [14]. An important point to note here is that the Hashin–Shtrikman–Walpole (HSW for short) estimates are independent (uncoupled).

Mutually dependent bounds on k_h and μ_h were introduced in CHERKAEV and GIBIANSKY [2] in case of plane stress elasticity, see also GIBIANSKY [4] for a discussion of the same problem from a different viewpoint, and it follows that admissible values of pairs (k_h, μ_h) occupy a region that is smaller than the one predicted by HSW principle. In their research, authors of both papers make use of the quasi-convexity theory-based method, developed by F. Murat and L. Tartar and independently, in alternative setting, by K. A. Lurie and A. V. Cherkaev. In the remainder we follow the latter approach based on the estimation of energy accumulated in the mixture subject to macroscopic fields of any type, see CHERKAEV [1, Ch. 8] or MILTON [11, Ch. 24, Ch. 25.1], where the Lurie–Cherkaev's variational approach is compared to the Murat-Tartar's idea of compensated compactness.

The main idea of calculations follows from the definition of the outer estimate of G_mU. Roughly speaking, constitutive tensors of basic materials are translated by a certain constant fourth-order tensor and then the classical harmonic mean bound is applied. Thus, following MILTON [10], the estimation technique is referred to as a "translation method". Its quality depends on the choice of a translator which in general can be any quasi-convex function, but here we restrict ourselves to quasi-affine quadratic functions, see e.g. CHERKAEV [1, Ch. 8.2] and MILTON [11, Ch. 24.3]. We refer the Reader to these works for the thorough derivation of the translation method and its application to modern mechanics.

The problem of bounding the effective properties of a multiphase composite is less developed if it is set in a theory of plates in bending. Some of its aspects are dealt with in LEWIŃSKI and TELEGA [7, Ch. 22], where HS estimates were derived by the translation method. LURIE and CHERKAEV [9] used the same approach to characterize the "G-closure" of the set of basic materials with equal shear moduli.

In the present paper we wish to investigate the Kirchhoff–Love theory for the analytical formulae, establishing (k_h, μ_h)-coupling of an isotropic assembly of two isotropic constituents whose moduli $k_\alpha, \mu_\alpha, \alpha = 1, 2$ are arbitrary. We are looking for correlative bounds which are tighter than the known HSW (uncoupled) estimates obtained in LEWIŃSKI and TELEGA [7], see also LI [8].

2. Basic equations of a composite thin plate in bending

The Kirchhoff–Love theory of thin plates can be discussed from different viewpoints, hence the exposition of its equations ranges from the classical development based on the kinematical assumptions imposed on a thin three-dimensional body to the precise, mathematical justification of the theory; for details we refer the Reader to e.g. TIMOSHENKO and WOINOWSKI–KRIEGER [13], CIARLET [3]. Below we recall the formulae necessary for further considerations.

Let $(\mathbf{e}_1, \mathbf{e}_2)$ denote the Cartesian basis in $\mathbb{R}^2$, also referred to as a "physical basis", and introduce an orthonormal basis:

$$\begin{aligned} \mathbf{E}_1 &= \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2\right), \qquad & \mathbf{E}_2 &= \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2\right), \\ \mathbf{E}_3 &= \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1\right), \qquad & \mathbf{E}_4 &= \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1\right), \end{aligned} \tag{2.1}$$

convenient for representing the second-order tensors related to $\mathbb{E} = \mathbb{R}^2 \otimes \mathbb{R}^2$, i.e. for any $\mathbf{f} = f_{\alpha\beta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, $\alpha, \beta = 1, 2$, we may write $\mathbf{f} = F_i\, \mathbf{E}_i$, where

$$\begin{aligned} F_1 &= \frac{1}{\sqrt{2}}(f_{11} + f_{22}), \qquad & F_2 &= \frac{1}{\sqrt{2}}(f_{11} - f_{22}), \\ F_3 &= \frac{1}{\sqrt{2}}(f_{12} + f_{21}), \qquad & F_4 &= \frac{1}{\sqrt{2}}(f_{12} - f_{21}). \end{aligned} \tag{2.2}$$

Note that subspaces $\mathbb{E}_s$ and $\mathbb{E}_a$ of symmetric and antisymmetric members of $\mathbb{E}$ respectively, admit bases $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\mathbf{E}_4$. Moreover, $\mathbf{E}_1$ corresponds to the one-dimensional subspace of hydrostatic tensors, while $\{\mathbf{E}_2,\ \mathbf{E}_3\}$ span the two-dimensional subspace of symmetric deviators.

Write $\Omega \in \mathbb{R}^2$ for the mid-plane of a plate and set w for a scalar field of its transverse displacement (deflection). According to the Kirchhoff kinematical assumption, the rotation angles of a segment perpendicular to Ω and components of the mid-plane curvature tensor $\boldsymbol{\kappa} = \kappa_{\alpha\beta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ are respectively denoted by

$$\vartheta_\alpha = -w_{,\alpha}, \qquad \kappa_{\alpha\beta} = \vartheta_{\alpha,\beta}, \tag{2.3}$$

where $\alpha, \beta = 1,2$, and $\kappa_{\alpha\beta}$ satisfy the compatibility condition

$$\kappa_{11,22} + \kappa_{22,11} - 2\,\kappa_{12,12} = 0, \tag{2.4}$$

where $(\cdot)_{,\alpha} = \partial\,(\cdot)\,/\partial x_\alpha$. Since $\boldsymbol{\kappa}$ is a symmetric tensor, thus in (2.1) we have

$$\boldsymbol{\kappa} = \frac{1}{\sqrt{2}} \begin{pmatrix} \vartheta_{1,1} + \vartheta_{2,2} \\ \vartheta_{1,1} - \vartheta_{2,2} \\ 2\,\vartheta_{1,2} \\ 0 \end{pmatrix}. \tag{2.5}$$

Assume that q stands for a scalar field of transversal loading applied to Ω and let $\boldsymbol{M} = M_{\alpha\beta}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ denote the symmetric tensor field of plate moments. Components of $\boldsymbol{M}$ are related by the local equlibrium equation

$$M_{\alpha\beta,\alpha\beta} = -q, \tag{2.6}$$

hence the stress functions Ψ_1, Ψ_2, see e.g. GOLDENVEIZER [5, Ch. 3.21], such that

$$M_{11} = \Psi_{2,2}, \qquad M_{22} = \Psi_{1,1}, \qquad 2M_{12} = 2M_{21} = -(\Psi_{1,2} + \Psi_{2,1}), \tag{2.7}$$

satisfy Eq. (2.6) for $q = 0$. In the sequel we refer to $\boldsymbol{\sigma} \in \mathbb{E}$ given by

$$\boldsymbol{\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{1,1} + \Psi_{2,2} \\ \Psi_{1,1} - \Psi_{2,2} \\ \Psi_{1,2} + \Psi_{2,1} \\ \Psi_{1,2} - \Psi_{2,1} \end{pmatrix}. \tag{2.8}$$

Suppose that the mid-plane Ω of a plate is built from microscopic cells ω and each cell is occupied by an arbitrary mixture of two isotropic elastic materials, whose bulk and shear moduli are denoted by

$$k_\alpha = \frac{E_\alpha\, t^3}{24\,(1-\nu_\alpha)}, \qquad \mu_\alpha = \frac{E_\alpha\, t^3}{24\,(1+\nu_\alpha)}, \tag{2.9}$$

$\alpha = 1,2$, where E_α, ν_α stand for Young's modulus and Poisson's ratio of material α respectively, and t denotes the thickness of a plate. Consequently, matrices

$$\begin{aligned} \boldsymbol{D}_\alpha &= \operatorname{diag}\lceil 2\,k_\alpha,\ 2\,\mu_\alpha,\ 2\,\mu_\alpha,\ 0\rfloor, \\ \boldsymbol{d}_\alpha &= \operatorname{diag}\lceil 1/(2\,k_\alpha),\ 1/(2\,\mu_\alpha),\ 1/(2\,\mu_\alpha),\ 0\rfloor, \end{aligned} \tag{2.10}$$

represent plate stiffness and compliance tensors. Area fractions of constituent phases in ω are given by $m_1(\omega)$, $m_2(\omega)$ such that

$$m_1(\omega) + m_2(\omega) = 1, \tag{2.11}$$

while their distribution in the direction perpendicular to Ω is homogeneous.

Our main interest lies in estimating the range of macroscopic elastic properties related to the microscopic mixture of basic materials in ω. By the theory of G-convergence applied to the case under study and due to the Dal Maso–Kohn–Raitums theorem, see LEWIŃSKI and TELEGA [7, Ch. 26], without any loss of generality we can restrict our considerations to the ω-periodic distributions of materials in Ω, where ω denotes a basic periodicity cell. Obviously, the effective tensor also reflects the microscopic layout of materials in ω and it is worth pointing out that this tensor is not necessarily isotropic. Following the idea of periodicity, in the remainder we assume that Ψ_1, Ψ_2 and ϑ_1, ϑ_2 are represented by their ω-periodic expansions in an appropriate Fourier space. Consequently, we write $\boldsymbol{\sigma}_0 = \langle \boldsymbol{\sigma} \rangle$ and $\boldsymbol{\kappa}_0 = \langle \boldsymbol{\kappa} \rangle$ for the constant macroscopic fields, where

$$\langle \mathbf{F} \rangle = m_1 \mathbf{F}_1 + m_2 \mathbf{F}_2 \tag{2.12}$$

for $\mathbf{F}_1$, $\mathbf{F}_2$ acting in ω. The effective stiffness and compliance tensors are respectively calculated by

$$\boldsymbol{\kappa}_0^T \boldsymbol{D}_h \boldsymbol{\kappa}_0 = \min_{\boldsymbol{\kappa} \in V_\kappa} \langle \boldsymbol{\kappa}^T \boldsymbol{D}\, \boldsymbol{\kappa} \rangle, \tag{2.13}$$

$$\boldsymbol{\sigma}_0^T \boldsymbol{d}_h\, \boldsymbol{\sigma}_0 = \inf_{\boldsymbol{\sigma} \in V_\sigma} \langle \boldsymbol{\sigma}^T \boldsymbol{d}\, \boldsymbol{\sigma} \rangle, \tag{2.14}$$

where

$$V_\kappa = \{ \boldsymbol{\kappa} : \boldsymbol{\kappa} \text{ is represented by (2.5), } \omega\text{-periodic and } \langle \boldsymbol{\kappa} \rangle = \boldsymbol{\kappa}_0 \}\,, \tag{2.15}$$

$$V_\sigma = \{ \boldsymbol{\sigma} : \boldsymbol{\sigma} \text{ is represented by (2.8), } \omega\text{-periodic and } \langle \boldsymbol{\sigma} \rangle = \boldsymbol{\sigma}_0 \}\,. \tag{2.16}$$

3. Outline of the estimation method

3.1. Quasi-affinity of the translation functional

Energy accumulated in a particle of a composite plate in bending can be understood as a measure of simultaneous action of $n \leq 3$ ($\dim \mathbb{E}_s = 3$) independent mean fields acting at ω and according to CHERKAEV [1, Ch. 16], complete derivation of effective property bounds involves consideration of all sums of primal and dual energy components. Following MILTON [10], for the convenience of calculations in the sequel, we transfer the problem to the higher-order tensorial space $\mathbb{E} \otimes \mathbb{E}_s$ thus defining the above-mentioned sum of energies as a single functional.

For a basis $\{\mathbf{E}_i\}$, $i = 1, 2, 3$, see (2.1), let

$$V_\Phi = \left\{ \boldsymbol{\Phi} : \boldsymbol{\Phi} = \mathbf{f}_{(1)} \otimes \mathbf{E}_1 + \mathbf{f}_{(2)} \otimes \mathbf{E}_2 + \mathbf{f}_{(3)} \otimes \mathbf{E}_3, \quad \text{where } \mathbf{f}_{(i)} \in \{V_\kappa, V_\sigma\} \right\} \tag{3.1}$$

denote a set of 12×1 supervectors and define a 12×12 constitutive supertensor

$$\boldsymbol{\Delta}_\alpha = \boldsymbol{C}_{(1)} \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{C}_{(2)} \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{C}_{(3)} \otimes \mathbf{E}_3 \otimes \mathbf{E}_3, \tag{3.2}$$

where $\boldsymbol{C}_{(i)}$ stands for a stiffness tensor $\boldsymbol{D}_\alpha$ if $\mathbf{f}_{(i)} \in V_\kappa$ or a compliance tensor $\boldsymbol{d}_\alpha$ if $\mathbf{f}_{(i)} \in V_\sigma$. It is easily seen that $\boldsymbol{\Delta}_\alpha$ is represented by a block-diagonal matrix. By similar notation we denote the translation supertensor:

$$\mathbf{T} = \boldsymbol{T}_{(ij)} \otimes \mathbf{E}_i \otimes \mathbf{E}_j, \qquad i, j = 1, 2, 3, \tag{3.3}$$

whose components $\boldsymbol{T}_{(ij)}$ will be derived in the remainder of this section. Consequently, we define the energy functional in the form

$$2\,U(\boldsymbol{\Phi}, \boldsymbol{\Delta}) = \sum_{i=1}^{3} B_\Delta(\mathbf{f}_{(i)}, \mathbf{f}_{(i)}), \tag{3.4}$$

where

$$B_\Delta(\mathbf{f}_{(i)}, \mathbf{f}_{(i)}) = \mathbf{f}_{(i)}^T\, \boldsymbol{C}_{(i)}\, \mathbf{f}_{(i)}. \tag{3.5}$$

Next, we introduce a quasi-affine function $T(\boldsymbol{\Phi})$, i.e. such that

$$\langle T\,(\boldsymbol{\Phi})\rangle = T\,(\langle \boldsymbol{\Phi}\rangle) \tag{3.6}$$

for any periodic field $\boldsymbol{\Phi} \in V_\Phi$. From (3.3) we deduce that $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{pmatrix} \boldsymbol{T}_{(11)} & \boldsymbol{T}_{(12)} & \boldsymbol{T}_{(13)} \\ \boldsymbol{T}_{(21)} & \boldsymbol{T}_{(22)} & \boldsymbol{T}_{(23)} \\ \boldsymbol{T}_{(31)} & \boldsymbol{T}_{(32)} & \boldsymbol{T}_{(33)} \end{pmatrix}, \tag{3.7}$$

hence $T(\boldsymbol{\Phi})$ admits the following notation:

$$T(\boldsymbol{\Phi}) = \sum_{i=1}^{3}\sum_{j=1}^{3} B_T\big(\langle \mathbf{f}_{(i)}\rangle, \langle \mathbf{f}_{(j)}\rangle\big), \tag{3.8}$$

where

$$B_T\big(\langle \mathbf{f}_{(i)}\rangle, \langle \mathbf{f}_{(j)}\rangle\big) = \langle \mathbf{f}_{(i)}\rangle^T\, \boldsymbol{T}_{(ij)}\, \langle \mathbf{f}_{(j)}\rangle. \tag{3.9}$$

Determination of $\boldsymbol{T}_{(ij)}$ satisfying (3.6) is the key issue of the translation method. To this end we make use of the well-known result for the function

$$J(\nabla u, \nabla v) = \det \begin{pmatrix} u_{,1} & v_{,1} \\ u_{,2} & v_{,2} \end{pmatrix}, \tag{3.10}$$

where u, v denote two potentials. According to CHERKAEV and GIBIANSKY [2], see also references therein, this function is quasi-affine, i.e.

$$\langle J(\nabla u, \nabla v)\rangle = J\big(\langle \nabla u\rangle,\ \langle \nabla v\rangle\big) \tag{3.11}$$

provided u, v are periodic. This result is valid for the linear combination of functions defined in (3.10), that is for any vector-valued periodic functions

$$\mathbf{g} = g_1\mathbf{e}_1 + g_2\mathbf{e}_2, \qquad \mathbf{h} = h_1\mathbf{e}_1 + h_2\mathbf{e}_2 \tag{3.12}$$

and for arbitrary parameters $\tau_1, \ldots, \tau_4$ in $\mathbb{R}$, the function

$$\tau_1\, J(g_1, h_1) + \tau_2\, J(g_1, h_2) + \tau_3\, J(g_2, h_1) + \tau_4\, J(g_2, h_2) \tag{3.13}$$

is also quasi-affine. By rewriting $\nabla\mathbf{g}$ and $\nabla\mathbf{h}$ in the form of (2.2), we deduce that (3.13) is compatible with (3.9), i.e.

$$B_T(\nabla\mathbf{g}, \nabla\mathbf{h}) = \nabla\mathbf{g}^T \begin{pmatrix} -t_1 & -t_2 & -t_3 & -t_4 \\ t_2 & t_1 & -t_4 & -t_3 \\ t_3 & t_4 & t_1 & t_2 \\ t_4 & t_3 & -t_2 & -t_1 \end{pmatrix} \nabla\mathbf{h} \tag{3.14}$$

and t_k, $k = 1, \ldots, 4$, are linearly dependent on τ_k. Obviously, $\nabla\mathbf{g}$ and $\nabla\mathbf{h}$ can be taken as two arbitrary fields belonging to V_κ or V_σ, whose components are defined as derivatives of periodic potential functions, see (2.15), (2.16). By applying (3.14) in (3.7) we deduce that $T(\mathbf{\Phi})$ admits 36 different translation parameters, but this number can be reduced to 4, as will be shown in the sequel.

3.2. Symmetry and isotropy of the translation matrix

Following MILTON [10], we take into account the invariance of $T(\mathbf{\Phi})$ with respect to rotation of the physical basis. Moreover, we limit our concern to symmetric translators since in the course of calculations, $\mathbf{T}$ admits certain properties of the constitutive supertensor. Our exposition extends the result in GIBIANSKY [4].

The symmetry of $\mathbf{T}$ is handled by setting

$$\boldsymbol{T}_{(ji)} = \boldsymbol{T}^T_{(ij)} \tag*{(3.15)$_1$}$$

in (3.7) and, if $i = j$,

$$t_2^{(ii)} = t_3^{(ii)} = t_4^{(ii)} = 0. \tag*{(3.15)$_2$}$$

To satisfy the isotropy of $T(\boldsymbol{\Phi})$ we will set the conditions of its invariance with respect to the rotation of the physical basis. For this purpose we assume

$$\mathbf{f}_{(1)} = \begin{pmatrix} q_1 \\ q_5 + q_7 \\ q_6 + q_8 \\ q_4 \end{pmatrix}, \qquad \mathbf{f}_{(2)} = \begin{pmatrix} q_5 - q_7 \\ q_2 + q_{11} \\ q_3 + q_{12} \\ q_9 \end{pmatrix}, \qquad \mathbf{f}_{(3)} = \begin{pmatrix} q_6 - q_8 \\ -q_3 + q_{12} \\ q_2 - q_{11} \\ q_{10} \end{pmatrix} \tag{3.16}$$

in (3.1) and we reshape $\boldsymbol{\Phi}$ to a 4×3 matrix form

$$\tilde{\boldsymbol{\Phi}} = \left(\mathbf{f}_{(1)}\ \mathbf{f}_{(2)}\ \mathbf{f}_{(3)}\right). \tag{3.17}$$

Next, we introduce a rotation operator $\mathcal{R}*$ defined as

$$\mathcal{R} * \boldsymbol{\Phi} = \boldsymbol{R}_4 \tilde{\boldsymbol{\Phi}} \boldsymbol{R}_3^T, \tag{3.18}$$

where

$$\begin{aligned} \boldsymbol{R}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\varphi) & -\sin(2\varphi) \\ 0 & \sin(2\varphi) & \cos(2\varphi) \end{pmatrix}, \\ \boldsymbol{R}_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & \sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.19}$$

Basis $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{12}$ composed of 4×3 matrices such that

$$\begin{aligned} \boldsymbol{B}_1 &= \mathbf{E}_1 \otimes \mathbf{E}_1, & \boldsymbol{B}_2 &= \mathbf{E}_2 \otimes \mathbf{E}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3, \\ \boldsymbol{B}_3 &= \mathbf{E}_3 \otimes \mathbf{E}_2 - \mathbf{E}_2 \otimes \mathbf{E}_3, & \boldsymbol{B}_4 &= \mathbf{E}_4 \otimes \mathbf{E}_1, \\ \boldsymbol{B}_5 &= \mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1, & \boldsymbol{B}_6 &= \mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1, \\ \boldsymbol{B}_7 &= \mathbf{E}_2 \otimes \mathbf{E}_1 - \mathbf{E}_1 \otimes \mathbf{E}_2, & \boldsymbol{B}_8 &= \mathbf{E}_3 \otimes \mathbf{E}_1 - \mathbf{E}_1 \otimes \mathbf{E}_3, \\ \boldsymbol{B}_9 &= \mathbf{E}_4 \otimes \mathbf{E}_2, & \boldsymbol{B}_{10} &= \mathbf{E}_4 \otimes \mathbf{E}_3, \\ \boldsymbol{B}_{11} &= \mathbf{E}_2 \otimes \mathbf{E}_2 - \mathbf{E}_3 \otimes \mathbf{E}_3, & \boldsymbol{B}_{12} &= \mathbf{E}_3 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_3 \end{aligned} \tag{3.20}$$

is convenient for representing of $\boldsymbol{\Phi} \in V_\Phi$ as a sum of orthogonal components, stable under rotation of a physical basis. We will show that

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathrm{inv}} + \boldsymbol{\Phi}_{2\varphi} + \boldsymbol{\Phi}_{4\varphi}, \tag{3.21}$$

where

$$\begin{aligned} \boldsymbol{\Phi}_{\mathrm{inv}} &= q_1 \boldsymbol{B}_1 + q_2 \boldsymbol{B}_2 + q_3 \boldsymbol{B}_3 + q_4 \boldsymbol{B}_4, \\ \boldsymbol{\Phi}_{2\varphi} &= q_5 \boldsymbol{B}_5 + q_6 \boldsymbol{B}_6 + q_7 \boldsymbol{B}_7 + q_8 \boldsymbol{B}_8 + q_9 \boldsymbol{B}_9 + q_{10} \boldsymbol{B}_{10}, \\ \boldsymbol{\Phi}_{4\varphi} &= q_{11} \boldsymbol{B}_{11} + q_{12} \boldsymbol{B}_{12} \end{aligned} \tag{3.22}$$

and $\mathbf{\Phi}_{\text{inv}}$ denotes a rotationally invariant supervector, whereas $\mathbf{\Phi}_{2\varphi}$ and $\mathbf{\Phi}_{4\varphi}$ stand for supervectors remaining stable if the physical basis is rotated by angles $\pi + 2k\pi$ and $\pi/2 + 2k\pi$, $k \in \mathbb{N}$, respectively. Application of (3.18) leads to

$$(3.23)_1 \qquad \mathcal{R} * (q_i \boldsymbol{B}_i) = q_i' \boldsymbol{B}_i = q_i \boldsymbol{B}_i$$

if $i = 1, \ldots, 4$,

$$(3.23)_2 \qquad \begin{aligned} \mathcal{R} * (q_i \boldsymbol{B}_i + q_{i+1} \boldsymbol{B}_{i+1}) &= q_i' \boldsymbol{B}_i + q_{i+1}' \boldsymbol{B}_{i+1} \\ &= (q_i \cos(2\varphi) - q_{i+1} \sin(2\varphi)) \boldsymbol{B}_i + (q_i \sin(2\varphi) + q_{i+1} \cos(2\varphi)) \boldsymbol{B}_{i+1} \end{aligned}$$

if $i = 5, 7, 9$ and

$$(3.23)_3 \qquad \begin{aligned} \mathcal{R} * (q_{11} \boldsymbol{B}_{11} + q_{12} \boldsymbol{B}_{12}) &= q_{11}' \boldsymbol{B}_{11} + q_{12}' \boldsymbol{B}_{12} \\ &= (q_{11} \cos(4\varphi) - q_{12} \sin(4\varphi)) \boldsymbol{B}_{11} + (q_{11} \sin(4\varphi) + q_{12} \cos(4\varphi)) \boldsymbol{B}_{12}, \end{aligned}$$

hence the decomposition (3.21) follows. The requirement of isotropy

$$(3.24) \qquad T(\mathbf{\Phi}) = T(\mathcal{R} * \mathbf{\Phi}),$$

is satisfied if $T(\mathbf{\Phi})$ is represented as a linear combination of invariants of supervectors $\mathbf{\Phi}_{\text{inv}}$, $\mathbf{\Phi}_{2\varphi}$ and $\mathbf{\Phi}_{4\varphi}$. An easy computation shows that

$$(3.25) \qquad \begin{aligned} i_{\text{inv}} &= \{q_1',\ q_2',\ q_3',\ q_4'\}, \\ i_{2\varphi} &= \{(q_5')^2 + (q_6')^2,\ (q_7')^2 + (q_8')^2,\ (q_9')^2 + (q_{10}')^2, \\ &\qquad q_5' q_7' + q_6' q_8',\ q_5' q_{10}' - q_6' q_9',\ q_7' q_{10}' - q_8' q_9', \\ &\qquad q_5' q_8' - q_6' q_7',\ q_5' q_9' + q_6' q_{10}',\ q_7' q_9' + q_8' q_{10}'\}, \\ i_{4\varphi} &= \{(q_{11}')^2 + (q_{12}')^2\} \end{aligned}$$

denote the sets of invariants with respect to the corresponding rotation angles. Moreover, only 6 elements of $i_{2\varphi}$ are independent, as

$$(3.26)_1 \qquad \left(q_5' q_8' - q_6' q_7'\right)^2 = \left[(q_5')^2 + (q_6')^2\right]\left[(q_7')^2 + (q_8')^2\right] - \left(q_5' q_7' + q_6' q_8'\right)^2,$$

$$(3.26)_2 \qquad \left(q_5' q_9' + q_6' q_{10}'\right)^2 = \left[(q_5')^2 + (q_6')^2\right]\left[(q_9')^2 + (q_{10}')^2\right] - \left(q_5' q_{10}' - q_6' q_9'\right)^2,$$

$$(3.26)_3 \qquad \left(q_7' q_9' + q_8' q_{10}'\right)^2 = \left[(q_7')^2 + (q_8')^2\right]\left[(q_9')^2 + (q_{10}')^2\right] - \left(q_7' q_{10}' - q_8' q_9'\right)^2.$$

In the sequel we take six invariants on the right-hand sides of (3.26) as independent and we observe the invariance of

$$(3.27) \qquad (q_5' + q_7' + q_{10}')^2 + (q_6' + q_8' - q_9')^2.$$

3.3. Summary of the procedure for determining the translation supertensor

We satisfy the requirements of quasi-affinity and symmetry of $T(\mathbf{\Phi})$ by applying (3.14) and (3.15) to (3.8). Next, we calculate $T(\mathbf{\Phi})$ for $\mathbf{f}_{(i)}$, $i = 1, 2, 3$, given by (3.16) and we reshape the result in terms of a 12×1 vector $\mathcal{F}(\mathbf{\Phi})$ divided into 5 following components

$$\mathcal{F}(\mathbf{\Phi}) = \big((q_1, \dots, q_4), (q_5, q_7, q_{10}), (q_6, q_8, -q_9), (q_{11}), (q_{12})\big)^T, \tag{3.28}$$

thus obtaining

$$T(\mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})^T\, \mathcal{G}(\mathbf{T})\, \mathcal{F}(\mathbf{\Phi}), \tag{3.29}$$

where $\mathcal{G}(\mathbf{T})$ denotes a matrix whose components conform with the requirement of isotropy, i.e. all components linking q_i, q_j, $i, j = 1, \dots, 12$, from different subvectors in (3.28) are equal to zero. We obtain $\mathcal{G}(\mathbf{T})$ in a block matrix form

$$\mathcal{G}(\mathbf{T}) = \begin{pmatrix} \mathcal{A}(\mathbf{T}) & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathcal{B}(\mathbf{T}) & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathcal{B}(\mathbf{T}) & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}(\mathbf{T}) & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}(\mathbf{T}) \end{pmatrix}, \tag{3.30}$$

where matrices

$$\begin{aligned} \mathcal{A}(\mathbf{T}) &= \begin{pmatrix} -t_1 & -2\,t_3 & 0 & 0 \\ & 2\,t_2 - 2\,t_4 & 0 & 0 \\ & & 2\,t_2 - 2\,t_4 & -2\,t_3 \\ & sym & & -t_1 \end{pmatrix}, \\ \mathcal{B}(\mathbf{T}) &= \begin{pmatrix} t_1 - t_2 + 2\,t_3 & t_1 + t_2 & -t_3 - t_4 \\ & t_1 - t_2 - 2\,t_3 & -t_3 + t_4 \\ sym & & -t_2 \end{pmatrix}, \\ \mathcal{C}(\mathbf{T}) &= 2\,t_2 + 2\,t_4 \end{aligned} \tag{3.31}$$

depend on 4 free parameters, which have to be optimally adjusted during the calculations of estimates for the effective constitutive properties. The non-zero components of matrices $\boldsymbol{T}^{(ij)}$ in (3.7), see also (3.14), are thus given by

$$\begin{aligned} &t_1^{(11)} = t_1, && t_2^{(12)} = t_3^{(13)} = -t_2^{(21)} = -t_3^{(31)} = t_3, \\ &t_1^{(22)} = t_1^{(33)} = t_2, && t_4^{(23)} = -t_4^{(32)} = t_4. \end{aligned} \tag{3.32}$$

Choosing different invariants in $i_{2\varphi}$ as independent would obviously result in different representation of matrices in (3.31). However, in such cases the number of free parameters in their description would decrease to 2, hence imposing redundant constraints in calculations of estimates.

3.4. Algorithm of the effective properties estimation routine

Taking

$$\boldsymbol{\Delta}_\alpha = \underline{\boldsymbol{\Delta}_\alpha - \mathbf{T}} + \mathbf{T} \tag{3.33}$$

and estimating the underlined term from below by harmonic mean, yields the translated constitutive supertensor

$$\boldsymbol{\Delta}_t = \langle (\boldsymbol{\Delta} - \mathbf{T})^{-1} \rangle^{-1} + \mathbf{T}. \tag{3.34}$$

Denoting an effective constitutive supertensor by

$$\boldsymbol{\Delta}_h = \boldsymbol{C}^h_{(1)} \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{C}^h_{(2)} \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{C}^h_{(3)} \otimes \mathbf{E}_3 \otimes \mathbf{E}_3, \tag{3.35}$$

we express the energy accumulated in a particle of a composite plate as

$$U(\boldsymbol{\Phi}_0, \boldsymbol{\Delta}_h) \geq U(\boldsymbol{\Phi}_0, \boldsymbol{\Delta}_t), \tag{3.36}$$

see (3.5), where $\boldsymbol{\Phi}_0 \in V_\Phi$ is a 9×1 supervector whose components are given in the form of (3.16) with $q_4 = q_9 = q_{10} = 0$, as they are symmetric tensors from $\mathbb{E}$. Translated constitutive supertensors are admissible in (3.34) if for any $\mathbf{T}$ they retain their symmetry and semi-positiveness, that is

$$\boldsymbol{\Delta}_\alpha - \mathbf{T} \geq \mathbf{0}, \qquad \alpha = 1, 2. \tag{3.37}$$

Arbitrariness of $\boldsymbol{\Phi}_0$ clearly forces

$$\boldsymbol{\Delta}_h \geq \langle (\boldsymbol{\Delta} - \mathbf{T})^{-1} \rangle^{-1} + \mathbf{T}, \tag{3.38}$$

hence determining the translation bound on the effective supertensor $\boldsymbol{\Delta}_h$.

Analogously to Sec. 3.3, formula (3.37) can be transformed into a system

$$\mathcal{A}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) \geq \mathbf{0}, \qquad \mathcal{B}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) \geq \mathbf{0}, \qquad \mathcal{C}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) \geq 0. \tag{3.39}$$

Indeed, for given $\alpha = 1$ or 2 we first calculate $U(\boldsymbol{\Phi}, \boldsymbol{\Delta}_\alpha)$ by (3.4) with $\mathbf{f}_{(i)}$, $i = 1, 2, 3$, and $\boldsymbol{\Delta}_\alpha$ set by (3.16) and (3.2), respectively. Next, we reshape this obtained result in terms of $\mathcal{F}(\boldsymbol{\Phi})$, see (3.28), hence determining

$$U(\boldsymbol{\Phi}, \boldsymbol{\Delta}_\alpha) = \mathcal{F}(\boldsymbol{\Phi})^T \, \mathcal{G}(\boldsymbol{\Delta}_\alpha) \, \mathcal{F}(\boldsymbol{\Phi}), \tag{3.40}$$

where $\mathcal{G}(\boldsymbol{\Delta}_\alpha)$ is a block-diagonal matrix with components $\mathcal{A}(\boldsymbol{\Delta}_\alpha)$, $\mathcal{B}(\boldsymbol{\Delta}_\alpha)$ and $\mathcal{C}(\boldsymbol{\Delta}_\alpha)$. Next, we conclude that

$$U(\boldsymbol{\Phi}, \boldsymbol{\Delta}_\alpha) - T(\boldsymbol{\Phi}) = \mathcal{F}^T(\boldsymbol{\Phi}) \, \mathcal{G}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) \, \mathcal{F}(\boldsymbol{\Phi}) > 0 \tag{3.41}$$

and this inequality has to be valid for any choice of $\mathbf{\Phi}$, hence establishing (3.39). However, it is obvious from the comparison of (2.5) and (3.16) that if any of the fields in $\mathbf{\Phi}$ is symmetric, then $q_4 = 0$ and/or $q_9 = 0$ and/or $q_{10} = 0$. If this is the case then we delete appropriate rows and columns from the representation of $\mathcal{G}(\mathbf{\Delta}_\alpha - \mathbf{T})$. By the observation made in Cherkaev [1, Ch. 8.3] and Milton [11, Ch. 25.2], the optimal translation parameters in $\mathbf{T}$ retain non-degeneracy of matrices in (3.39) and they are extremal, i.e. such that

$$\sum_{\alpha=1}^{2} \big(\operatorname{rank}(\mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T})) + \operatorname{rank}(\mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})) + \operatorname{rank}(\mathcal{C}(\mathbf{\Delta}_\alpha - \mathbf{T}))\big) \tag{3.42}$$

achieves its minimal value.

We will also make use of Y-transformation of a tensor, see e.g. Cherkaev [1, Ch. 16.2] and Milton [11, Ch. 24.10]. The Y-transformed estimates of $\mathbf{\Delta}_h$ do not depend on volume fractions of materials in ω and (3.38) assumes a simple form

$$Y\left(\mathbf{\Delta}_h\right) + \mathbf{T} \geq \mathbf{0}, \tag{3.43}$$

valid however only if the tensor $\mathbf{\Delta}_1 - \mathbf{\Delta}_2$ is nonsingular. To deal with this incovenience we follow Cherkaev and Gibiansky [2] in introducing matrices $\mathbf{P}$ and $\mathbf{Q}$ as projectors on the non-degenerated and degenerated subspaces of $\mathbf{\Delta}_1 - \mathbf{\Delta}_2$ respectively and rewriting (3.43) in the form

$$Y_P(\mathbf{\Delta}_h) \geq \mathbf{0}, \tag*{(3.44)$_1$}$$

where

$$Y_P(\mathbf{\Delta}_h) = Y(\mathbf{P}\,\mathbf{\Delta}_h\,\mathbf{P}^T) + \mathbf{P}\,\mathbf{T}\,\mathbf{P}^T - (\mathbf{P}\,\mathbf{T}\,\mathbf{Q}^T)(\mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T)^{-1}(\mathbf{Q}\,\mathbf{T}\,\mathbf{P}^T). \tag*{(3.44)$_2$}$$

Full construction of ((3.44)$_2$) is omitted here and we refer the Reader to Cherkaev and Gibiansky [2] for detailed exposition of this formula. Let us only mention that shapes of matrices $\mathbf{P}$ and $\mathbf{Q}$ depend on the fields defining the supervector $\mathbf{\Phi}$. Influence of this fact on the calculations is clarified in Section 4.

For isotropic $\mathbf{\Delta}_h$, the Y-transformation can be applied directly to scalar components of the effective tensor. The following three equalities:

$$\begin{aligned}
y(x_h, x_1, x_2, m_1, m_2) &= \frac{x_h - \langle x^{-1}\rangle^{-1}}{\langle x\rangle - x_h}(m_2 x_1 + m_1 x_2), \\
y(x_h, x_1, x_2, m_1, m_2) &= \frac{1}{y\left(x_h^{-1}, x_1^{-1}, x_2^{-1}, m_1, m_2\right)}, \\
y(a\,x_h, a\,x_1, a\,x_2, m_1, m_2) &= a\,y(x_h, x_1, x_2, m_1, m_2),
\end{aligned} \tag{3.45}$$

see (2.12) for $\langle\cdot\rangle$, hold true for any $a \in \mathbb{R}$ and for x_h being the effective modulus of isotropy. In the sequel we write $y(x_h)$ for short when no confusion can arise.

Formula $((3.44)_1)$ is easily transformable to the form

$$\mathcal{A}(Y_P(\boldsymbol{\Delta}_h)) \geq \mathbf{0}, \qquad \mathcal{B}(Y_P(\boldsymbol{\Delta}_h)) \geq \mathbf{0}, \qquad \mathcal{C}(Y_P(\boldsymbol{\Delta}_h)) \geq 0 \tag{3.46}$$

by the procedure similar to the one developed in the prequel for (3.37). Let us assume that the projectors $\mathbf{P}$ and $\mathbf{Q}$ are given and optimal translation parameters, denoted by t_i^*, $i = 1, \ldots, 4$, are calculated with help of (3.42), hence defining (3.7) through (3.32). This, in turn, allows for calculating of the projections of $\boldsymbol{\Delta}_h$ and $\mathbf{T}$ in $(3.44)_2$. Obviously, the dimensions of the non-reduced supertensors conform with those of $\boldsymbol{\Delta}_\alpha$. Then, for $\mathbf{f}^0_{(i)}$, $i = 1, 2, 3$, given by (3.16), we calculate $U(\boldsymbol{\Phi}_0, Y_P(\boldsymbol{\Delta}_h))$ by (3.4) and we reshape the result in terms of $\mathcal{F}(\boldsymbol{\Phi})$, see (3.28), hence establishing (3.46). Examination of the determinants of these matrices will lead to the most restrictive bounds on effective properties of a composite.

4. Calculating bounds on effective isotropic moduli

4.1. Summary of known bounds

The estimation method outlined in Sec. 3 is general, therefore capable of predicting bounds on any anisotropic effective tensor. However, in the remainder we restrict our concern to the simplest isotropic case. For this purpose, we take into consideration that the response of a composite material to any deviatoric field has to be identical, hence we limit the class of admissible supervectors to

$$\boldsymbol{\Phi} = \mathbf{f}^{(H)} \otimes \mathbf{E}_1 + \mathbf{f}^{(D)} \otimes \mathbf{E}_2 + \mathbf{f}^{(D)} \otimes \mathbf{E}_3, \tag{4.1}$$

where $\mathbf{f}^{(H)}$ and $\mathbf{f}^{(D)}$ respectively denote hydrostatic and deviatoric ω-periodic fields. The well-known HSW uncoupled bounds on the bulk and shear moduli are obtained by estimating the energy in specific directions. Indeed, taking

$$\boldsymbol{\Phi}_0 = \boldsymbol{\kappa}_0^{(H)} \otimes \mathbf{E}_1 \quad \text{and} \quad \boldsymbol{\Phi}_0 = \boldsymbol{\sigma}_0^{(H)} \otimes \mathbf{E}_1 \tag{4.2}$$

respectively, leads to the lower and upper HSW bounds on k_h. The effective shear modulus μ_h is estimated by exposing the composite to the simultaneous action of two deviatoric fields of the same type , i.e.

$$\boldsymbol{\Phi}_0 = \boldsymbol{\kappa}_0^{(D)} \otimes \mathbf{E}_2 + \boldsymbol{\kappa}_0^{(D)} \otimes \mathbf{E}_3 \quad \text{and} \quad \boldsymbol{\Phi}_0 = \boldsymbol{\sigma}_0^{(D)} \otimes \mathbf{E}_2 + \boldsymbol{\sigma}_0^{(D)} \otimes \mathbf{E}_3. \tag{4.3}$$

The ranges of Y-transformed effective moduli of an isotropic composite plate in bending were found in LEWIŃSKI and TELEGA [7, Ch. 23.4]. They are given by

$$y(k_h) \in [\mu_2, \mu_1]\,, \qquad y(\mu_h) \in [2\,k_{\min} + \mu_2, 2\,k_{\max} + \mu_1]\,, \tag{4.4}$$

where $k_{\min} = \min\{k_1, k_2\}$, $k_{\max} = \max\{k_1, k_2\}$. Taking

$$k_1 \geq k_2, \qquad \mu_1 \geq \mu_2, \qquad y(\mu_h) \in [2\,k_2 + \mu_2, 2\,k_1 + \mu_1]\,, \tag{4.5}$$

in a "well-ordered materials" case leads to the Hashin–Shtrikman estimates for $y(\mu_h)$ in (4.4). In the sequel, these bounds are referred to as $y(\mu_h) \in [HS_l, HS_u]$ whereas for "badly-ordered materials" we assume without any loss of generality that

$$k_1 \leq k_2, \qquad \mu_1 \geq \mu_2, \qquad y(\mu_h) \in [2\,k_1 + \mu_2, 2\,k_2 + \mu_1]\,, \tag{4.6}$$

which gives $y(\mu_h) \in [W_l, W_u]$ denoting the Walpole bounds in (4.4).

Placing no restrictions on the direction of searching for extremal values of energy potentials in (3.4), leads to coupled estimates on pairs (k_h, μ_h) of an isotropic composite, as will be shown in the sequel.

4.2. Coupled estimates for "well-ordered materials"

4.2.1. The upper bound Let us consider a supervector

$$\mathbf{\Phi} = \boldsymbol{\kappa} \otimes \mathbf{E}_1 + \boldsymbol{\sigma} \otimes \mathbf{E}_2 + \boldsymbol{\sigma} \otimes \mathbf{E}_3, \tag{4.7}$$

and the corresponding constitutive supertensors of basic materials ($\alpha = 1, 2$):

$$\mathbf{\Delta}_\alpha = \boldsymbol{D}_\alpha \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{d}_\alpha \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{d}_\alpha \otimes \mathbf{E}_3 \otimes \mathbf{E}_3. \tag{4.8}$$

We note that (4.7) corresponds to $q_4 = 0$ in (3.16), as $\boldsymbol{\kappa}$ is a symmetric field. Procedure described in Sec. 3.4 yields the following form of matrices in (3.39):

$$(4.9)_1 \quad \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T}) = \begin{pmatrix} 2\,k_\alpha + t_1 & 2\,t_3 & 0 \\ & \dfrac{1}{\mu_\alpha} - 2t_2 + 2t_4 & 0 \\ sym & & \dfrac{1}{\mu_\alpha} - 2t_2 + 2t_4 \end{pmatrix},$$

$$(4.9)_2 \quad \mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})$$
$$= \begin{pmatrix} 2\mu_\alpha + \dfrac{1}{2k_\alpha} - t_1 + t_2 - 2t_3 & 2\,\mu_\alpha - \dfrac{1}{2k_\alpha} - t_1 - t_2 & t_3 + t_4 \\ & 2\mu_\alpha + \dfrac{1}{2k_\alpha} - t_1 + t_2 + 2t_3 & t_3 - t_4 \\ sym & & t_2 \end{pmatrix},$$

$$(4.9)_3 \quad \mathcal{C}(\mathbf{\Delta}_\alpha - \mathbf{T}) = \frac{1}{\mu_\alpha} - 2t_2 - 2t_4.$$

Next, we attempt to fulfill the requirement set in (3.42) by assuming

$$t_4 = \frac{1}{2\mu_1} - t_2. \tag{4.10}$$

This gives $\mathcal{C}(\mathbf{\Delta}_1 - \mathbf{T}) = 0$ and $\mathcal{C}(\mathbf{\Delta}_2 - \mathbf{T}) > 0$ in (4.9_3) and we see that $\det \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T})$ and $\det \mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})$ are expressed by $(\mathrm{A}.1)_1$ and (A.2). Since $t_3 \in \mathbb{R}$, then inequalities $g_2(\mathcal{H}_\alpha, t_1, t_2) \geq 0$ restrict the domain of pairs (t_1, t_2) to

$$S_1 = \left\{ (t_1, t_2) : (t_1, t_2) \in [-2k_2, 2\mu_2] \times \left[\frac{k_1}{2\mu_1(\mu_1 + 2\,k_1)}, \frac{1}{2\mu_1} \right] \right\}. \tag{4.11}$$

We check by inspection that $g_1(\mathcal{H}_\alpha, t_1, t_2) > 0$ for any $(t_1, t_2) \in S_1$ hence we limit our search to those translations whose parameters (t_1, t_2, t_3) correspond to

$$g_2(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0. \tag{4.12}$$

This task can be reduced to the following algorithm based on a simple geometrical interpretation: examine each intersection of all planes defined as

$$\pi_2(\mathcal{H}_\alpha) : S_1 \mapsto g_2(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0. \tag{4.13}$$

Straightforward calculations lead to the conclusion that the set of mappings $\{\pi_2(\mathcal{A}_2), \pi_2(\mathcal{B}_1), \pi_2(\mathcal{B}_2)\}$ yield optimal values of translation parameters

$$\begin{aligned}
t_1^* &= -2\frac{(k_1\mu_2 HS_l - k_2\mu_1 HS_u)W_u + (\mu_2)^2\mu_1(k_1 - k_2)}{(\mu_2 HS_l - \mu_1 HS_u)W_u + 2k_2\mu_2(k_1 - k_2)}, \\
t_2^* &= \frac{2k_1\mu_1 W_u - 2k_2\mu_2 HS_u - \mu_1(k_1 - k_2)t_1^*}{4\mu_1(\mu_1 - \mu_2)HS_u W_u}, \\
t_3^* &= \sqrt{(2k_2 + t_1^*)\left(\frac{1}{4\mu_2} + \frac{1}{4\mu_1} - t_2^*\right)}, \\
t_4^* &= \frac{1}{2\mu_1} - t_2^*.
\end{aligned} \tag{4.14}$$

Matrix $\mathbf{\Delta}_1 - \mathbf{\Delta}_2$ has two zero eigenvalues, hence we introduce the projectors

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P} \end{pmatrix}_{9\times 11}, \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{0} & \boldsymbol{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{Q} \end{pmatrix}_{2\times 11}, \tag{4.15_1}$$

where $\boldsymbol{I}$ denotes a 3×3 unity matrix and

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.15_2}$$

Next we apply the algorithm defining matrices in (3.46) thus obtaining

$$(4.16)_1 \quad \mathcal{A}(Y_P(\mathbf{\Delta}_h)) = \begin{pmatrix} 2y(k_h) - t_1^* & -2t_3^* & 0 \\ & \dfrac{1}{y(\mu_h)} + 2t_2^* - 2t_4^* & 0 \\ sym & & \dfrac{1}{y(\mu_h)} + 2t_2^* - 2t_4^* \end{pmatrix},$$

$$(4.16)_2 \quad \mathcal{B}(Y_P(\mathbf{\Delta}_h)) = \begin{pmatrix} \begin{array}{l} 2y(\mu_h) + \dfrac{1}{2y(k_h)} + \\ \quad + t_1^* - t_2^* + 2t_3^* + \dfrac{(t_3^* + t_4^*)^2}{t_2^*} \end{array} & \begin{array}{l} 2\,y(\mu_h) - \dfrac{1}{2\,y(k_h)} + \\ \quad + t_1^* + t_2^* + \dfrac{(t_3^*)^2 - (t_4^*)^2}{t_2^*} \end{array} \\ sym & \begin{array}{l} 2y(\mu_h) + \dfrac{1}{2y(k_h)} + \\ \quad + t_1^* - t_2^* - 2t_3^* + \dfrac{(t_3^* - t_4^*)^2}{t_2^*} \end{array} \end{pmatrix},$$

$$(4.16)_3 \quad \mathcal{C}(Y_P(\mathbf{\Delta}_h)) = \frac{1}{y(\mu_h)} + 2t_2^* + 2t_4^*.$$

The most restrictive upper coupled estimate on pairs $(y(k_h), y(\mu_h))$ is given by

$$\det \mathcal{A}(Y_P(\mathbf{\Delta}_h)) = 0 \tag{4.17}$$

and reads

$$y(k_h) \in [\mu_2, \mu_1], \qquad y(\mu_h) \le h_1(y(k_h)), \tag{4.18}$$

where

$$h_1(y(k_h)) = \frac{1}{2} \frac{2y(k_h) - t_1^*}{(2\,y(k_h) - t_1^*)(t_4^* - t_2^*) + 2(t_3^*)^2}. \tag{4.19}$$

4.2.2. The lower bound. Next, let us consider a supervector

$$\mathbf{\Phi} = \boldsymbol{\sigma} \otimes \mathbf{E}_1 + \boldsymbol{\kappa} \otimes \mathbf{E}_2 + \boldsymbol{\kappa} \otimes \mathbf{E}_3, \tag{4.20}$$

and the corresponding constitutive super-tensors of basic materials $(\alpha = 1, 2)$

$$\mathbf{\Delta}_\alpha = \boldsymbol{d}_\alpha \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{D}_\alpha \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{D}_\alpha \otimes \mathbf{E}_3 \otimes \mathbf{E}_3. \tag{4.21}$$

Field $\boldsymbol{\kappa}$ is symmetric, hence $q_9 = q_{10} = 0$ in (3.16). Matrices in (3.39) take a form

$$(4.22)_1 \quad \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T}) = \begin{pmatrix} \dfrac{1}{2k_\alpha} + t_1 & 2t_3 & 0 & 0 \\ & 4\mu_\alpha - 2t_2 + 2t_4 & 0 & 0 \\ & & 4\mu_\alpha - 2t_2 + 2t_4 & 2\,t_3 \\ & sym & & t_1 \end{pmatrix},$$

$(4.22)_2$
$$\mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T}) = \begin{pmatrix} \dfrac{1}{2\mu_\alpha} + 2k_\alpha - t_1 + t_2 - 2t_3 & \dfrac{1}{2\mu_\alpha} - 2k_\alpha - t_1 - t_2 \\ sym & \dfrac{1}{2\mu_\alpha} + 2k_\alpha - t_1 + t_2 + 2t_3 \end{pmatrix},$$

$(4.22)_3$
$$\mathcal{C}(\mathbf{\Delta}_\alpha - \mathbf{T}) = 4\mu_\alpha - 2t_2 - 2t_4.$$

By setting

$$t_4 = 2\mu_2 - t_2, \tag{4.23}$$

we obtain $\det \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T})$ and $\det \mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})$ through $(A.1)_1$, $(A.1)_2$ and (A.3). Similarly to Section 4.2.1., we restrict the range of (t_1, t_2) to

$$S_2 = \left\{ (t_1, t_2) : (t_1, t_2) \in \left[0, \frac{1}{2\,\mu_1}\right] \times [-2\,k_2, 2\,\mu_2] \right\}. \tag{4.24}$$

We next examine all intersections of planes

$$\begin{aligned} \pi_1(\mathcal{H}_\alpha) &: S_2 \mapsto g_1(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0, \\ \pi_2(\mathcal{H}_\alpha) &: S_2 \mapsto g_2(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0, \end{aligned} \tag{4.25}$$

hence the set of mappings $\{\pi_1(\mathcal{A}_2), \pi_2(\mathcal{B}_1), \pi_2(\mathcal{B}_2)\}$ yield

$$\begin{aligned} t_1^* &= \frac{1}{2}\frac{k_1 - k_2}{(k_1 + \mu_2)\mu_1 - (k_2 + \mu_2)\mu_2}, \\ t_2^* &= 2\frac{k_1\mu_2 - k_2\,\mu_1 - 2\mu_1\mu_2(k_1 - k_2)t_1^*}{\mu_1 - \mu_2}, \\ t_3^* &= \sqrt{t_1^*(2\mu_2 - t_2^*)}, \\ t_4^* &= 2\mu_2 - t_2^*. \end{aligned} \tag{4.26}$$

Matrix $\mathbf{\Delta}_1 - \mathbf{\Delta}_2$ has one zero eigenvalue, hence the projectors on the subspace of non-degenerate matrices are given by $(4.15)_2$ and

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{pmatrix}_{9\times 10}, \quad \mathbf{Q} = \begin{pmatrix} \boldsymbol{Q} & \mathbf{0} & \mathbf{0} \end{pmatrix}_{1\times 10}. \tag{4.27}$$

The procedure put forward in Sec. 3.4 leads to (3.46) with matrices

$$(4.28)_1 \quad \mathcal{A}(Y_P(\boldsymbol{\Delta}_h)) = \begin{pmatrix} \dfrac{1}{2y(k_h)} - t_1^* & -2\,t_3^* & 0 \\ & 4y(\mu_h) + 2t_2^* - 2t_4^* & 0 \\ sym & & 4y(\mu_h) + 2t_2^* - 2t_4^* + \dfrac{4(t_3^*)^2}{t_1^*} \end{pmatrix},$$

$$(4.28)_2 \quad \mathcal{B}(Y_P(\boldsymbol{\Delta}_h, \mathbf{T})) = \begin{pmatrix} \dfrac{1}{2y(\mu_h)} + 2y(k_h) + t_1^* - t_2^* + 2t_3^* & \dfrac{1}{2y(\mu_h)} - 2y(k_h) + t_1^* + t_2^* \\ sym & \dfrac{1}{2y(\mu_h)} + 2y(k_h) + t_1^* - t_2^* - 2t_3^* \end{pmatrix},$$

$$(4.28)_3 \quad \mathcal{C}(Y_P(\boldsymbol{\Delta}_h, \mathbf{T})) = 4y(\mu_h) + 2t_2^* + 2t_4^*.$$

The most restrictive lower coupled estimate on pairs $(y(k_h), y(\mu_h))$ is due to

$$(4.29) \qquad \det \mathcal{A}(Y_P(\boldsymbol{\Delta}_h)) = 0$$

and reads

$$(4.30) \qquad y(k_h) \in [\mu_2, \mu_1], \qquad h_2(y(k_h)) \le y(\mu_h),$$

where

$$(4.31) \qquad h_2(y(k_h)) = \frac{1}{2}\frac{\left(\dfrac{1}{2y(k_h)} - t_1^*\right)(t_4^* - t_2^*) + 2t_3^*}{\dfrac{1}{2\,y(k_h)} - t_1^*}.$$

New estimates attain HSW points, see Fig. 1, that is

$$(4.32) \qquad \begin{aligned} h_1(\mu_2) &= 2k_2 + \mu_1, \qquad & h_1(\mu_1) &= 2k_1 + \mu_1, \\ h_2(\mu_2) &= 2k_2 + \mu_2, \qquad & h_2(\mu_1) &= 2k_1 + \mu_2 \end{aligned}$$

and it is worth pointing out that the ordering of HSW estimates strongly depend on k_α, μ_α, $\alpha = 1, 2$. For chosen characteristics of the constituent materials, $k_1 = 3$, $\mu_1 = 3$, $k_2 = 1$, $\mu_2 = 1$, order of the Walpole estimates, see (4.6), is inversed.

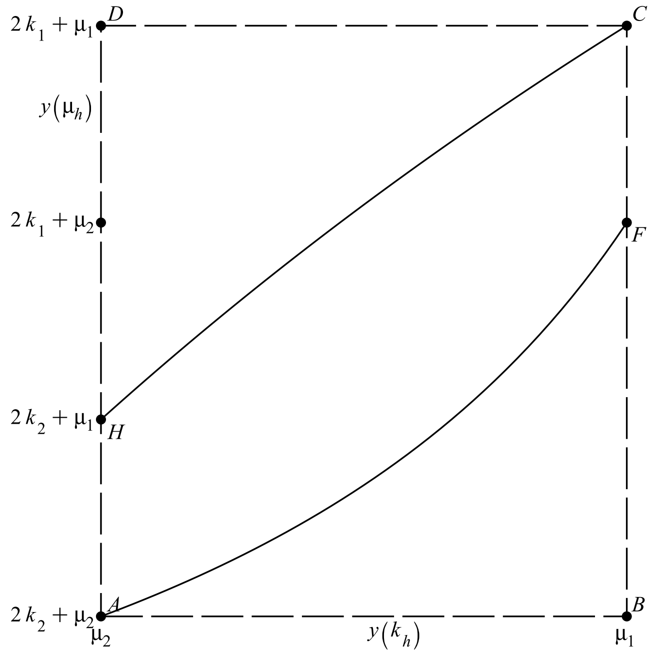

Fig. 1. "Well-ordered materials", $k_1 = 3$, $\mu_1 = 3$, $k_2 = 1$, $\mu_2 = 1$. Coupled estimates on effective stiffnesses $(y(k_h), y(\mu_h))$ are shown by solid lines H-C and A-F; the Hashin–Shtrikman rectangle – by dashed line A-B-C-D; the Walpole points correspond to H, F.

4.3. Coupled estimates for "badly-ordered materials"

4.3.1. The upper bound In case of "badly-ordered materials", see (4.6), we first consider a supervector

$$\boldsymbol{\Phi} = \boldsymbol{\sigma} \otimes \mathbf{E}_1 + \boldsymbol{\sigma} \otimes \mathbf{E}_2 + \boldsymbol{\sigma} \otimes \mathbf{E}_3, \tag{4.33}$$

and corresponding constitutive supertensors of basic materials $(\alpha = 1, 2)$

$$\boldsymbol{\Delta}_\alpha = \boldsymbol{d}_\alpha \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{d}_\alpha \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{d}_\alpha \otimes \mathbf{E}_3 \otimes \mathbf{E}_3. \tag{4.34}$$

Using notation (3.39) we obtain

$$(4.35)_1 \qquad \mathcal{A}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) = \begin{pmatrix} \dfrac{1}{2k_\alpha} + t_1 & 2t_3 & 0 & 0 \\ & \dfrac{1}{\mu_\alpha} - 2t_2 + 2t_4 & 0 & 0 \\ & & \dfrac{1}{\mu_\alpha} - 2t_2 + 2t_4 & 2t_3 \\ sym & & & t_1 \end{pmatrix},$$

(4.35)$_2$ $\mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T}) =$

$$\begin{pmatrix} \dfrac{1}{2\mu_\alpha} + \dfrac{1}{2k_\alpha} - t_1 + t_2 - 2t_3 & \dfrac{1}{2\mu_\alpha} - \dfrac{1}{2k_\alpha} - t_1 - t_2 & t_3 + t_4 \\ & \dfrac{1}{2\mu_\alpha} + \dfrac{1}{2k_\alpha} - t_1 + t_2 + 2t_3 & t_3 - t_4 \\ sym & & t_2 \end{pmatrix},$$

(4.35)$_3$ $$\mathcal{C}(\mathbf{\Delta}_\alpha - \mathbf{T}) = \frac{1}{\mu_\alpha} - 2t_2 - 2t_4.$$

By setting

$$t_4 = \frac{1}{2\mu_1} - t_2 \tag{4.36}$$

we obtain $\det \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T})$ and $\det \mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})$ through (A.1)$_1$, (A.1$_2$) and (A.4). By restricting the range of (t_1, t_2) to

$$S_3 = \left\{(t_1, t_2) : (t_1, t_2) \in \left[0, \frac{1}{2\,\mu_1}\right] \times \left[\frac{k_2}{2\,\mu_1\,(\mu_1 + 2\,k_2)}, \frac{1}{2\,\mu_1}\right]\right\} \tag{4.37}$$

and examining all intersections of planes

$$\begin{aligned} &\pi_1(\mathcal{H}_\alpha) : S_3 \mapsto g_1(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0, \\ &\pi_2(\mathcal{H}_\alpha) : S_3 \mapsto g_2(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0, \end{aligned} \tag{4.38}$$

we conclude that the set of mappings $\{\pi_1(\mathcal{A}_1), \pi_2(\mathcal{B}_1), \pi_2(\mathcal{B}_2)\}$ yield

$$\begin{aligned} t_1^* &= \frac{1}{2}\frac{(k_2 - k_1)\mu_1}{2\mu_1(\mu_1 + k_1)W_u - 2\mu_2(\mu_1 + k_2)HS_u}, \\ t_2^* &= \frac{k_2\mu_1 HS_u - k_1\mu_2 W_u - 2(\mu_1)^2\mu_2(k_2 - k_1)t_1^*}{2\mu_1(\mu_1 - \mu_2)HS_u W_u}, \\ t_3^* &= \sqrt{t_1^*\left(\frac{1}{2\,\mu_1} - t_2^*\right)}, \\ t_4^* &= \frac{1}{2\mu_1} - t_2^*. \end{aligned} \tag{4.39}$$

Matrix $\mathbf{\Delta}_1 - \mathbf{\Delta}_2$ has three zero eigenvalues, so projectors are due to (4.15)$_2$ and

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P} \end{pmatrix}_{9\times 12}, \qquad \mathbf{Q} = \begin{pmatrix} \boldsymbol{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{Q} \end{pmatrix}_{3\times 12}. \tag{4.40}$$

The procedure put forward in Sec. 3.4 leads to (3.46) with matrices

$$(4.41)_1 \quad \mathcal{A}(Y_P(\boldsymbol{\Delta}_h))$$

$$(4.41)_2 \quad = \begin{pmatrix} \dfrac{1}{2y(k_h)} - t_1^* & -2t_3^* & 0 \\ & \dfrac{1}{y(\mu_h)} + 2t_2^* - 2t_4^* & 0 \\ sym & & \dfrac{1}{y(\mu_h)} + 2t_2^* + \dfrac{4(t_3^*)^2}{t_1^*} - 2t_4^* \end{pmatrix},$$

$$(4.41)_3 \quad \mathcal{B}(Y_P(\boldsymbol{\Delta}_h))$$

$$(4.41)_4 \quad = \begin{pmatrix} \begin{array}{c} \dfrac{1}{2\,y(\mu_h)} + \dfrac{1}{2\,y(k_h)} + \\ +t_1^* - t_2^* + 2\,t_3^* + \dfrac{(t_3^*+t_4^*)^2}{t_2^*} \end{array} & \begin{array}{c} \dfrac{1}{2\,y(\mu_h)} - \dfrac{1}{2\,y(k_h)} + \\ +t_1^* + t_2^* + \dfrac{(t_3^*)^2-(t_4^*)^2}{t_2^*} \end{array} \\ sym & \begin{array}{c} \dfrac{1}{2\,y(\mu_h)} + \dfrac{1}{2\,y(k_h)} + \\ +t_1^* - t_2^* - 2\,t_3^* + \dfrac{(t_3^*-t_4^*)^2}{t_2^*} \end{array} \end{pmatrix},$$

$$(4.41)_5 \quad \mathcal{C}(Y_P(\boldsymbol{\Delta}_h)) = \frac{1}{y(\mu_h)} + 2\,t_2^* + 2\,t_4^*.$$

The most restrictive upper estimate on pairs $(y(k_h), y(\mu_h))$ is determined by

$$(4.42) \qquad \det \mathcal{A}(Y_P(\boldsymbol{\Delta}_h)) = 0$$

and reads

$$(4.43) \qquad y(k_h) \in [\mu_2, \mu_1]\,, \qquad y(\mu_h) \le h_3\,(y(k_h))\,,$$

where

$$(4.44) \qquad h_3\,(y(k_h)) = \frac{1}{2}\,\frac{\dfrac{1}{2\,y(k_h)} - t_1^*}{\left(\dfrac{1}{2\,y(k_h)} - t_1^*\right)(t_4^* - t_2^*) + 2\,(t_3^*)^2}.$$

4.3.2. The lower bound Finally, let us take a supervector

$$(4.45) \qquad \boldsymbol{\Phi} = \boldsymbol{\kappa} \otimes \mathbf{E}_1 + \boldsymbol{\kappa} \otimes \mathbf{E}_2 + \boldsymbol{\kappa} \otimes \mathbf{E}_3,$$

and the corresponding constitutive super-tensors of basic materials ($\alpha = 1, 2$)

$$(4.46) \qquad \boldsymbol{\Delta}_\alpha = \boldsymbol{D}_\alpha \otimes \mathbf{E}_1 \otimes \mathbf{E}_1 + \boldsymbol{D}_\alpha \otimes \mathbf{E}_2 \otimes \mathbf{E}_2 + \boldsymbol{D}_\alpha \otimes \mathbf{E}_3 \otimes \mathbf{E}_3.$$

Next, we observe that $q_4 = q_9 = q_{10} = 0$ in (3.16), because $\boldsymbol{\kappa}$ is a symmetric field. Using notation (3.39) we obtain

$$
\begin{aligned}
&\mathcal{A}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) = \begin{pmatrix} 2k_\alpha + t_1 & 2t_3 & 0 \\ & 4\mu_\alpha - 2t_2 + 2t_4 & 0 \\ sym & & 4\mu_\alpha - 2t_2 + 2t_4 \end{pmatrix}, \\
&\mathcal{B}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) = \\
&\qquad \begin{pmatrix} 2\mu_\alpha + 2k_\alpha - t_1 + t_2 - 2t_3 & 2\mu_\alpha - 2k_\alpha - t_1 - t_2 \\ sym & 2\mu_\alpha + 2k_\alpha - t_1 + t_2 + 2t_3 \end{pmatrix}, \\
&\mathcal{C}(\boldsymbol{\Delta}_\alpha - \mathbf{T}) = 4\mu_\alpha - 2t_2 - 2t_4.
\end{aligned}
\tag{4.47}
$$

By setting

$$t_4 = 2\mu_2 - t_2 \tag{4.48}$$

we obtain $\det \mathcal{A}(\boldsymbol{\Delta}_\alpha - \mathbf{T})$ and $\det \mathcal{B}(\boldsymbol{\Delta}_\alpha - \mathbf{T})$ through $(\mathrm{A}.1)_1$ and (A.5). By restricting the range of (t_1, t_2) to

$$S_4 = \{(t_1, t_2) : (t_1, t_2) \in [-2k_1, 2\mu_2] \times [-2k_1, 1\mu_2]\} \tag{4.49}$$

and examining all intersections of planes

$$\pi_2(\mathcal{H}_\alpha) : S_4 \mapsto g_2(\mathcal{H}_\alpha, t_1, t_2) - t_3^2 = 0, \tag{4.50}$$

we conclude that the set of mappings $\{\pi_2(\mathcal{A}_1), \pi_2(\mathcal{B}_1), \pi_2(\mathcal{B}_2)\}$ yield

$$
\begin{aligned}
t_1^* &= 2\,\frac{k_1(\mu_1 HS_u - \mu_2 HS_l) - 2\mu_1\mu_2(k_2 - k_1)}{\mu_2 W_l - \mu_1 W_u - 2k_1(k_2 - k_1)}, \\
t_2^* &= \frac{2k_2\mu_2 - 2k_1\mu_1 - (k_2 - k_1)t_1^*}{\mu_1 - \mu_2}, \\
t_3^* &= \sqrt{(2k_1 + t_1^*)(\mu_1 + \mu_2 - t_2^*)}, \\
t_4^* &= 2\mu_2 - t_2^*.
\end{aligned}
\tag{4.51}
$$

Matrix $\boldsymbol{\Delta}_1 - \boldsymbol{\Delta}_2$ is positive-definite provided $k_1 \neq k_2$ and $\mu_1 \neq \mu_2$, thus the projector $\mathbf{P}$ is given by a unity matrix, hence matrices in (3.46) take the form

$$\mathcal{A}(Y_P(\boldsymbol{\Delta}_h)) = \begin{pmatrix} 2y(k_h) - t_1^* & -2t_3^* & 0 \\ & 4y(\mu_h) + 2t_2^* - 2t_4^* & 0 \\ sym & & 4y(\mu_h) + 2t_2^* - 2t_4^* \end{pmatrix}, \tag{4.52$_1$}$$

$(4.52)_2$ $\mathcal{B}(Y_P(\mathbf{\Delta}_h))$

$$= \begin{pmatrix} 2y(\mu_h) + 2y(k_h) + t_1^* - t_2^* + 2t_3^* & 2y(\mu_h) - 2y(k_h) + t_1^* + t_2^* \\ sym & 2y(\mu_h) + 2y(k_h) + t_1^* - t_2^* - 2t_3^* \end{pmatrix},$$

$(4.52)_3$ $\mathcal{C}(Y_P(\mathbf{\Delta}_h)) = 4y(\mu_h) + 2t_2^* + 2t_4^*.$

The most restrictive lower estimate on pairs $(y(k_h), y(\mu_h))$ is obtained by

$$\det \mathcal{A}(Y_P(\mathbf{\Delta}_h)) = 0 \tag{4.53}$$

and reads

$$y(k_h) \in [\mu_2, \mu_1], \qquad h_4(y(k_h)) \le y(\mu_h), \tag{4.54}$$

where

$$h_4(y(k_h)) = \frac{1}{2}\frac{(2\,y(k_h) - t_1^*)(t_4^* - t_2^*) + 2(t_3^*)^2}{2y(k_h) - t_1^*}. \tag{4.55}$$

New estimates attain HSW points, see Fig. 2, that is

$$\begin{aligned} h_3(\mu_2) &= 2k_2 + \mu_1, \qquad h_3(\mu_1) = 2k_1 + \mu_1, \\ h_4(\mu_2) &= 2k_2 + \mu_2, \qquad h_4(\mu_1) = 2k_1 + \mu_2 \end{aligned} \tag{4.56}$$

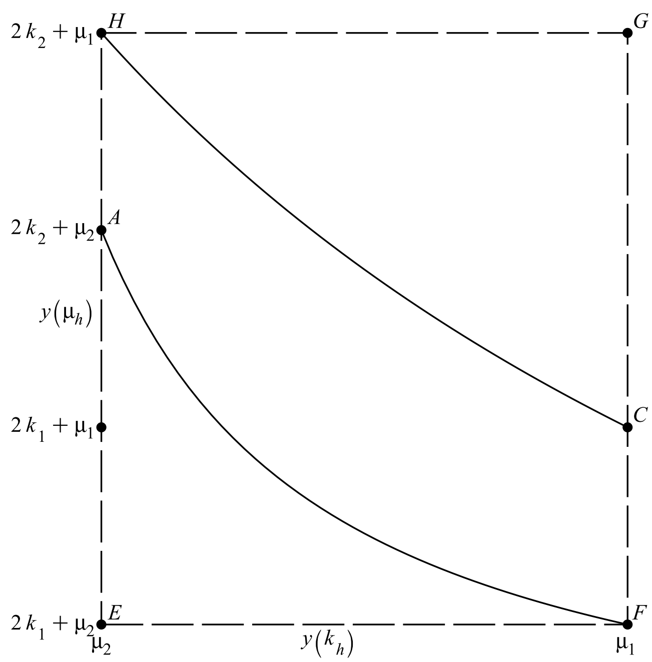

Fig. 2. "Badly-ordered materials", $k_1 = 1, \mu_1 = 3, k_2 = 3, \mu_2 = 1$. Coupled estimates on effective stiffnesses $(y(k_h), y(\mu_h))$ are shown by solid lines H-C and A-F; the Walpole rectangle – by dashed line E-F-G-H; the Hashin–Shtrikman points correspond to A, C.

and, as it was mentioned in case of "well-ordered materials", the ordering of HSW estimates strongly depend on k_α, μ_α, $\alpha = 1, 2$. For chosen characteristics of the constituent materials, $k_1 = 1$, $\mu_1 = 3$, $k_2 = 3$, $\mu_2 = 1$, order of the Hashin-Shtrikman estimates, see (4.5), is inversed.

5. Disscussion of the results and conclusions

5.1. Obtaining 2D elasticity bounds by the statical-geometrical analogy

The link between the results obtained in Sec. 4.2 and 4.3 for the theory of thin plates in bending and those calculated in CHERKAEV and GIBIANSKY [2] for two-dimensional elasticity, exists through the correspondence between the equations of both theories (KL and PS for short), known as the static-geometric analogy. It was discovered in the 1940's by A.I. Lurie and A.L. Goldenveizer in the context of the theory of thin shells. For a deeper discussion of this topic we refer the Reader to GOLDENVEIZER [5, Ch. 5.36.] or NAGHDI [12, Ch. 7.].

Write $\boldsymbol{N}$ for the in-plane force tensor of two-dimensional elasticity. Its components can be expressed in terms of the Airy function χ derivatives, see e.g. GOLDENVEIZER [5, Ch. 3.21]. Thus we can substitute $w \rightsquigarrow \chi$ in the definition of $\boldsymbol{\kappa}$, see (2.5), by observing that the compatibility condition (2.4) is preserved for $\kappa_{\alpha\beta} = -(\sqrt{12}/t)\epsilon_{\lambda\alpha}\epsilon_{\mu\beta}N_{\lambda\mu}$, $\alpha, \beta, \lambda, \mu = 1, 2$; $\epsilon_{\alpha\beta}$ denotes a permutation symbol and t stands for the thickness of the plate. Set $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$ for the in-plane displacement field and $\boldsymbol{\varepsilon}$ for the corresponding strain tensor. Applying (purely formal) relation $M_{\alpha\beta} = (t/\sqrt{12})\epsilon_{\lambda\alpha}\epsilon_{\mu\beta}\varepsilon_{\lambda\mu}$ yields the equilibrium equation (2.6) for $q = 0$ in terms of $\boldsymbol{\varepsilon}$. By introducing the scaling coefficient $\sqrt{12}/t$ in the relations above, we avoid the field measurement unit mismatch. Next, we write $\Psi_{\alpha,\beta} = \epsilon_{\lambda\alpha}\epsilon_{\mu\beta}u_{\lambda,\mu}$ in (2.8). The stiffness properties interchange according to

$$
\begin{aligned}
&k_1^{KL} \rightsquigarrow k_1^{PS} = \frac{1}{k_2^{KL}}, \qquad \mu_1^{KL} \rightsquigarrow \mu_1^{PS} = \frac{1}{\mu_2^{KL}}, \qquad k_2^{KL} \rightsquigarrow k_2^{PS} = \frac{1}{k_1^{KL}}, \\
&\mu_2^{KL} \rightsquigarrow \mu_2^{PS} = \frac{1}{\mu_1^{KL}}, \qquad k_h^{KL} \rightsquigarrow k_h^{PS} = \frac{1}{k_h^{KL}}, \qquad \mu_h^{KL} \rightsquigarrow \mu_h^{PS} = \frac{1}{\mu_h^{KL}},
\end{aligned}
\tag{5.1}
$$

thus we conclude that bounding effective constitutive properties of a Kirchhoff–Love plate is formally equivalent to the two-dimensional elasticity problem with materials whose stiffness moduli are properly redefined. This observation was made in LEWIŃSKI and TELEGA [7, Ch. 23.4], for Hashin–Shtrikman estimates obtained independently for bending and in-plane load cases, but it also remains valid for Walpole bounds. Consequently, the coupled estimates of effective properties k_h^{PS}, μ_h^{PS} of two-dimensional elasticity can be re-established by means of functions $h_1 \dots h_4$ derived in Sections 4.2 and 4.3. By straightforward computation in cases of "well-" and "badly-ordered materials", we respectively obtain

$$
(5.2)\qquad
\begin{aligned}
y(\mu_h^{PS}) &\in \left[h_1^{-1}\left(\frac{1}{y(k_h^{PS})}, \frac{1}{k_2}, \frac{1}{k_1}, \frac{1}{\mu_2}, \frac{1}{\mu_1}\right), h_2^{-1}\left(\frac{1}{y(k_h^{PS})}, \frac{1}{k_2}, \frac{1}{k_1}, \frac{1}{\mu_2}, \frac{1}{\mu_1}\right)\right],\\
y(\mu_h^{PS}) &\in \left[h_3^{-1}\left(\frac{1}{y(k_h^{PS})}, \frac{1}{k_1}, \frac{1}{k_2}, \frac{1}{\mu_2}, \frac{1}{\mu_1}\right), h_4^{-1}\left(\frac{1}{y(k_h^{PS})}, \frac{1}{k_1}, \frac{1}{k_2}, \frac{1}{\mu_2}, \frac{1}{\mu_1}\right)\right],
\end{aligned}
$$

and one can easily check that (5.2) coincide with estimates calculated in CHERKAEV and GIBIANSKY [2].

It is worth pointing out that although the Cherkaev–Gibiansky estimates for two-dimensional (plane stress) elasticity and those reported in the present paper can be derived by the same method, the correlation between them strongly relies on the statical-geometrical analogy. In other words, both results are complementary in the sense of the above-mentioned purely mathematical formalism, linking two physically different phenomena of the theory of elasticity.

5.2. Conclusions

Simple cross-property bounds for $\left(k_h, \mu_h^{KL}, \mu_h^{PS}\right)$ related to the thin plate subject to simultaneous bending and in-plane loads, follow from the results obtained in Sec. 4 and 5.1. They are valid only if bending and membrane effects remain uncoupled, e.g. like in the theory of shallow shells. New bounds are tighter than those obtained by aggregating HSW bounds given by the paralellepiped:

$$
(5.3)\qquad
\begin{aligned}
H = \big\{(k_h, \mu_h^{KL}, \mu_h^{PS}) &: y(k_h) \in [\mu_2, \mu_1],\\
&y(\mu_h^{KL}) \in [2k_{\min}^{KL} + \mu_2^{KL}, 2k_{\max}^{KL} + \mu_1^{KL}], \text{ see (4.4)},\\
&y(\mu_h^{PS}) \in \left[2k_{\min}^{PS} + \mu_2^{PS}, 2k_{\max}^{PS} + \mu_1^{PS}\right], \text{ see (5.1)}\big\},
\end{aligned}
$$

see Figs. 3, 4. Superscripts in k_h can be dropped in the description of H because, up to the scaling factor mentioned in the previous section, the range of admissible values for effective bulk coefficient (or its Y-transformation) is the same, regardless of the underlying theory.

In cases of "well-" and "badly-ordered materials", we respectively set

$$
(5.4)\qquad
\begin{aligned}
H_w = \big\{(k_h, \mu_h^{KL}, \mu_h^{PS}) &: y(k_h) \in [\mu_2, \mu_1],\\
&y(\mu_h^{KL}) \in [h_2(y(k_h)), h_1(y(k_h))], \text{ see (4.18), (4.30)},\\
&y(\mu_h^{PS}) \text{ as in } (5.2)_1\big\},
\end{aligned}
$$

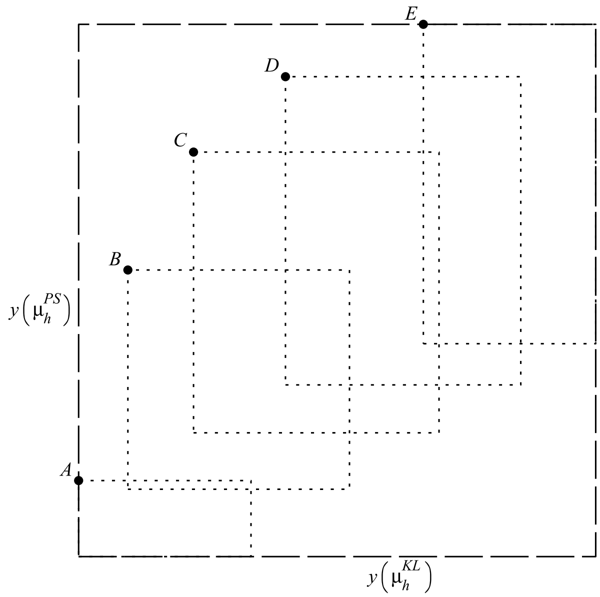

Fig. 3. "Well-ordered materials", $k_1 = 3$, $\mu_1 = 3$, $k_2 = 1$, $\mu_2 = 1$. Coupled bounds on shear moduli μ_h^{KL}, μ_h^{PS}. Dotted line rectangles A, B, C, D, E respectively correspond to cross-sections of H_w for $k_h = \mu_2 + \{0, 0.25(\mu_1 - \mu_2), 0.5(\mu_1 - \mu_2), 0.75(\mu_1 - \mu_2), 1.0(\mu_1 - \mu_2)\}$. Dashed line rectangle illustrate the cross-section of the HSW paralellepiped H.

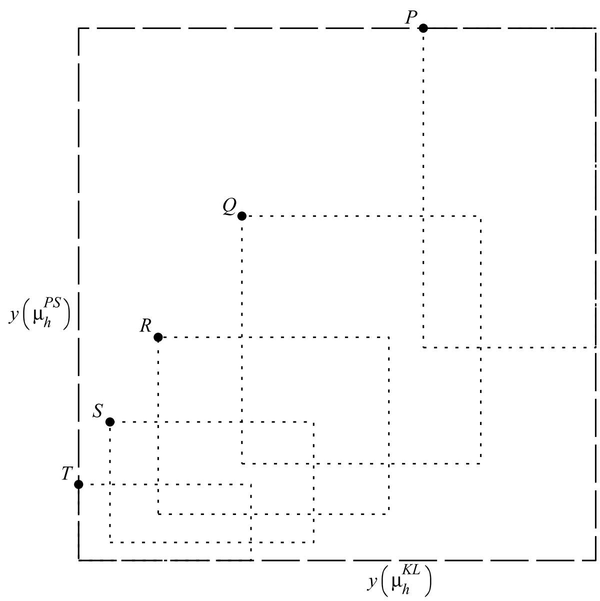

Fig. 4. "Badly-ordered materials", $k_1 = 1$, $\mu_1 = 3$, $k_2 = 3$, $\mu_2 = 1$. Coupled bounds on shear moduli μ_h^{KL}, μ_h^{PS}. Dotted line rectangles P, Q, R, S, T correspond respectively to cross-sections of H_b for $k_h = \mu_2 + \{0, 0.25(\mu_1 - \mu_2), 0.5(\mu_1 - \mu_2), 0.75(\mu_1 - \mu_2), 1.0(\mu_1 - \mu_2)\}$. Dashed line rectangles illustrate the cross-section of the HSW paralellepipeds H.

$$H_b = \big\{(k_h, \mu_h^{KL}, \mu_h^{PS}) : y(k_h) \in [\mu_2, \mu_1], \quad y(\mu_h^{KL}) \in [h_4(y(k_h)), h_3(y(k_h))], \text{ see } (4.43), (4.54), \quad y(\mu_h^{PS}) \text{ as in } (5.2)_2\big\}. \tag{5.5}$$

Appendix

Two general equations ($\alpha = 1, 2$, $\mathcal{H}_\alpha \in \{\mathcal{A}(\mathbf{\Delta}_\alpha), \mathcal{B}(\mathbf{\Delta}_\alpha)\}$)

$$\det \mathcal{H}(\mathbf{\Delta}_\alpha - \mathbf{T}) = g_1(\mathcal{H}_\alpha, t_1, t_2)\,[g_2(\mathcal{H}_\alpha, t_1, t_2) - (t_3)^2], \tag{A.1$_1$}$$

$$\det \mathcal{H}(\mathbf{\Delta}_\alpha - \mathbf{T}) = [g_1(\mathcal{H}_\alpha, t_1, t_2) - (t_3)^2]\,[g_2(\mathcal{H}_\alpha, t_1, t_2) - (t_3)^2], \tag{A.1$_2$}$$

express $\det \mathcal{A}(\mathbf{\Delta}_\alpha - \mathbf{T})$, $\det \mathcal{B}(\mathbf{\Delta}_\alpha - \mathbf{T})$ in Sec. 4. The formulae are shown in the following order: $(\text{A.X})_1 \rightarrow \det \mathcal{A}(\mathbf{\Delta}_1 - \mathbf{T})$, $(\text{A.X})_2 \rightarrow \det \mathcal{A}(\mathbf{\Delta}_2 - \mathbf{T})$, $(\text{A.X})_3 \rightarrow \det \mathcal{B}(\mathbf{\Delta}_1 - \mathbf{T})$, $(\text{A.X})_4 \rightarrow \det \mathcal{B}(\mathbf{\Delta}_2 - \mathbf{T})$, where $\text{X} = 2, 3, 4, 5$.

Determinants of matrices in Section 4.2.1

$$16\left(\frac{1}{2\mu_1} - t_2\right)\left[(t_1 + 2k_1)\left(\frac{1}{2\mu_1} - t_2\right) - (t_3)^2\right], \tag{A.2$_1$}$$

$$16\left(\frac{1}{4\mu_2} + \frac{1}{4\mu_1} - t_2\right)\left[(t_1 + 2k_2)\left(\frac{1}{4\mu_2} + \frac{1}{4\mu_1} - t_2\right) - (t_3)^2\right], \tag{A.2$_2$}$$

$$\frac{2\,(\mu_1 + 2k_1)}{k_1\mu_1}\left[(2\mu_1 - t_1)\left(t_2 - \frac{k_1}{2\mu_1\,(\mu_1 + 2k_1)}\right) - (t_3)^2\right], \tag{A.2$_3$}$$

$$\frac{2(\mu_1 + 2k_2)}{k_2\mu_1}\left[(2\mu_2 - t_1)\left(t_2 - \frac{k_2}{2\mu_1\,(\mu_1 + 2k_2)}\right) - (t_3)^2\right]. \tag{A.2$_4$}$$

It is easily seen that all determinants correspond to $(\text{A.1})_1$.

Determinants of matrices in Section 4.2.2

$$16[t_1(\mu_1 + \mu_2 - t_2) - (t_3)^2]\left[(\mu_1 + \mu_2 - t_2)\left(\frac{1}{2k_1} + t_1\right) - (t_3)^2\right], \tag{A.3$_1$}$$

$$16\,[t_1(2\mu_2 - t_2) - (t_3)^2]\left[(2\mu_2 - t_2)\left(\frac{1}{2k_2} + t_1\right) - (t_3)^2\right], \tag{A.3$_2$}$$

$$4\left[\left(\frac{1}{2\mu_1} - t_1\right)(t_2 + 2k_1) - (t_3)^2\right], \tag{A.3$_3$}$$

$$4\left[\left(\frac{1}{2\mu_2} - t_1\right)(t_2 + 2k_2) - (t_3)^2\right]. \tag{A.3$_4$}$$

Formulae $(\text{A.3})_{1,2}$ correspond to $(\text{A.1})_1$ and $(\text{A.3})_{1,2}$ are related to $(\text{A.1})_1$.

Determinants of matrices in Section 4.3.1

$$(\mathrm{A}.4)_1 \quad 16\left[t_1\left(\frac{1}{2\mu_1}-t_2\right)-(t_3)^2\right]\left[\left(\frac{1}{2k_1}+t_1\right)\left(\frac{1}{2\mu_1}-t_2\right)-(t_3)^2\right],$$

$$(\mathrm{A}.4)_2 \quad 16\left[t_1\left(\frac{1}{4\mu_1}+\frac{1}{4\mu_2}-t_2\right)-(t_3)^2\right] \times\left[\left(\frac{1}{2k_1}+t_1\right)\left(\frac{1}{4\mu_1}+\frac{1}{4\mu_2}-t_2\right)-(t_3)^2\right],$$

$$(\mathrm{A}.4)_3 \quad \frac{2\left(\mu_1+2k_1\right)}{k_1\mu_1}\left[\left(\frac{1}{2\mu_1}-t_1\right)\left(t_2-\frac{k_1}{2\mu_1\left(\mu_1+2k_1\right)}\right)-(t_3)^2\right],$$

$$(\mathrm{A}.4)_4 \quad \frac{2\left(\mu_1+2k_2\right)}{k_2\mu_1}\left[\left(\frac{1}{2\mu_2}-t_1\right)\left(t_2-\frac{k_1}{2\mu_1\left(\mu_1+2k_2\right)}\right)-(t_3)^2\right].$$

Formulae $(\mathrm{A}.4)_{1,2}$ correspond to $(\mathrm{A}.1)_2$ and $(\mathrm{A}.4)_{3,4}$ are related to $(\mathrm{A}.1)_1$.

Determinants of matrices in Section 4.3.1

$$(\mathrm{A}.5)_1 \quad 16(\mu_1+\mu_2-t_2)((2k_1+t_1)(\mu_1+\mu_2-t_2)-(t_3)^2),$$

$$(\mathrm{A}.5)_2 \quad 16(2\mu_2-t_2)((2k_2+t_1)(2\mu_2-t_2)-(t_3)^2),$$

$$(\mathrm{A}.5)_3 \quad 4((2\mu_1-t_1)(t_2+2k_1)-(t_3)^2),$$

$$(\mathrm{A}.5)_4 \quad 4((2\mu_2-t_1)(t_2+2k_2)-(t_3)^2).$$

It is easily seen that all formulae correspond to $(\mathrm{A}.1)_1$.

References

1. A.V. Cherkaev, *Variational Methods for Structural Optimization*, Springer, New York, 2000.
2. A.V. Cherkaev, L.V. Gibiansky, *Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite*, J. Mech. Phys. Solids, **41**, (5), 937–980, 1993.
3. P.G. Ciarlet, *Mathematical Elasticity. Vol. II: Theory of Plates*, North-Holland. Elsevier, Amsterdam, 1997.
4. L.V. Gibiansky, *Effective properties of a plane two-phase elastic composites: coupled bulk-shear moduli bounds*, [in:] Homogenization, V. Berdichevsky, V. Jikov, G. Papanicolau [Eds.], Series on Advances in Mathematics for Applied Sciences, Vol. 50, World Scientific, Singapore, 1999.
5. A.L. Goldenveizer, *The Theory of Thin Elastic Shells*, 2nd. ed., Nauka, Moscow, 1976 [in Russian].
6. Z. Hashin, S. Shtrikman, *A variational approach to the theory of the elastic behaviour of multiphase materials*, J. Mech. Phys. Solids, **11**, 127–140, 1963.

7. T. LEWIŃSKI, J.J. TELEGA, *Plates, Laminates and Shells. Asymptotic Analysis and Homogenization*, Series on Advances in Mathematics for Applied Sciences, Vol. 52, World Scientific, Singapore, 2000.

8. S. LI, *The micromechanics theory of classical plates: a congruous estimate of overall elastic stiffness*, Int. J. Solids Struct., **37**, 5599–5628, 2000.

9. K.A. LURIE, A.V. CHERKAEV, *G-closure of some particular sets of admissible material characteristics for the problem of bending of thin elastic plates*, J. Optimiz. Theory Appl., **42**, 2, 305-316, 1984.

10. G.W. MILTON, *On characterizing the set of possible effective tensors of composites: The variational method and the translation method*, Comm. Pure Appl. Math., **43**, 63–125, 1990.

11. G.W. MILTON, *The Theory of Composites*, Cambridge University Press, Cambridge, 2002.

12. P.M. NAGHDI, *Foundations of elastic shell theory*, [in:] I.N. SNEDDON, R. HILL [Eds.], Progress in Solid Mechanics, vol. IV, North-Holland, Amsterdam, 1963.

13. S. TIMOSHENKO, S. WOINOWSKY-KRIEGER, *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, 1959.

14. L.J. WALPOLE, *On bounds for the overall elastic moduli of inhomogeneous systems*, J. Mech. Phys. Solids, **14**, 151–162, 1966.

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