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Cowin–Mehrabadi Theorem in six dimensions

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The Cowin–Mehrabadi Theorem concerning normals to the planes of symmetry of an anisotropic material is generalized to six dimensions. Commutation of the reflection matrix with the 6×6 matrix representing the elasticity tensor in the six-dimensional formulation of the elasticity tensor, provides the condition for the existence of a plane of symmetry. This condition implies the existence of at least two isochoric states for every class except the triclinic one. A simple proof is presented of the fact that an axis of symmetry A_n, with $n > 4$ must be an axis of isotropy.

Key words: planes of symmetry, anisotropic material, necessary and sufficient condition.

1. Introduction

Lord Kelvin's description of the properties of an elastic material in terms of eigenvalues and eigenvectors of the elasticity tensor was made in the middle of the nineteenth century [1], but it was entirely forgotten until it was independently rediscovered by Rychlewski [2] and by Mehrabadi and Cowin [3] (also see [4, 5]). The main idea of [2, 3] is to represent the elasticity tensor $\mathbf{C}$, which has rank 4 in three dimensions, by a tensor $\hat{\mathbf{c}}$ of rank 2 in six dimensions. Eigenvectors of $\hat{\mathbf{c}}$ are 6×1 column vectors but they can be interpreted as tensors of rank 2 in three dimensions. This formulation has several advantages, one of them being that the coordinate transformations of the elasticity tensor are accomplished by means of matrix multiplications and standard results of linear algebra become applicable. For example, if the elasticity tensor is invariant under a coordinate transformation, then the matrix $\hat{Q}$ corresponding to that transformation must commute with the 6×6 matrix $\hat{\mathbf{c}}$ representing the tensor. Recently Ahmad and Khan have used this fact to construct matrix representations for $\hat{\mathbf{c}}$ belonging to various symmetry classes [6]. The six-dimensional representation of the elasticity tensor has found many applications (see, for example, [8, 9], and references therein). For an orthotropic material, Blinowski and Ostrowska-Maciejewska have found expressions for the Young's modulus and Poisson's ratio in terms of eigenvalues and eigenvectors of the elasticity tensor [7]. They also found the general representation of the rotation matrix in six dimensions. In [10], MEHRABADI *et al.* have found the six-dimensional representation of the rotation in terms of the axis of rotation $\mathbf{n}$ and the angle of rotation θ and NORRIS [11] has used this representation to derive the coaxiality condition for the strain energy to be a minimum under a state of uniform stress.

A tensor of rank 2 in three dimensions is called a *pure shear* if both its trace and determinant vanish. A traceless tensor of rank 2 will be called an *isochoric* tensor. Of course, a pure shear is isochoric but the converse is not necessarily true. BLINOWSKI and RYCHLEWSKI have discussed properties of the set of pure shears in [12]. They proved the following result (Theorem 4.2, p. 489) which we can rightly name after them:

THEOREM 1 (Blinowski–Rychlewski Theorem). *An elastic material is a symmetric one only if at least two of its proper states are pure shears belonging to some subspace of shears with common direction* P_A.

In our terminology it means that the elasticity tensor for an elastic material possessing a plane of symmetry must have at least two eigenvectors which are pure shears.

In this short note, we shall use the six-dimensional formulation of the elasticity tensor to elucidate two interesting properties of elastic materials, namely:

- An eigenvector of the elasticity tensor represents a state of stress tensor which is proportional to a strain tensor. The top three components of these tensors represent the normal stresses and strains. Vanishing of the sum of normal strains implies that the rate of change of volume is zero i.e. the strain tensor represents an *isochoric* or an *equivoluminal* state. We shall show that, for all materials which possess a plane of symmetry, at least two such states of strain exist. This result is less general than Theorem 1. However, our method is capable of providing a simple proof of this theorem.
- A geometrical argument using the 'law of rational indices' establishes the result that if a crystal possesses an n-fold axis of symmetry, A_n, then n must be such that $\cos(2\pi/n)$ is a rational number [13, Chap. 2]. This allows $n = 2, 3, 4$ and 6 but forbids $n = 5$. However, this argument does not imply that an *arbitrary* rotation about the A_6 axis should leave the system invariant. On the other hand, Hermann's theorem [14] states that if a tensor of rank r possesses an axis of symmetry A_p with $p > r$, then A_p is an axis of isotropy for that tensor. Hermann's proof uses sophisticated mathematics to prove his theorem. In this note we shall give an elementary proof of the result: $\hat{\mathbf{c}}$ can have at most four distinct coaxial planes of symmetry and an n-fold axis of symmetry, A_n, with $n > 4$, must be an axis of isotropy.

To prove the above results we need to generalize the following Theorem of COWIN and MEHRABADI [15], to six dimensions.

THEOREM 2. *A set of necessary and sufficient conditions for a unit vector* $\mathbf{n}$ *to be a normal to a plane of symmetry is that it should be a common eigenvector of the following tensors:*

$$\begin{aligned} U_{ij} &= C_{ijkk}, \\ V_{ij} &= C_{ikjk}, \\ Q_{ik}(\mathbf{n}) &= C_{ijks}n_j n_s, \\ Q_{ik}(\mathbf{m}) &= C_{ijks}m_j m_s, \end{aligned}$$

where $\mathbf{m}$ *is any vector perpendicular to* $\mathbf{n}$.

2. Six-dimensional formulation

With respect to a Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, let T_{ij} and E_{ij} respectively denote the stress tensor and the strain tensor. They are related through the generalized Hooke's law

$$T_{ij} = C_{ijkl}E_{kl}. \tag{2.1}$$

Eq. (2.1) is a constitutive equation for *linear elasticity.* The general relation connecting the stress to the strain tensor can be more complicated. MEHRABADI and COWIN have defined an orthonormal basis in a six-dimensional space [3] which can be written concisely as

$$\hat{\mathbf{e}}_{\alpha(i,j)} = 2^{-1/(2-\delta_{ij})}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \tag{2.2}$$

where $\alpha(i,j) = i\delta_{ij} + (9 - i - j)$ and δ_{ij} is the Kronecker delta. Define $\hat{T}_{\alpha(i,j)} = 2^{1/(2-\delta_{ij})}T_{ij}$ and $\hat{E}_{\alpha(i,j)} = 2^{1/(2-\delta_{ij})}E_{ij}$ where i, j take values from 1 to 3 and α from 1 to 6. With respect to the basis $\{\hat{\mathbf{e}}_\alpha,\ \alpha = 1, \ldots, 6\}$, Eq. (2.1) becomes

$$\hat{T}_\alpha = \hat{c}_{\alpha\beta}\hat{E}_\beta, \qquad \alpha, \beta = 1, \ldots, 6, \tag{2.3}$$

where $\hat{c}_{\alpha\beta}$ has the matrix representation

$$\hat{c}_{\alpha\beta} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{pmatrix}, \tag{2.4}$$

where, on the right-hand side, the well-known two-index notation

$$11 \leftrightarrow 1, \quad 22 \leftrightarrow 2, \quad 33 \leftrightarrow 3, \quad 23 \leftrightarrow 4, \quad 13 \leftrightarrow 5, \quad 12 \leftrightarrow 6,$$

has been used, i.e. $c_{11} = C_{1111}$, $c_{45} = C_{2313}$, etc.

Mehrabadi and Cowin [3] have shown that if the three-dimensional basis vectors transform as

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \qquad i, j = 1, 2, 3, \tag{2.5}$$

then the basis vectors in six dimensions transform as

$$\hat{\mathbf{e}}'_\alpha = \hat{Q}_{\alpha\beta}\hat{\mathbf{e}}_\beta, \qquad \alpha, \beta = 1, \ldots, 6, \tag{2.6}$$

where the matrix representation of $\hat{Q}$ in terms of components of Q is as follows:

$$\hat{Q} = \begin{pmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & \sqrt{2}Q_{12}Q_{13} & \sqrt{2}Q_{11}Q_{13} & \sqrt{2}Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & \sqrt{2}Q_{22}Q_{23} & \sqrt{2}Q_{21}Q_{23} & \sqrt{2}Q_{22}Q_{21} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & \sqrt{2}Q_{33}Q_{32} & \sqrt{2}Q_{33}Q_{31} & \sqrt{2}Q_{31}Q_{32} \\ \sqrt{2}Q_{21}Q_{31} & \sqrt{2}Q_{22}Q_{32} & \sqrt{2}Q_{23}Q_{33} & Q_{22}Q_{33}+Q_{23}Q_{32} & Q_{21}Q_{33}+Q_{31}Q_{23} & Q_{21}Q_{32}+Q_{31}Q_{22} \\ \sqrt{2}Q_{11}Q_{31} & \sqrt{2}Q_{12}Q_{32} & \sqrt{2}Q_{13}Q_{33} & Q_{12}Q_{33}+Q_{32}Q_{13} & Q_{11}Q_{33}+Q_{13}Q_{31} & Q_{11}Q_{32}+Q_{31}Q_{12} \\ \sqrt{2}Q_{11}Q_{21} & \sqrt{2}Q_{12}Q_{22} & \sqrt{2}Q_{13}Q_{23} & Q_{12}Q_{23}+Q_{22}Q_{13} & Q_{11}Q_{23}+Q_{21}Q_{13} & Q_{11}Q_{22}+Q_{21}Q_{12} \end{pmatrix}. \tag{2.7}$$

With $\hat{Q}$ defined as above, Eq. (2.5) becomes a tensor equation in six dimensions. We shall use this fact to find a generalized version of Theorem 2.

3. Cowin–Mehrabadi Theorem in six dimensions

Consider a plane L with normal $\mathbf{n}$ and the tensor $\boldsymbol{\Omega}$ of rank two in three dimensions:

$$\Omega_{ij} = \delta_{ij} - 2n_i n_j. \tag{3.1}$$

Let $\mathbf{m}$ be a unit vector in the plane L. Since

$$\Omega_{ij}n_j = -n_i, \qquad \text{and} \qquad \Omega_{ij}m_j = m_j, \tag{3.2}$$

it follows that the transformation associated with $\boldsymbol{\Omega}$ is a reflection in the plane L. If L is a plane of symmetry for a material, then the tensor $\mathbf{C}$ must be invariant under this transformation i.e.

$$\Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{ls}C_{pqrs} = C_{ijkl}, \tag{3.3}$$

[16, Ch. 2]. The matrix form of (3.1) is

$$\boldsymbol{\Omega}(\mathbf{n}) = \begin{pmatrix} 1-2n_1^2 & -2n_1n_2 & -2n_1n_3 \\ -2n_1n_2 & 1-2n_2^2 & -2n_2n_3 \\ -2n_1n_3 & -2n_2n_3 & 1-2n_3^2 \end{pmatrix}. \tag{3.4}$$

The transformation matrix $\hat{\mathbf{N}}(\mathbf{n})$ corresponding to the reflection in plane L, in six dimensions, will be in the same relation to the matrix (3.4) as the matrix $\hat{\mathbf{Q}}$ of (2.7) is to the 3×3 matrix $\mathbf{Q}$, whose components Q_{ij} appear in Eq. (2.5). Since $\Omega_{11} = 1-2n_1^2$, $\Omega_{12} = \Omega_{21} = -2n_1n_2$, $\Omega_{22} = 1-2n_2^2$, $\Omega_{13} = \Omega_{31} = -2n_1n_3$, $\Omega_{23} = \Omega_{32} = -2n_2n_3$, $\Omega_{33} = 1-2n_3^2$, the matrix $\hat{\mathbf{N}}$ is easily obtained from $\hat{\mathbf{Q}}$ by simply replacing in (2.7) Q_{ij} by Ω_{ij}, $i, j = 1, 2, 3$. Since $\boldsymbol{\Omega}$ is symmetric, so is $\hat{\mathbf{N}}$. The orthogonality of $\boldsymbol{\Omega}$ implies orthogonality of $\hat{\mathbf{N}}$, i.e.

$$\hat{\mathbf{N}}\,\hat{\mathbf{N}}^T = \hat{\mathbf{N}}^2 = \mathbf{I}, \tag{3.5}$$

where $\mathbf{I}$ is the unit matrix in six dimensions. The condition of invariance under reflection in the plane P now becomes

$$\hat{\mathbf{N}}\,\hat{\mathbf{c}}\,\hat{\mathbf{N}}^T = \hat{\mathbf{c}}. \tag{3.6}$$

Since $\hat{\mathbf{N}}^T = \hat{\mathbf{N}}^{-1}$, the above condition becomes

$$\hat{\mathbf{N}}\,\hat{\mathbf{c}} = \hat{\mathbf{c}}\,\hat{\mathbf{N}}. \tag{3.7}$$

Therefore we have the following

THEOREM 3. *An anisotropic material has a plane of symmetry with normal* $\mathbf{n}$ *if and only if*

$$\hat{\mathbf{N}}\,\hat{\mathbf{c}} = \hat{\mathbf{c}}\,\hat{\mathbf{N}}.$$

The above Theorem is a formal extension of the Cowin–Mehrabadi Theorem [15] to six dimensions. It may, in principle, be employed to find the orientation of the normals to the planes of symmetry of an anisotropic material, but this task is more easily accomplished by using TING's results such as the following [17]:

THEOREM 4. *A necessary and sufficient condition for* $\mathbf{n}$ *to be normal to a symmetry plane is that* $\mathbf{n}$ *should be an eigenvector of* $\mathbf{U}, \mathbf{V}, \mathbf{Q}(\mathbf{n})$ *and* $\mathbf{Q}(\mathbf{m})$ *for any* $\mathbf{m}$.

4. Applications of Theorem 3

4.1. Isochoric property of eigenvectors

It is easily verified that the matrix $\hat{\mathbf{N}}$, corresponding to an arbitrary $\mathbf{n}$, has eigenvalues 1, with multiplicity four and -1 with multiplicity two. Also $X_1 =$

$(1,1,1,0,0,0)^T$ is an eigenvector belonging to 1. Denote the two eigenvectors belonging to -1 by X_2 and X_3. Since $\hat{\mathbf{N}}$ is a real symmetric matrix, X_1 is orthogonal to both X_2 and X_3. Now suppose that $\mathbf{n}$ is a normal to a plane of symmetry of $\hat{\mathbf{c}}$, hence $\hat{\mathbf{N}}(\mathbf{n})$ commutes with $\hat{\mathbf{c}}$. Consequently, the eigenspace of $\hat{\mathbf{N}}$ spanned by X_2 and X_3 will be invariant with respect to $\hat{\mathbf{c}}$ implying that if X is a vector in that eigenspace, then $\hat{\mathbf{c}}X$ will also belong to the subspace. Therefore $\hat{\mathbf{c}}$ will have two eigenvectors in the subspace [18, Ch. 2]. Since X_1 is orthogonal to every vector in the subspace, it is orthogonal to these two eigenvectors of $\hat{\mathbf{c}}$. This proves that $\hat{\mathbf{c}}$ possesses at least two eigenvectors whose top three components add up to zero.

In [3] the contribution of Lord Kelvin to eigentensors associated with elastic symmetries has been reviewed. The authors list three properties A, B, C discussed by him of these tensors to which they add a fourth property:

PROPERTY D. For any elastic symmetry, the traces of the stress and strain tensors of identical form, or the traces of the squares of the stress and strain eigentensors of identical form, are directly proportional.

We can add to this list

PROPERTY E. For any elastic symmetry, except triclinic, there are at least two states in which the mean pressure vanishes and the corresponding deformation is isochoric.

We shall now use the result of Theorem 3 to prove the Blinowski–Rychlewski Theorem. Since the trace and the determinant of a second rank tensor are invariants, we can choose a coordinate system to simplify (3.4) and $\hat{\mathbf{N}}$ as a consequence. To this end, choose x_3-axis along $\mathbf{n}$, the normal to a plane of symmetry. Now

$$\mathbf{\Omega}(\mathbf{n}) = \mathrm{diag}(1,1,-1),$$

and

$$\hat{\mathbf{N}}(\mathbf{n}) = \mathrm{diag}(1,1,1,-1,-1,1). \tag{4.1}$$

The matrix has a two-fold eigenvalue -1 with eigenvectors $X_4 = (0,0,0,1,0,0)^T$ and $X_5 = (0,0,0,0,1,0)^T$. Since $\mathbf{n} = (0,0,1)^T$ is assumed to be a normal to a plane of symmetry, $\hat{\mathbf{N}}(\mathbf{n})$ commutes with $\hat{\mathbf{c}}$, therefore $\hat{\mathbf{c}}$ must have *two* eigenvectors in the eigenspace spanned by X_4 and X_5, i.e. eigenvectors of the form $(0,0,0,a,b,0)^T$ and $(0,0,0,b,-a,0)^T$ for some a, b. Both of these states are pure shears. This proves the Theorem.

4.2. An axis of symmetry A_n, with $n > 4$ implies isotropy

Suppose that a crystal possesses an n-fold axis of symmetry A_n. If $n \geq 3$, there are n coaxial planes of symmetry with A_n as the common axis [19, Ch. 4].

The normals to these planes will all lie in the plane perpendicular to A_n. Let X_1 and X_2 be arbitrary but independent vectors in this plane. Define the vector

$$X(\theta) = X_1 \cos\theta + X_2 \sin\theta. \tag{4.1}$$

We can, in principle, find the reflection matrix $\hat{\mathbf{N}}(X)$, the commutator $W(\theta) = \hat{\mathbf{N}}(X)\hat{\mathbf{c}} - \hat{\mathbf{c}}\,\hat{\mathbf{N}}(X)$ and the function $f(\theta) = \mathrm{Tr}[W^T(\theta)W(\theta)]$. It is clear that $f(\theta)$ will be of the form

$$f(\theta) = a_0 + a_2\cos(2\theta + \alpha_2) + a_4\cos(4\theta + \alpha_4) + a_6\cos(6\theta + \alpha_6) + a_8\cos(8\theta + \alpha_8), \tag{4.2}$$

where $a_0, \ldots, a_8$ and $\alpha_2, \ldots, \alpha_8$ are constants depending on the components of $\hat{\mathbf{c}}$ and the vectors X_1 and X_2. The zeros of $f(\theta)$ in $[0, \pi)$ will determine the normals to the planes of symmetry. Now let $n = 3$ and rotate the crystal about the A_3 axis through an angle $\pi/3$. The system is invariant with respect to this rotation but θ in (4.1) is replaced by $\theta - \pi/3$. This requires

$$f(\theta) = f(\theta - \pi/3),$$

which is possible only if $a_2 = a_4 = a_8 = 0$. Thus $f(\theta)$ reduces to

$$f(\theta) = a_0 + a_6\cos(6\theta + \alpha_6). \tag{4.3}$$

A similar reasoning for $n = 4$ will reduce (4.2) to

$$f(\theta) = a_0 + a_8\cos(8\theta + \alpha_8). \tag{4.4}$$

Continuing with this line of argument, it becomes clear that $n \geq 5$ will require $f(\theta)$ to vanish *identically*, i.e. $f(\theta) = 0$ is satisfied for all θ. Hence A_n, for $n > 4$, is an axis of transverse isotropy.

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