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# On instantaneous invariants in dual Lorentzian space kinematics

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In this work, we study the concepts of canonical systems and instantaneous invariants in dual Lorentzian space. It gives a systematic approximation about determining the instantaneous invariants. Important for the characterization of higher-order intrinsic properties of a ruled surface, the instantaneous invariants are derived on the basis of line coordinates.

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## 1. Introduction

Instantaneous invariants have been introduced by Bottema [4]. They are geometric invariants under the group of similitudes. The concepts of instantaneous invariants and canonical systems are extended to spatial kinematics by Veldkamp [11]. In this work, Veldkamp developed instantaneous invariants with the aid of point coordinates being useful for characterization of a space curve, which is the path trajectory of a point in a rigid body in spatial motion, thus making an important contribution to the kinematic geometry. He also developed instantaneous invariants via the line coordinates to characterize a ruled surface, which is the path trajectory of a line embedded in a rigid body in spatial motion. So, the instantaneous invariants developed have been employed for a kinematic curvature theory for ruled surfaces [10]. Screw notations make possible the development of a systematic procedure to determine the dual number instantaneous invariants for a given rigid body in spatial motion [8].

## 2. Settings

If $u$ and $u^\circ$ are real numbers, the combination $\widehat{u} = u + \varepsilon u^\circ$ with $\varepsilon^2 = 0$ is called a dual number, where $\varepsilon$ is the dual unit [12]. The set of dual numbers is denoted by $\mathbb{D}$. Then the set

$$\mathbb{D}^3 = \{\widehat{x} = x + \varepsilon x^\circ \mid x, x^\circ \in \mathbb{R}^3\}$$

is a module over the ring $\mathbb{D}$, which is called a $\mathbb{D}$-module as a dual space. The elements of $\mathbb{D}^3$ are called dual vectors [5].

Let $\mathbb{D}^3_1$ be a dual Lorentzian space with the inner product

$$\langle \widehat{x}, \widehat{y} \rangle = \langle x, y \rangle + \varepsilon(\langle x, y^\circ \rangle + \langle x^\circ, y \rangle),$$

where the inner product of the vectors $x$ and $y$ is

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

and the vector product

$$\widehat{x} \times \widehat{y} = x \times y + \varepsilon(x \times y^\circ + x^\circ \times y),$$

where the vector product of the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is

$$x \times y = \left( \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, - \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$

A dual vector $\widehat{x}$ of $\mathbb{D}^3_1$ is said to be spacelike if $\langle x, x \rangle > 0$ or $x = 0$, timelike if $\langle x, x \rangle < 0$ and lightlike or null if $\langle x, x \rangle = 0$ and $x \neq 0$ [1, 2, 3]. The norm of a dual vector $\widehat{x}$ in $\mathbb{D}^3_1$ is defined to be $\|\widehat{x}\| = \sqrt{|\langle \widehat{x}, \widehat{x} \rangle|}$. A $3 \times 3$ matrix $\widehat{A}$ is called the orthogonal matrix in sense of Lorentzian if it is $\widehat{A}^T = \widehat{S}\widehat{A}^{-1}\widehat{S}$, where the matrix $\widehat{S}$ is a signature matrix and will be denoted by, [9],

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now, consider a rigid body moving in free space. Assume any inertial reference frame $\{F\}$ fixed in the space and a frame $\{M\}$ fixed to the body. At each instance, the position and the orientation of the rigid body is uniquely described by a rigid body displacement from frame $\{F\}$ to frame $\{M\}$ [7].

A unit vector bounded on a line is called the unit screw and denoted by $\widehat{R}$. Let $\widehat{R}_F$ indicate the unit screw $\widehat{R}$ in the frame $\{F\}$. The three dual number components of $\widehat{R}_F$ can be written as a column matrix $(\widehat{R})_F$. Since the dual number components of $\widehat{R}_F$ consist of six real numbers, the well-known Plücker coordinates of the line in $\{F\}$, the terms: unit screw and line, are synonymous in this study.

The instantaneous state of motion of a rigid body can be described by a dual screw velocity $\widehat{\Omega}$. The dual screw velocity of a rigid body specified in a system $\{M\}$ is denoted by $\widehat{\Omega}_M$. Also this dual screw velocity may be written as a column matrix $(\widehat{\Omega})_M$ with dual number elements. The real part of this dual screw velocity physically represents the angular velocity of the rigid body, while the dual part represents the linear velocity of the origin of the system $\{M\}$.

## 3. Instantaneous motion

The motion of a rigid body at each instance can be uniquely described by a dual screw velocity

$$\widehat{\Omega} = \widehat{w}\widehat{s} = (w + \varepsilon w^\circ)\widehat{s} = w\left(1 + \varepsilon\frac{w^\circ}{w}\right)\widehat{s} = w(1 + \varepsilon p)\widehat{s}. \tag{3.1}$$

Equation (3.1) represents an instantaneous screw motion. This representation consists of a rotation with the angular velocity $w$ about, and a translation with velocity $w^\circ$ – along the instantaneous screw axis (ISA) specified by the unit screw $\widehat{s}$. Commonly, the direction of ISA is defined in the positive sense of angular velocity. $p$, which is defined by $p = w^\circ/w$ , is the instantaneous screw pitch of the motion. In this study, we are going to present the way of determination of ISA for a given rigid body motion and to introduce the geometric concept of moving and fixed axodes.

In the study of the instantaneous kinematics of a rigid body, let us choose two frames of reference: frame $\{M\}$ attached to a moving rigid body, and frame $\{F\}$ fixed in a dual Lorentzian 3-space. Let $(\widehat{R})_M$ be an arbitrary line, embedded in the body and let $(\widehat{R})_F$, specified in the fixed frame $\{F\}$, be the line which is coincident with $(\widehat{R})_M$ at this instant. Hence the motion of the rigid body may be described as

$$(\widehat{R})_F = [\widehat{A}](\widehat{R})_M, \tag{3.2}$$

where

$$[\widehat{A}] = \begin{bmatrix} \widehat{a}_1 & \widehat{b}_1 & \widehat{c}_1 \\ \widehat{a}_2 & \widehat{b}_2 & \widehat{c}_2 \\ \widehat{a}_3 & \widehat{b}_3 & \widehat{c}_3 \end{bmatrix} \tag{3.3}$$

is referred to as a screw matrix with dual number elements, which is an orthogonal matrix in Lorentzian sense. Therefore, since $[\widehat{A}]^T = [\widehat{S}][\widehat{A}]^{-1}[\widehat{S}]$ and $[\widehat{A}]^{-1} = [\widehat{S}][\widehat{A}]^T[\widehat{S}]$,

$$[\widehat{A}][\widehat{A}]^{-1} = [\widehat{A}][\widehat{S}][\widehat{A}]^T[\widehat{S}] = [\widehat{A}]^{-1}[\widehat{A}] = [\widehat{S}][\widehat{A}]^T[\widehat{S}][\widehat{A}] = \widehat{I}. \tag{3.4}$$

From Eq. (3.4) we can write the inverse transformation of Eq. (3.2) as

$$(\widehat{R})_M = [\widehat{S}][\widehat{A}]^T[\widehat{S}](\widehat{R})_F. \tag{3.5}$$

Here the elements of a screw matrix determined by the instantaneous position of $\{M\}$ relative to $\{F\}$ are functions of a real parameter $t$. We will assume that these functions have derivatives of any order. Hence the time derivatives of any order of the screw matrix are denoted by $[\overset{(n)}{\widehat{A}}]$, $n = 1, 2, 3, \ldots$

Taking the derivative of Eq. (3.2) with respect to $t$, we get

$$(\dot{\widehat{R}})_F = [\dot{\widehat{A}}](\widehat{R})_M. \tag{3.6}$$

Hence we can write

$$(\dot{\widehat{R}})_F = [\widehat{\Omega}]_F(\widehat{R})_F, \tag{3.7}$$

$$(\dot{\widehat{R}})_M = [\widehat{\Omega}]_M(\widehat{R})_M, \tag{3.8}$$

where $(\dot{\widehat{R}})_F$ and $(\dot{\widehat{R}})_M$ denote the absolute time derivatives of the unit screw $\widehat{R}$ expressed in $\{F\}$ and $\{M\}$, respectively. By definition, we obtain $[\widehat{\Omega}]_F$ and $[\widehat{\Omega}]_M$ as below:

$$\begin{aligned} [\widehat{\Omega}]_F &= \begin{bmatrix} 0 & \widehat{w}_{F_3} & -\widehat{w}_{F_2} \\ -\widehat{w}_{F_3} & 0 & \widehat{w}_{F_1} \\ -\widehat{w}_{F_2} & \widehat{w}_{F_1} & 0 \end{bmatrix}, \\ [\widehat{\Omega}]_M &= \begin{bmatrix} 0 & \widehat{w}_{M_3} & -\widehat{w}_{M_2} \\ -\widehat{w}_{M_3} & 0 & \widehat{w}_{M_1} \\ -\widehat{w}_{M_2} & \widehat{w}_{M_1} & 0 \end{bmatrix}, \end{aligned} \tag{3.9}$$

where $[\widehat{\Omega}]_F$ and $[\widehat{\Omega}]_M$ are two skew-symmetric matrices in Lorentzian sense [6]. Also here $\widehat{w}_{F_1}$, $\widehat{w}_{F_2}$ and $\widehat{w}_{F_3}$ are elements of $(\widehat{\Omega})_F$ while $\widehat{w}_{M_1}$, $\widehat{w}_{M_2}$ and $\widehat{w}_{M_3}$ are elements of $(\widehat{\Omega})_M$. The column matrices $(\widehat{\Omega})_F$ and $(\widehat{\Omega})_M$ are the dual screw velocity $\widehat{\Omega}$ of the rigid body specified in $\{F\}$ and $\{M\}$, respectively.

Taking into consideration equations (3.6), (3.7) and with the aid of Eq. (3.2) we can write

$$[\widehat{\Omega}]_F[\widehat{A}](\widehat{R})_M = [\dot{\widehat{A}}](\widehat{R})_M.$$

From Eq. (3.4) we can obtain

$$[\widehat{\Omega}]_F = [\dot{\widehat{A}}][\widehat{S}][\widehat{A}]^T[\widehat{S}]. \tag{3.10}$$

On the other hand, since $(\dot{\widehat{R}})_F = [\widehat{A}](\dot{\widehat{R}})_M$, we have

$$[\widehat{A}][\widehat{\Omega}]_M(\widehat{R})_M = [\dot{\widehat{A}}](\widehat{R})_M$$

from Eqs. (3.6) and (3.8). Therefore we get

$$[\widehat{\Omega}]_M = [\widehat{S}][\widehat{A}]^T[\widehat{S}][\dot{\widehat{A}}]. \tag{3.11}$$

If we do the matrix multiplication of the right-hand sides of Eqs. (3.10) and (3.11), the matrices $(\widehat{\Omega})_F$ and $(\widehat{\Omega})_M$ are found in terms of $[\widehat{A}]$ and $[\dot{\widehat{A}}]$ as below:

$$(\widehat{\Omega})_F = \begin{bmatrix} \widehat{w}_{F_1} \\ \widehat{w}_{F_2} \\ \widehat{w}_{F_3} \end{bmatrix} = \begin{bmatrix} \dot{\widehat{a}}_3\widehat{a}_2 + \dot{\widehat{b}}_3\widehat{b}_2 - \dot{\widehat{c}}_3\widehat{c}_2 \\ -\dot{\widehat{a}}_3\widehat{a}_1 - \dot{\widehat{b}}_3\widehat{b}_1 + \dot{\widehat{c}}_3\widehat{c}_1 \\ \dot{\widehat{a}}_1\widehat{a}_2 + \dot{\widehat{b}}_1\widehat{b}_2 - \dot{\widehat{c}}_1\widehat{c}_2 \end{bmatrix}, \tag{3.12}$$

$$(\widehat{\Omega})_M = \begin{bmatrix} \widehat{w}_{M_1} \\ \widehat{w}_{M_2} \\ \widehat{w}_{M_3} \end{bmatrix} = \begin{bmatrix} \dot{\widehat{c}}_1\widehat{b}_1 + \dot{\widehat{c}}_2\widehat{b}_2 - \dot{\widehat{c}}_3\widehat{b}_3 \\ -\dot{\widehat{c}}_1\widehat{a}_1 - \dot{\widehat{c}}_2\widehat{a}_2 + \dot{\widehat{c}}_3\widehat{a}_3 \\ \dot{\widehat{b}}_1\widehat{a}_1 + \dot{\widehat{b}}_2\widehat{a}_2 - \dot{\widehat{b}}_3\widehat{a}_3 \end{bmatrix}. \tag{3.13}$$

The dual modulus $\widehat{w}$ given in Eq. (3.1) can be expressed as

$$\widehat{w} = \sqrt{|(\widehat{w}_{F_1})^2 + (\widehat{w}_{F_2})^2 - (\widehat{w}_{F_3})^2|} = \sqrt{|(\widehat{w}_{M_1})^2 + (\widehat{w}_{M_2})^2 - (\widehat{w}_{M_3})^2|}. \tag{3.14}$$

The real part $w$ obtained by expansion in Eq. (3.14) is the magnitude of the angular velocity of the body, while the dual part $w^\circ$ is the velocity of the body translating along the timelike ISA. Hence using Eq. (3.1), we can express the timelike ISA $\widehat{s}$ in $\{F\}$ and $\{M\}$ as:

$$(\widehat{s})_F = \frac{(\widehat{\Omega})_F}{\widehat{w}}, \tag{3.15}$$

$$(\widehat{s})_M = \frac{(\widehat{\Omega})_M}{\widehat{w}}. \tag{3.16}$$

We may observe from Eqs. (3.15) and (3.16) that $(\widehat{s})_F$ and $(\widehat{s})_M$ represent two ruled surfaces. The ruled surface $(\widehat{s})_M$ embedded in the body is called the moving axode and the ruled surface $(\widehat{s})_F$ fixed in space is called the fixed axode. Thus we conclude that a rigid body in general motion may be uniquely described by a moving axode rolling and sliding simultaneously on a fixed axode. At any instant, there is one and only one coincident generator of the moving and fixed axodes. The coincident generator is the instantaneous screw axis of the motion.

## 4. ISA trihedron $\{g\}$ and screw matrix $[\widehat{L}]$

Let us denote the derivative of ISA as:

$$\dot{\widehat{s}} = \frac{d\widehat{s}}{dt} = \frac{d\widehat{s}}{d\widehat{Q}} \frac{d\widehat{Q}}{dt},$$

where $d\widehat{Q}$ is the dual arc element of the fixed axode and $d\widehat{s}/d\widehat{Q}$ is a unit screw intersecting the ISA $\widehat{s}$ orthogonally. Now we may define the trihedron $\{g\}$ associated with the fixed axode by a set of base screws, which are three orthogonally intersecting unit screws:

$$\begin{aligned} \widehat{g}_3 &= \widehat{s}, \\ \widehat{g}_1 &= \frac{d\widehat{s}}{d\widehat{Q}} = \frac{\dot{\widehat{s}}}{d\widehat{Q}/dt} = \frac{\dot{\widehat{s}}}{\|\dot{\widehat{s}}\|}, \\ \widehat{g}_2 &= \widehat{g}_3 \times \widehat{g}_1, \end{aligned} \tag{4.1}$$

where $\widehat{g}_3$ is the timelike generator, $\widehat{g}_1$ is the spacelike central normal and $\widehat{g}_2$ is the spacelike central tangent of the fixed axode. The origin of $\{g\}$ is located at the center point of the timelike generator. Likewise, the basic unique screws of the generator trihedron $\{h\}$ of the moving axode can be defined as:

$$\begin{aligned} \widehat{h}_3 &= \widehat{s}, \\ \widehat{h}_1 &= \frac{\dot{\widehat{s}}}{\|\dot{\widehat{s}}\|}, \\ \widehat{h}_2 &= \widehat{h}_3 \times \widehat{h}_1, \end{aligned}$$

with $\widehat{h}_3$ as the timelike generator, $\widehat{h}_1$ as the spacelike central normal and $\widehat{h}_2$ as the spacelike central tangent of the moving axode.

Using the operator rule, (see [7]), on $\widehat{s}$ in both $\{F\}$ and $\{M\}$, we have

$$\begin{aligned} \frac{d\widehat{s}}{dt} &= \dot{\widehat{s}}_{|M} + (\widehat{\Omega})_M \times \widehat{s} = \dot{\widehat{s}}_{|M} + (\widehat{s}\widehat{w}) \times \widehat{s} = \dot{\widehat{s}}_{|M}, \\ \frac{d\widehat{s}}{dt} &= \dot{\widehat{s}}_{|F} + (\widehat{\Omega})_F \times \widehat{s} = \dot{\widehat{s}}_{|F} + (\widehat{s}\widehat{w}) \times \widehat{s} = \dot{\widehat{s}}_{|F} \end{aligned}$$

so that

$$\dot{\widehat{s}}_{|M} = \dot{\widehat{s}}_{|F}$$

or

$$\widehat{g}_1 = \widehat{h}_1.$$

This means that the trihedron $\{h\}$ always coincides with the trihedron $\{g\}$. Therefore we will use one notation $\{g\}$ and refer to it as the ISA trihedron. Let us assume that the screw matrix $[\widehat{L}]$ defines the transformation between the ISA trihedron $\{g\}$ and the fixed system $\{F\}$. Then we can write

$$(\widehat{R})_F = [\widehat{L}](\widehat{R})_g. \tag{4.2}$$

Taking into consideration Eqs. (3.2), (3.4) and (4.2), we can express the transformation between the ISA trihedron and the system $\{M\}$ as

$$(\widehat{R})_M = [\widehat{S}][\widehat{A}]^T[\widehat{S}][\widehat{L}](\widehat{R})_g. \tag{4.3}$$

The elements of $[\widehat{L}]$ determined by the position of $\{g\}$ relative to $\{F\}$ are given by the dual components of $\widehat{g}_1$, $\widehat{g}_2$ and $\widehat{g}_3$, expressed in $\{F\}$ denoted, respectively, by $\widehat{g}_{1F}$, $\widehat{g}_{2F}$ and $\widehat{g}_{3F}$.

Now we will determine the elements of $[\widehat{L}]$. Because $\widehat{g}_{3F}$ is coincident with the timelike ISA, from Eq. (3.15) we have

$$\widehat{g}_{3F} = \widehat{s}_F = \frac{1}{\widehat{w}}(\widehat{\Omega})_F. \tag{4.4}$$

Taking into consideration the second equation of Eq. (4.1) in $\{F\}$, we get

$$\widehat{g}_{1F} = \frac{\dot{\widehat{s}}_F}{\|\dot{\widehat{s}}_F\|}, \tag{4.5}$$

where

$$\dot{\widehat{s}}_F = \frac{1}{\widehat{w}}(\dot{\widehat{\Omega}})_F - \frac{\dot{\widehat{w}}}{\widehat{w}^2}(\widehat{\Omega})_F, \tag{4.6}$$

which is obtained by differentiation of Eq. (4.4) with respect to the real parameter $t$. Finally, we get $\widehat{g}_{2F}$ as below

$$\widehat{g}_{2F} = \widehat{g}_{3F} \times \widehat{g}_{1F}. \tag{4.7}$$

$(\widehat{\Omega})_F$ given in Eq. (3.12) and $\widehat{w}$ given in Eq. (3.14) are analytical functions of the time parameter $t$. So their time derivatives exist. Therefore the set of base screws $\widehat{g}_{1F}$, $\widehat{g}_{2F}$ and $\widehat{g}_{3F}$ can be expressed as column matrices:

$$\widehat{g}_{1F} = \begin{bmatrix} \widehat{l}_1 \\ \widehat{l}_2 \\ \widehat{l}_3 \end{bmatrix}, \qquad \widehat{g}_{2F} = \begin{bmatrix} \widehat{k}_1 \\ \widehat{k}_2 \\ \widehat{k}_3 \end{bmatrix}, \qquad \widehat{g}_{3F} = \begin{bmatrix} \widehat{n}_1 \\ \widehat{n}_2 \\ \widehat{n}_3 \end{bmatrix}. \tag{4.8}$$

Hence we can construct the screw matrix

$$[\widehat{L}] = \begin{bmatrix} \widehat{l}_1 & \widehat{k}_1 & \widehat{n}_1 \\ \widehat{l}_2 & \widehat{k}_2 & \widehat{n}_2 \\ \widehat{l}_3 & \widehat{k}_3 & \widehat{n}_3 \end{bmatrix}, \tag{4.9}$$

where all the elements are the functions of parameter $t$.

## 5. Canonical systems and screw matrix $[\widehat{B}]$

In the study of instantaneous kinematics, we are interested only in the motion of a rigid body at a particular instant ($t = t_0$). The position of the rigid body at $t = 0$ is referred to as the zero position. When a moving reference frame is at the zero position, its symbol carries a subscript 0, for example $\{g\}_0$. A screw matrix with a subscript 0 indicates that it is evaluated at $t = 0$, for example: $[\widehat{A}]_0 = [\widehat{A}]_{t=0}$.

Now let us define two new systems. One of the system $\{X\}$ is attached to the body and the other system $\{Y\}$ is fixed in space so that both $\{X\}_0$ and $\{Y\}_0$ at zero position are coincident with the ISA trihedron $\{g\}_0$. We know from VELDKAMP [11], that the two systems $\{X\}$ and $\{Y\}$ are canonical systems.

The systems $\{F\}$ and $\{Y\}$ are fixed in the dual Lorentzian space. Hence we can write the relationship between the systems $\{F\}$ and $\{Y\}$ as below:

$$(\widehat{R})_F = [\widehat{L}]_0(\widehat{R})_Y. \tag{5.1}$$

Taking into consideration Eqs. (3.2) and (3.4) we can express the transformation between the systems $\{M\}$ and $\{X\}$ attached to the same rigid body as follows:

$$(\widehat{R})_M = [\widehat{S}][\widehat{A}]_0^T[\widehat{S}][\widehat{L}]_0(\widehat{R})_X. \tag{5.2}$$

Here we note that the systems $\{X\}$ and $\{Y\}$ at a general instant $t$ do not coincide. The transformation between them can be defined by the screw matrix. Therefore we can write

$$(\widehat{R})_Y = [\widehat{B}](\widehat{R})_X \tag{5.3}$$

for an arbitrary line $\widehat{R}$ in the body. If we use orthogonal properties of screw matrices $[\widehat{L}]_0$ and $[\widehat{A}]_0$, we can get the screw matrix $[\widehat{B}]$ with the aid of Eq. (3.2)

$$[\widehat{B}] = [\widehat{S}][\widehat{L}]_0^T[\widehat{S}][\widehat{A}][\widehat{S}][\widehat{A}]_0^T[\widehat{S}][\widehat{L}]_0. \tag{5.4}$$

If we take the $n$th-order derivative of $[\widehat{B}]$, we obtain

$$[\overset{(n)}{\widehat{B}}] = [\widehat{S}][\widehat{L}]_0^T[\widehat{S}][\overset{(n)}{\widehat{A}}][\widehat{S}][\widehat{A}]_0^T[\widehat{S}][\widehat{L}]_0. \tag{5.5}$$

Let us denote $[\widehat{A}]_n = [\overset{(n)}{\widehat{A}}]_{t=0}$. Then we obtain the general expression for derivatives of $[\widehat{B}]$ at the zero position

$$[\widehat{B}]_n = [\widehat{S}][\widehat{L}]_0^T[\widehat{S}][\widehat{A}]_n[\widehat{S}][\widehat{A}]_0^T[\widehat{S}][\widehat{L}]_0 \tag{5.6}$$

with $n = 1, 2, 3, \ldots$; while for $n = 0$ we get

$$[\widehat{B}]_0 = [\widehat{I}]. \tag{5.7}$$

This means that the canonical systems $\{X\}_0$ and $\{Y\}_0$ are actually coincident.

We shall drop the subscript $X$ in subsequent derivations if the screw and column matrices are expressed in the fixed canonical system $\{X\}$, for example: $\widehat{s} = \widehat{s}_X$, $\dot{\widehat{R}} = \dot{\widehat{R}}_X$, $(\widehat{\Omega}) = (\widehat{\Omega})_X$. In addition, we shall introduce notations suggested by Veldkamp: the values of functions and their time derivatives, evaluated at $t = 0$, are to be denoted by the last subscript of the function symbol. For example, $\widehat{R}_0 = \widehat{R}(0)$, $\widehat{s}_1 = \dot{\widehat{s}}(0)$, $\widehat{w}_{30} = \widehat{w}_3(0)$, $\widehat{\alpha}_{21} = \dot{\widehat{\alpha}}_2(0)$, ...

The base screws $\widehat{x}_{10}$, $\widehat{x}_{20}$ and $\widehat{x}_{30}$ of $\{Y\}_0$ are defined by Eq. (4.1) since the canonical systems at zero position coincide with the ISA trihedron $\{g\}_0$. With the ISA $\widehat{s}_0$ and $\widehat{x}_{30}$ coinciding, we have

$$(\widehat{s})_0 = [0\ 0\ 1]^T, \tag{5.8}$$

and $\widehat{s}_1$ is coincident with $\widehat{x}_{10}$

$$(\widehat{s})_1 = \widehat{s}_{11}[1\ 0\ 0]^T, \tag{5.9}$$

in which $s_{11}$ is always positive by convention.

If the velocity screw $\widehat{\Omega}$ of the rigid body specified in system $\{Y\}$ is

$$(\widehat{\Omega}) = [\widehat{w}_1\ \widehat{w}_2\ \widehat{w}_3]^T \tag{5.10}$$

we have, in view of Eq. (5.8) at the zero position,

$$(\widehat{\Omega})_0 = [0\ 0\ \widehat{w}_{30}]^T, \tag{5.11}$$

which represents a screw motion about the ISA $\widehat{s}_0$. Note that the real part of $\widehat{w}_{30}$ is always positive by convention.

Differentiating Eq. (3.1) with respect to $t$, we express the resultant at the zero position as

$$(\widehat{\Omega})_1 = \widehat{w}_{31}(\widehat{s})_0 + \widehat{w}_{30}(\widehat{s})_1. \tag{5.12}$$

Substituting Eqs. (5.8) and (5.9) into Eq. (5.12), we obtain the column matrix

$$(\widehat{\Omega})_1 = [\widehat{w}_{11}\ 0\ \widehat{w}_{31}]^T, \tag{5.13}$$

where $\widehat{w}_{11} = \widehat{w}_{30}\widehat{s}_{11}$.

## 6. Instantaneous invariants

Let $(\widehat{R})_X$ be a line in the rigid body. From Eq. (5.3), we get the $n$th-order time derivative of $\widehat{R}$ in the system $\{Y\}$ as follows:

$$\overset{(n)}{(\widehat{R})} = \overset{(n)}{[\widehat{B}]}(\widehat{R})_X = \overset{(n)}{[\widehat{B}]}[\widehat{S}][\widehat{B}]^T[\widehat{S}](\widehat{R}).$$

Since $[\widehat{B}]_0^T = \widehat{I}$, we get the expression for derivatives of $(\widehat{R})$ at the zero position

$$(\widehat{R})_n = [\widehat{B}]_n(\widehat{R})_0. \tag{6.1}$$

Similarly, we can express the derivative of $(\widehat{R})$ in the system $\{Y\}$ as

$$(\dot{\widehat{R}}) = [\widehat{\Omega}](\widehat{R}). \tag{6.2}$$

Hence Eq. (6.2) at the zero position can be written as

$$(\widehat{R})_1 = [\widehat{\Omega}]_0(\widehat{R})_0. \tag{6.3}$$

If we take the derivative of $(\widehat{R})_1$ at zero position, we find

$$(\widehat{R})_2 = [\widehat{\Omega}]_1(\widehat{R})_0 + [\widehat{\Omega}]_0(\widehat{R})_1.$$

Here, taking into consideration Eq. (6.3), $(\widehat{R})_2$ can be written as

$$(\widehat{R})_2 = \{[\widehat{\Omega}]_1 + [\widehat{\Omega}]_0[\widehat{\Omega}]_0\}(\widehat{R})_0. \tag{6.4}$$

In the Lorentzian sense, the skew-symmetric matrices $[\widehat{\Omega}]_0$ and $[\widehat{\Omega}]_1$ in Eqs. (6.3) and (6.4) are

$$[\widehat{\Omega}]_0 = \begin{bmatrix} 0 & \widehat{w}_{30} & 0 \\ -\widehat{w}_{30} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\widehat{\Omega}]_1 = \begin{bmatrix} 0 & \widehat{w}_{31} & 0 \\ -\widehat{w}_{31} & 0 & \widehat{w}_{11} \\ 0 & \widehat{w}_{11} & 0 \end{bmatrix}. \tag{6.5}$$

If we equate Eq. (6.1) with Eqs. (6.3) and (6.4) for $n = 1, 2$, respectively, we have

$$[\widehat{B}]_1 = [\widehat{\Omega}]_0 = \begin{bmatrix} 0 & \widehat{w}_{30} & 0 \\ -\widehat{w}_{30} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.6}$$

$$[\widehat{B}]_2 = \begin{bmatrix} -\widehat{w}_{30}^2 & \widehat{w}_{31} & 0 \\ -\widehat{w}_{31} & -\widehat{w}_{30}^2 & \widehat{w}_{11} \\ 0 & \widehat{w}_{11} & 0 \end{bmatrix}. \tag{6.7}$$

Since the geometric properties of the motion of a rigid body are independent of the rate at which the motion is performed, any given motion can be modified in such a manner that the magnitude of its angular velocity about the ISA will be kept continuously equal to unity. It follows that

$$\widehat{w}_{30} = 1 + \varepsilon \widehat{w}^{\circ}_{31} = 1 + \varepsilon p_0, \tag{6.8}$$

$$\widehat{w}_{31} = \varepsilon \widehat{w}^{\circ}_{31} = \varepsilon p_1, \tag{6.9}$$

where $p_0$ is the pitch and $p_1$ is the rate of change of the pitch at zero position.

Now let us assume that the screw matrix $[\widehat{B}]$ is

$$[\widehat{B}] = \begin{bmatrix} \widehat{\alpha}_1 & \widehat{\beta}_1 & \widehat{\gamma}_1 \\ \widehat{\alpha}_2 & \widehat{\beta}_2 & \widehat{\gamma}_2 \\ \widehat{\alpha}_3 & \widehat{\beta}_3 & \widehat{\gamma}_3 \end{bmatrix}. \tag{6.10}$$

The derivatives of any order of $[\widehat{B}]$ at zero position are

$$[\widehat{B}]_n = \begin{bmatrix} \widehat{\alpha}_{1n} & \widehat{\beta}_{1n} & \widehat{\gamma}_{1n} \\ \widehat{\alpha}_{2n} & \widehat{\beta}_{2n} & \widehat{\gamma}_{2n} \\ \widehat{\alpha}_{3n} & \widehat{\beta}_{3n} & \widehat{\gamma}_{3n} \end{bmatrix} \tag{6.11}$$

with $n = 1, 2, 3, \dots$. Since the screw matrix $[\widehat{B}]$ is orthogonal in the Lorentzian sense, we can write

$$[\widehat{B}][\widehat{S}][\widehat{B}]^T[\widehat{S}] = \widehat{I}. \tag{6.12}$$

If we take the derivative of this equation in succession with respect to $t$, and substitute $t = 0$ into the resultant expressions, we get the following result

$$[\widehat{B}]_1 + [\widehat{S}]_0[\widehat{B}]_1^T[\widehat{S}]_0 = 0, \tag{6.13}$$

$$[\widehat{B}]_2 + [\widehat{S}]_0[\widehat{B}]_2^T[\widehat{S}]_0 = -2[\widehat{B}]_1[\widehat{S}]_0[\widehat{B}]_1^T[\widehat{S}]_0, \tag{6.14}$$

$$[\widehat{B}]_3 + [\widehat{S}]_0[\widehat{B}]_3^T[\widehat{S}]_0 = -3[\widehat{B}]_2[\widehat{S}]_0[\widehat{B}]_1^T[\widehat{S}]_0 - 3[\widehat{B}]_1[\widehat{S}]_0[\widehat{B}]_2^T[\widehat{S}]_0. \tag{6.15}$$

Taking into consideration Eqs. (6.6), (6.7) and using Eqs. (6.13), (6.14) and (6.15), we find that $[\widehat{B}]_1$, $[\widehat{B}]_2$ and $[\widehat{B}]_3$ are

$$[\widehat{B}]_1 = \begin{bmatrix} 0 & -\widehat{\alpha}_{21} & 0 \\ \widehat{\alpha}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.16}$$

$$[\widehat{B}]_2 = \begin{bmatrix} -(\widehat{\alpha}_{12} + 2\widehat{\alpha}_{21}^2) & -\widehat{\alpha}_{22} & 0 \\ \widehat{\alpha}_{22} & -(\widehat{\beta}_{22} + 2\widehat{\alpha}_{21}^2) & \widehat{\beta}_{32} \\ 0 & \widehat{\beta}_{32} & 0 \end{bmatrix}, \tag{6.17}$$

$$[\widehat{B}]_3 = \begin{bmatrix} -6\widehat{\alpha}_{21}\widehat{\alpha}_{22}-\widehat{\alpha}_{13} & -3\widehat{\alpha}_{21}(\widehat{\beta}_{22}-\widehat{\alpha}_{12})-\widehat{\alpha}_{23} & -3\widehat{\alpha}_{21}\widehat{\beta}_{32}+\widehat{\alpha}_{33} \\ -3\widehat{\alpha}_{21}(\widehat{\beta}_{22}-\widehat{\alpha}_{12})-\widehat{\beta}_{13} & -6\widehat{\alpha}_{21}\widehat{\alpha}_{22}-\widehat{\beta}_{23} & \widehat{\beta}_{33} \\ 3\widehat{\alpha}_{21}\widehat{\beta}_{32}-\widehat{\gamma}_{13} & \widehat{\gamma}_{23} & -\widehat{\gamma}_{33} \end{bmatrix}. \tag{6.18}$$

If we go on like this, the elements for $[\widehat{B}]_n$ can be derived. Taking into consideration Eqs. (6.6), (6.8) and (6.16), the dual element in $[\widehat{B}]_1$ is found as

$$\widehat{\alpha}_{21} = -\widehat{w}_{30} = -(1+\varepsilon\widehat{w}_{31}) = -(1-\varepsilon\alpha^{\circ}_{22}), \tag{6.19}$$

which indicates that a single scalar parameter $\widehat{\alpha}^{\circ}_{21}$ characterizes completely, to the first order, the geometric properties of the motion of a rigid body at any given instant.

To determine the elements of $[\widehat{B}]_2$, two additional dual elements $\widehat{\alpha}_{22}$ and $\widehat{\beta}_{32}$ are found. So, from Eqs. (6.7) and (6.9) we have

$$\widehat{\alpha}_{22} = -\widehat{w}_{31} = -\varepsilon\widehat{w}_{31} = \varepsilon\alpha^{\circ}_{22}. \tag{6.20}$$

Thus, to characterize the geometric properties of the motion of a rigid body up to the second order, a total of four scalar parameters are required, which we must know, in addition to $\widehat{\alpha}^{\circ}_{21}$, $\widehat{\alpha}^{\circ}_{22}$ and another dual scalar parameter $\widehat{\beta}_{32}$. We may deduce from Eq. (6.18) that the geometric properties of the motion of a rigid body referred to any given instant up to the $n$th order ($n \geq 3$), characterized by a total of $[6(n-3)+4]$ scalar parameters, are referred to as the nth order instantaneous invariants.

**Example.** The dual hyperbolic angle between the two gear axes $\widehat{F}_3$ and $\widehat{M_3}$ is $\widehat{\delta} = \delta + \varepsilon\delta^{\circ}$. The ISA forms a dual hyperbolic angle $\widehat{\xi} = \xi + \varepsilon\xi^{\circ}$ with $\widehat{F}_3$ and a dual hyperbolic angle $\widehat{\eta} = \eta + \varepsilon\eta^{\circ}$ with $\widehat{M_3}$. So it follows that $\widehat{\delta} = \widehat{\xi} + \widehat{\eta}$, as shown in Fig. 1. The reference frame $\{\widehat{F}\}$ with base screws $\widehat{F}_1$, $\widehat{F}_2$, $\widehat{F}_3$ is attached to the fixed gear. Therefore ISA $\widehat{s}$ may be expressed as

$$(\widehat{s})_F = \begin{bmatrix} -\sin\phi\sinh\widehat{\xi} \\ \cos\phi\sinh\widehat{\xi} \\ \cosh\widehat{\xi} \end{bmatrix}. \tag{6.21}$$

The motion of the gear 2 is given by

$$(\widehat{\Omega})_F = \widehat{w}(\widehat{s})_F = \widehat{w} \begin{bmatrix} -\sin\phi\sinh\widehat{\xi} \\ \cos\phi\sinh\widehat{\xi} \\ \cosh\widehat{\xi} \end{bmatrix}. \tag{6.22}$$

Frame $\{\widehat{M}\}$ with base screws $\widehat{M_1}$, $\widehat{M_2}$, $\widehat{M_3}$ and origin $M$ is attached to the planet gear. We can write the transformation matrix between the frame $\{M\}$ and the fixed frame $\{F\}$ as

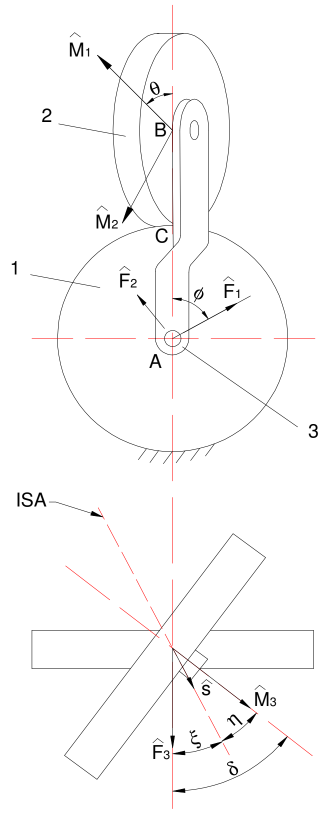


FIG. 1. Epicyclic hypoid gear train.

$$[\widehat{A}] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\widehat{\delta} & \sinh\widehat{\delta} \\ 0 & \sinh\widehat{\delta} & \cosh\widehat{\delta} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which, after expansion, yields

$$[\widehat{A}] = \begin{bmatrix} \widehat{a}_1 & \widehat{b}_1 & -\sin\phi\sinh\widehat{\delta} \\ \widehat{a}_2 & \widehat{b}_2 & \cos\phi\sinh\widehat{\delta} \\ \sin\phi\sinh\widehat{\delta} & \cos\phi\sinh\widehat{\delta} & \cosh\widehat{\delta} \end{bmatrix}, \tag{6.23}$$

where

$$\begin{aligned} \widehat{a}_1 &= \cos\phi\cos\theta - \sin\phi\sin\theta\cosh\widehat{\delta}, \\ \widehat{a}_2 &= \sin\phi\cos\theta + \cos\phi\sin\theta\cosh\widehat{\delta}, \\ \widehat{b}_1 &= -\cos\phi\sin\theta - \sin\phi\cos\theta\cosh\widehat{\delta}, \\ \widehat{b}_2 &= -\sin\phi\sin\theta + \cos\phi\cos\theta\cosh\widehat{\delta}. \end{aligned} \tag{6.24}$$

Differentiating Eq. (6.23) with respect to time we have

$$[\dot{\widehat{A}}] = \begin{bmatrix} -\dot{\phi}\widehat{a}_2 + \dot{\theta}\widehat{b}_1 & -\dot{\phi}\widehat{b}_2 - \dot{\theta}\widehat{a}_1 & \dot{\theta}\cos\theta\sinh\widehat{\delta} \\ \dot{\phi}\widehat{a}_1 + \dot{\theta}\widehat{b}_2 & \dot{\phi}\widehat{b}_1 - \dot{\theta}\widehat{a}_2 & -\dot{\theta}\sin\theta\sinh\widehat{\delta} \\ \dot{\theta}\cos\theta\sinh\widehat{\delta} & -\dot{\theta}\sin\theta\sinh\widehat{\delta} & 0 \end{bmatrix}, \tag{6.25}$$

where $\dot{\phi}$ is the angular velocity of carrier 3 and $\dot{\theta}$ is the angular velocity of the planet gear 2 relative to carrier 3.

Substituting the elements in Eqs. (6.23) and (6.25) into Eq. (3.12) we may write the velocity screw of the planet gear specified in $\{F\}$ as

$$(\widehat{\Omega})_F = \begin{bmatrix} \dot{\theta}\sin\phi\sinh\widehat{\delta} \\ -\dot{\theta}\cos\phi\sinh\widehat{\delta} \\ -\dot{\phi} - \dot{\theta}\cosh\widehat{\delta} \end{bmatrix}. \tag{6.26}$$

Equating Eqs. (6.22) and (6.26) we obtain

$$N_{12} = \frac{\dot{\theta}}{\dot{\phi}} = \frac{\sinh\xi}{\sinh\eta}, \tag{6.27}$$

$$p = \xi^\circ \coth\xi - \delta^\circ \coth\delta, \tag{6.28}$$

where $N_{12}$ is the gear ratio and $p$ is the screw pitch.

Using Eq. (6.21) for $\widehat{g}_{3F}$, and with the aid of Eqs. (4.5) and (4.7), we may construct the matrix $[\widehat{L}]$ as

$$[\widehat{L}] = \begin{bmatrix} -\cos\phi & \sin\phi\cosh\widehat{\xi} & -\sin\phi\sinh\widehat{\xi} \\ -\sin\phi & -\cos\phi\cosh\widehat{\xi} & \cos\phi\sinh\widehat{\xi} \\ 0 & -\sinh\widehat{\xi} & \cosh\widehat{\xi} \end{bmatrix}. \tag{6.29}$$

Taking the zero position at the instant when $\phi = \theta = 0$, we obtain from Eqs. (5.4), (6.23) and (6.29)

$$\begin{aligned} [\widehat{B}] &= [\widehat{S}][\widehat{L}]_0^T[\widehat{S}][\widehat{A}][\widehat{S}][\widehat{A}]_0^T[\widehat{S}][\widehat{L}]_0, \\ \widehat{\alpha}_2 &= a_2\cosh\widehat{\xi} - \sinh\widehat{\xi}\sinh\widehat{\delta}\sin\theta, \\ \widehat{\beta}_3 &= \sinh\widehat{\eta}(\cosh\widehat{\xi}\cosh\widehat{\delta} - \cos\phi\sinh\widehat{\xi}\sinh\widehat{\delta}) \\ &\qquad + \cosh\widehat{\eta}(b_2\sinh\widehat{\xi} - \cos\theta\cosh\widehat{\xi}\sinh\widehat{\delta}), \\ \widehat{\gamma}_1 &= b_1\sinh\widehat{\eta} + \sin\phi\sinh\widehat{\delta}\cosh\widehat{\eta}. \end{aligned}$$

Differentiating these equations in succession and setting $\phi = \theta = 0$, we obtain the instantaneous invariants to characterize the geometric properties of the motion of the planet gear 2 up to the third order.

## 7. Conclusion

The concepts of canonical systems and instantaneous invariants in dual Lorentzian space are presented. This gives a systematic approximation about determining the instantaneous invariants. The instantaneous invariants are derived on bases of line coordinates, which is important for the characterization of higher order intrinsic properties of a ruled surface.

As mentioned in Introduction, the use of instantaneous invariants to describe a rigid body in constrained spatial motion was introduced by Veldkamp. However, he did not provide a systematic procedure for computing the instantaneous invariants for a given rigid body motion. So, our work aims to close this gap by introducing the definition of instantaneous invariants in dual form in the sense of Lorentzian.

We hope that our work may contribute to the application of dual Lorentzian spherical motions, four-bar mechanisms, theory of mechanism synthesis for higher order approximations, gear theory and spatial mechanisms in engineering design.

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