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Wiener–Hopf analysis of diffraction of acoustic waves by a soft/hard half-plane

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In this paper, firstly, the far field due to a line source scattering of acoustic waves by a soft/hard half-plane is investigated. It is observed that if the line source is shifted to a large distance, the results differ from those of [16] by a multiplicative factor. Subsequently, the scattering due to a point source is also examined using the results of line source excitation. Both the problems are solved using the Wiener–Hopf technique and the steepest descent method. Some graphs showing the effects of various parameters on the diffracted field produced by the line source incidence are also plotted.

Key words: diffraction, Wiener–Hopf technique, line source diffraction, point source diffraction, steepest descent method.

1. Introduction

Numerous former investigations have been devoted to the study of classical problems of line source and point source diffractions of electromagnetic and acoustic waves by various types of half-planes. To name a few only, e.g. the line source diffraction of electromagnetic waves by a perfectly conducting half-plane was investigated by Jones [1]. Later on, Jones [2] considered the problem of line source diffraction of acoustic waves by a hard half-plane attached to a wake in still air as well as when the medium is convective. Rawlins [3] studied the diffraction of cylinderical waves from a line source by an absorbing half-plane in the presence of subsonic flow. Ahmad [4] considered the line source diffraction of acoustic waves by an absorbing half-plane using Myre's condition. Hussain [5] analyzed the line source diffraction of electromagnetic waves by a perfectly conducting half-plane in a bi-isotropic medium. Recently Ayub *et al.* [6] studied the magnetic line source difffraction of electromagnetic waves by an impedance step.

The phenomenon of point scource diffraction has also been investigated by various authors, e.g., VLAAR [7], BALASUBRAMANYAM [8], CHATTOPADHYAY *et al.* [9] and AYUB *et al.* [10–12]. This consideration is important because a point scource is regarded as a fundamental radiating device [13] and the solutions of the point source problems are called fundamental solutions of the given differential equation [14]. Keeping in view the studies mentioned above, in this paper we have studied two problems of line scource and point scource diffraction of acoustic waves by a soft/hard half-plane.

The continued interest in the problem of diffraction from a surface satisfying soft (pressure release) boundary condition on upper side and hard(rigid) boundary condition on the lower side, is due to the fact that it constitutes the simplest half-plane problem that can be formulated as a system of coupled Wiener–Hopf (WH) equations that cannot be decoupled trivially. RAWLINS [15] pointed out that two unusual features arose in this boundary value problem and adopted an ad-hoc method for the solution of this problem. After the lapse of many years, BÜYÜKAKSOY [16] not only reconsidered the problem solved by [15] but also proposed an appropriate method for the solution of the said boundary value problem. The Wiener–Hopf technique provides a powerful approach for considering diffraction of waves by a single half-plane [17]. This was fairly extended to diffraction by two parallel plates. However, there were problems in dealing with other configurations and mixed boundary value problems which were first attacked using the matrix formulation of the WH equations. A comprehensive procedure for tackling the matrix version of these equations is not yet available. The matrix appearing in the present problem is the same as in [16] and has been factorized by BÜYÜKAKSOY [16] by using the DANIELE–KHARAPKOV methods [18, 19]. A detailed survey of matrix factorization methods may also be found in the paper of BÜYÜKAKSOY and SERBEST [20]. Diffraction from a two-part surface is an important topic in diffraction theory and it constitutes a canonical boundary value problem for diffraction because of abrupt changes in the material properties [21]. BÜYÜKAKSOY [16] has considered the diffraction of plane waves by a soft/hard surface. Using the Jones method [17], a beautiful account of which is also given in his book [13], we have extended the analysis of [16] to the case of line source and point source diffraction of waves by a soft/hard half-plane.

The introduction of line source changes the incident field and the method of solution requires a careful analysis in calculating the diffracted field. The consideration of point source diffraction will help us to understand acoustic diffraction and will go a step further to complete the discussion for the soft/hard half-plane. The mathematical importance of this work lies in the fact that the introduction of point source introduces another variable. The difficulty that arises in the solution of the integrals occur in the inverse transform. These integrals are normally difficult to handle because of the presence of the branch points and are only amenable to solution using asymptotic approximations. The analytic solution of these integrals is obtained and the far-field solution is presented.

To the best of our knowledge, no attempt concerning the line source and point source diffraction of acoustic waves from a soft/hard half-plane is available in the literature. Finally, the results of plane wave situation [16] can be achieved when the line source is removed to infinity, which agrees well with the already known results [1, 10, 11]. Some graphs showing the variation of various parameters on the diffracted field have also been plotted.

2. The line source diffraction

2.1. Formulation of the problem

Consider the scattering of an acoustic wave due to a line source located at the position (x_0, y_0) by a soft/hard half-plane located at $x > 0$, $y = 0$, so that the edges lie along the z-axis. Thus we can assume that the field is independent of the z-axis, and let θ_0 be the angle of incidence. The geometry of the problem is depicted in Fig. 1.

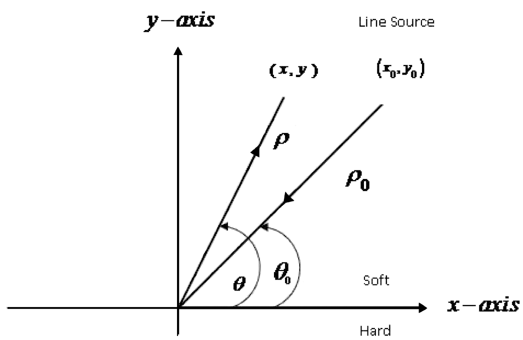

FIG. 1. Geometry of the problem.

For harmonic acoustic vibrations of the time dependence $e^{-i\omega t}$, which are assumed and suppressed, we require a solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi_t(x, y) = \delta(x - x_0)\delta(y - y_0), \tag{2.1}$$

where ψ_t is the total velocity potential, and the boundary and continuity conditions are given as follows:

$$\psi_t(x,0^+) = 0, \qquad x > 0, \tag{2.2}$$

$$\frac{\partial \psi_t(x,0^-)}{\partial y} = 0, \qquad x > 0, \tag{2.3}$$

$$\psi_t(x,0^+) = \psi_t(x,0^-), \qquad x < 0, \tag{2.4}$$

and

$$\frac{\partial \psi_t(x,0^+)}{\partial y} = \frac{\partial \psi_t(x,0^-)}{\partial y}, \qquad x < 0. \tag{2.5}$$

For a unique solution of the problem, it is required that the radiation condition

$$\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\psi_t \to 0 \qquad \text{as } r = (x^2+y^2)^{1/2} \to \infty, \tag{2.6}$$

must be satisfied. Following [2, 3], for the analysis purpose, it is convevient to express the total field for the line source incidence as

$$\psi_t\,(x,y) = \begin{cases} \psi_i(x,y) + \psi(x,y) & y > 0, \\ \psi(x,y) & y < 0, \end{cases} \tag{2.7}$$

where $\psi_i(x,y)$ accounts for the inhomogeneous source term and $\psi(x,y)$ represents the diffracted field. In Eq. (2.7), $\psi_i(x,y)$ is the incident field satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi_i(x,y) = \delta(x-x_0)\delta(y-y_0), \tag{2.8}$$

and the diffracted field $\psi(x,y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi(x,y) = 0. \tag{2.9}$$

For analytic convenience we shall assume that the wave number k has small imaginary part for which $k = k_r + ik_i$, where k_r and k_i are both positive and $k_i \to 0^+$ is the loss factor of the medium. The appropriate Fourier transform pair is

$$\widehat{\psi}(\alpha,y) = \int_{-\infty}^{\infty} \psi(x,y)e^{i\alpha x}dx, \tag*{(2.10)$_1$}$$

and

$$\psi\,(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\psi}\,(\alpha,y)\,e^{-i\alpha x}d\alpha. \tag*{(2.10)$_2$}$$

Using Eq. $(2.10)_1$, the solution of inhomogeneous Eq. (2.8) can be written as [17]

$$\widehat{\psi}_i(\alpha, y) = \frac{1}{2iK(\alpha)} e^{i\alpha x_0 + iK(\alpha)|y-y_0|}, \tag{2.11}$$

where $K(\alpha) = \sqrt{k^2 - \alpha^2}$. Defining $K(\alpha)$, the square root function, should be that branch which reduces to $+k$ when $\alpha = 0$ when the complex plane is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$.

For the diffracted field, the solution of homogeneous Eq. (2.9) satisfying the radiation condition can formally be written as

$$\widehat{\psi}(\alpha, y) = \begin{cases} A_1(\alpha) e^{iK(\alpha)y} & y > 0, \\ A_2(\alpha) e^{-iK(\alpha)y} & y < 0, \end{cases} \tag{2.12}$$

where $A_1(\alpha)$ and $A_2(\alpha)$ are the unknown coeffecients to be determined.

Taking the Fourier transform of the boundary and continuity conditions (2.2)–(2.5), we arrive at

$$\widehat{\psi}_+(\alpha, 0^+) = -\int_0^\infty \psi_i(x, 0^+) e^{i\alpha x} dx, \tag{2.13}$$

$$\frac{\partial \widehat{\psi}_+(\alpha, 0^-)}{\partial y} = -\int_0^\infty \frac{\partial \psi_i(x, 0^-)}{\partial y} e^{i\alpha x} dx, \tag{2.14}$$

$$\widehat{\psi}_-(\alpha, 0^+) = \widehat{\psi}_-(\alpha, 0^-), \tag{2.15}$$

$$\frac{\partial \widehat{\psi}_-(\alpha, 0^+)}{\partial y} = \frac{\partial \widehat{\psi}_-(\alpha, 0^-)}{\partial y}. \tag{2.16}$$

For a unique solution of the problem, the edge conditions require that ψ_t and its normal derivative must be bounded near $x = 0$, and these must be of the following orders [16]:

$$\begin{aligned} \psi_t(x, 0) &= -1 + O(x^{1/4}), && x \to 0, \\ \frac{\partial \psi_t(x, 0)}{\partial y} &= O(x^{-3/4}), && x \to 0. \end{aligned} \tag{2.17}$$

The substitution of solution (2.12) into boundary and continuity conditions (2.13)–(2.16) will yield

$$A_1(\alpha) = -\int_0^\infty \psi_i(x, 0^+) e^{i\alpha x} dx + \widehat{\psi}_{-1}(\alpha), \tag{2.18}$$

$$K(\alpha)A_2(\alpha) = -i\int_0^{\infty} \frac{\partial \psi_i(x,0^-)}{\partial y} e^{i\alpha x} dx + \widehat{\psi}_{-2}(\alpha), \tag{2.19}$$

$$A_1(\alpha) - A_2(\alpha) = \widehat{\psi}_{+1}(\alpha), \tag{2.20}$$

$$A_1(\alpha) + A_2(\alpha) = \frac{\widehat{\psi}_{+2}(\alpha)}{K(\alpha)}, \tag{2.21}$$

where $\widehat{\psi}_{-\ 1,2}(\alpha)$ and $\widehat{\psi}_{+\ 1,2}(\alpha)$ are defined as follows:

$$\widehat{\psi}_{-1}(\alpha) = \int_{-\infty}^{0} \psi(x,0^+) e^{i\alpha x} dx, \tag{2.22}$$

$$\widehat{\psi}_{-2}(\alpha) = i\int_{-\infty}^{0} \frac{\partial \psi(x,0^-)}{\partial y} e^{i\alpha x} dx, \tag{2.23}$$

$$\widehat{\psi}_{+1}(\alpha) = \int_0^{\infty} [\psi(x,0^+) - \psi(x,0^-)] e^{i\alpha x} dx, \tag{2.24}$$

$$\widehat{\psi}_{+2}(\alpha)(\alpha) = -i\int_0^{\infty} \left[\frac{\partial \psi(x,0^+)}{\partial y} - \frac{\partial \psi(x,0^-)}{\partial y}\right] e^{i\alpha x} dx. \tag{2.25}$$

Due to the analytical properties of the Fourier integrals, $\widehat{\psi}_{-1,2}(\alpha)$ and $\widehat{\psi}_{+1,2}(\alpha)$ are regular functions of α in the half-planes $\operatorname{Im}\alpha > \operatorname{Im} k\cos\theta_0$ and $\operatorname{Im}\alpha < \operatorname{Im} k$, respectively. By using the edge conditions in (2.17), we can show that when $|\alpha| \to \infty$ in their respective regions of regularity, we have:

$$\widehat{\psi}_{-1}(\alpha) = -\frac{1}{i\alpha} + O(\alpha^{-5/4}), \tag{2.26}$$

$$\widehat{\psi}_{+1}(\alpha) = O(\alpha^{-5/4}), \tag{2.27}$$

$$\widehat{\psi}_{\pm 2}(\alpha) = O(\alpha^{-1/4}). \tag{2.28}$$

The elimination of $A_1(\alpha)$ and $A_2(\alpha)$ between Eqs. (2.18)–(2.21) leads to the following matrix Wiener–Hopf equation:

$$\mathbf{G}(\alpha)\Psi_+(\alpha) = 2\Psi_-(\alpha) - 2\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}, \tag{2.29}$$

where $\mathbf{G}(\alpha)$, $\Psi_\pm(\alpha)$, $q(\alpha)$ and $r(\alpha)$ are given as follows:

$$\mathbf{G}(\alpha) = \begin{bmatrix} 1 & 1/K(\alpha) \\ -K(\alpha) & 1 \end{bmatrix}, \tag{2.30}$$

$$\Psi_\pm(\alpha) = \begin{bmatrix} \widehat{\psi}_{\pm 1}(\alpha) \\ \widehat{\psi}_{\pm 2}(\alpha) \end{bmatrix}, \tag{2.31}$$

$$q(\alpha) = \int_0^\infty \psi_i(x, 0^+) e^{i\alpha x} dx, \tag{2.32}$$

$$r(\alpha) = i \int_0^\infty \frac{\partial \psi_i(x, 0^-)}{\partial y} e^{i\alpha x} dx. \tag{2.33}$$

2.2. Solution of the matrix Wiener–Hopf equation

Incidently, the kernel matrix $\mathbf{G}(\alpha)$, which can be written as

$$\mathbf{G}(\alpha) = \mathbf{I} + \frac{1}{K(\alpha)} \begin{bmatrix} 0 & 1 \\ -(k^2 - \alpha^2) & 0 \end{bmatrix}, \tag{2.34}$$

is the same as in [16], where $\mathbf{I}$ is the unit matrix. The matrix $\mathbf{G}(\alpha)$ has been factorized by BÜYÜKAKSOY [16] by using the KHARAPKOV method [19] and the result is as follows:

$$\mathbf{G}_+(\alpha) = 2^{1/4} \begin{bmatrix} \cosh \chi(\alpha) & \sinh \chi(\alpha)/\gamma(\alpha) \\ \gamma(\alpha) \sinh \chi(\alpha) & \cosh \chi(\alpha) \end{bmatrix}, \tag{$(2.35)_1$}$$

with

$$\mathbf{G}_-(\alpha) = \mathbf{G}_+(-\alpha), \tag{$(2.35)_2$}$$

where

$$\chi(\alpha) = -\frac{i}{4} \arccos \frac{\alpha}{k}, \qquad \chi(-\alpha) = -\frac{i}{4} \left[\pi - \arccos \frac{\alpha}{k}\right], \tag{$(2.35)_3$}$$

and

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2}. \tag{$(2.35)_4$}$$

Also as $|\alpha| \to \infty$, we note that

$$\mathbf{G}_\pm(\alpha) \sim O(4k)^{-1/4} \begin{bmatrix} (\pm\alpha)^{1/4} & (\pm\alpha)^{-3/4} \\ (\pm\alpha)^{5/4} & (\pm\alpha)^{1/4} \end{bmatrix}. \tag{2.36}$$

Using the factorization of the kernel matrix, Eq. (2.29) can be rearranged as

$$\mathbf{G}_+ \Psi_+(\alpha) = 2 \left[\mathbf{G}_-(\alpha)\right]^{-1} \Psi_-(\alpha) - 2 \left[\mathbf{G}_-(\alpha)\right]^{-1} \begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}. \tag{2.37}$$

Equation (2.37) is the matrix Wiener–Hopf equation. To make it regular in the upper and lower half-planes we need to split the term

$$[\mathbf{G}_-(\alpha)]^{-1}\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix}.$$

To achieve this end, we shall apply the additive decomposition theorem [17], which results in

$$[\mathbf{G}_-(\alpha)]^{-1}\begin{bmatrix} q(\alpha) \\ r(\alpha) \end{bmatrix} = \begin{bmatrix} T(\alpha) \\ S(\alpha) \end{bmatrix} = \begin{bmatrix} T_+ + T_- \\ S_+ + S_- \end{bmatrix} = \begin{bmatrix} T_+ \\ S_+ \end{bmatrix} + \begin{bmatrix} T_- \\ S_- \end{bmatrix}, \tag{2.38}$$

where

$$\begin{aligned} T_\pm(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{T(\xi)}{(\xi-\alpha)}d\xi \\ &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{2^{-\frac{1}{4}}[\cosh\chi(-\xi)q(\xi) - r(\xi)\sinh\chi(-\xi)/\gamma(-\xi)]}{(\xi-\alpha)}d\xi, \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} S_\pm(\alpha) &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{S(\xi)}{(\xi-\alpha)}d\xi \\ &= \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{2^{-\frac{1}{4}}[-q(\xi)\gamma(-\xi)\sinh\chi(-\xi) + r(\xi)\cosh\chi(-\xi)]}{(\xi-\alpha)}d\xi. \end{aligned} \tag{2.40}$$

Hence Eq. (2.37) can finally be arranged as follows:

$$\mathbf{G}_+\Psi_+(\alpha) + 2\begin{bmatrix} T_+(\alpha) \\ S_+(\alpha) \end{bmatrix} = 2\,[\mathbf{G}_-(\alpha)]^{-1}\,\Psi_-(\alpha) - 2\begin{bmatrix} T_-(\alpha) \\ S_-(\alpha) \end{bmatrix}. \tag{2.41}$$

The left-hand side of Eq. (2.41) is regular in the upper half-plane $\mathrm{Im}(\alpha) > \mathrm{Im}(k\cos\theta_0)$ and the right-hand side is regular in the lower half-plane $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$, and hence by the analytic continuation principle they define an entire matrix-valued function $\mathbf{P}^*(\alpha)$. By taking into account the order relations (2.26)–(2.28), Eqs. $(2.35)_2$ and (2.36), we conclude from the extended Liouville's theorem that $\mathbf{P}^*$ is a constant matrix of the form

$$\mathbf{P}^* = p^*\begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{2.42}$$

Thus the solution of Eq. (2.41) becomes

$$[\mathbf{G}_-(\alpha)]^{-1}\Psi_-(\alpha)-\begin{bmatrix}T_-(\alpha)\\ S_-(\alpha)\end{bmatrix}=\begin{bmatrix}0\\ p^*\end{bmatrix}, \tag{2.43}$$

where the unknown constant p^* can be determined as follows.

From Eq. (2.43), we obtain

$$\Psi_-(\alpha)=\mathbf{G}_-(\alpha)\begin{bmatrix}T_-(\alpha)\\ p^*+S_-(\alpha)\end{bmatrix}. \tag{2.44}$$

By considering the order relation in Eq. (2.36), the unknown constant p^* can be specified with the help of Eq. (2.44) by following [21],

$$\begin{bmatrix}\widehat{\psi}_{-1}(\alpha)\\ \widehat{\psi}_{-2}(\alpha)\end{bmatrix}\approx(4k)^{-\frac{1}{4}}\left[p^*-\widetilde{T}_-\right]\begin{bmatrix}(-\alpha)^{-\frac{3}{4}}\\ (-\alpha)^{-\frac{1}{4}}\end{bmatrix}+O\begin{bmatrix}(-\alpha)^{-\frac{5}{4}}\\ (-\alpha)^{-\frac{1}{4}}\end{bmatrix} \tag{2.45}$$

with

$$\widetilde{T}_-=\lim_{\alpha\to\infty}\alpha T_-. \tag{2.46}$$

The correct behaviour of $\widehat{\psi}_{-1}(\alpha)$ and $\widehat{\psi}_{-2}(\alpha)$ is recovered if we choose

$$p^*-\widetilde{T}_-=0. \tag{2.47}$$

Hence the expressions, from Eq. (2.44) for $\widehat{\psi}_{-1}(\alpha)$ and $\widehat{\psi}_{-2}(\alpha)$, are given as

$$\begin{aligned}\widehat{\psi}_{-1}(\alpha)&\\=&\left[\cosh\chi(-\alpha)\left\{-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[\cosh\chi(-\xi)q(\xi)-r(\xi)\sinh\chi(-\xi)/\gamma(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right.\\&\left.+\frac{\sinh\chi(-\alpha)}{\gamma(-\alpha)}\left\{2^{\frac{1}{4}}p^*-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[-q(\xi)\gamma(-\xi)\sinh\chi(-\xi)+r(\xi)\cosh\chi(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right],\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}\widehat{\psi}_{-2}(\alpha)&\\=&\left[\gamma(-\alpha)\sinh\chi(-\alpha)\left\{-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[\cosh\chi(-\xi)q(\xi)-r(\xi)\sinh\chi(-\xi)/\gamma(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right.\\&\left.+\cosh\chi(-\alpha)\left\{2^{\frac{1}{4}}p^*-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{[-q(\xi)\gamma(-\xi)\sinh\chi(-\xi)+r(\xi)\cosh\chi(-\xi)]}{(\xi-\alpha)}d\xi\right\}\right].\end{aligned} \tag{2.49}$$

2.3. The far-field solution

Now by subsituting Eq. (2.48) into Eq. (2.18) and then the resulting equation in Eq. (2.12) and taking the inverse Fourier transform, we obtain for $y > 0$:

$$\begin{aligned}(2.50)\qquad & \psi(x,y) \\ = & \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Bigg\{\Bigg[\cosh\chi(-\alpha)\Bigg\{-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[\cosh\chi(-\xi)q(\xi)-r(\xi)\sinh\chi(-\xi)/\gamma(-\xi)]}{(\xi-\alpha)}d\xi\Bigg\} \\ & +\frac{\sinh\chi(-\alpha)}{\gamma(-\alpha)}\Bigg\{2^{\frac{1}{4}}p^{*}-\frac{1}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{[-q(\xi)\gamma(-\xi)\sinh\chi(-\xi)+r(\xi)\cosh\chi(-\xi)]}{(\xi-\alpha)}d\xi\Bigg\}\Bigg]\Bigg\} \\ & \times e^{iK(\alpha)y-i\alpha x}d\alpha-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Bigg\{\int\limits_{0}^{\infty}\psi_i(x,0^{+})e^{i\alpha x}dx\Bigg\}e^{iK(\alpha)y-i\alpha x}d\alpha.\end{aligned}$$

To determine the far-field behaviour of the diffracted field we introduce the following substitutions:

$$x=\rho\cos\theta,\qquad y=\rho\sin\theta\qquad(0<\theta<\pi), \tag{2.51}$$

$$x_0=\rho_0\cos\theta_0,\qquad y_0=\rho_0\sin\theta_0\qquad(\pi<\theta_0<0), \tag{2.52}$$

and the transformation

$$\alpha=-k\cos(\theta+i\zeta), \tag{2.53}$$

where ζ, given in Eq. (2.53), is real. The contour of integration over α in Eq. (2.50) goes into the branch of hyperbola around $-ik$ if $\pi/2<\theta<\pi$. We further observe that, in deforming the contour into a hyperbola, the pole $\alpha=\xi$ may be crossed. If we also make the transformation $\xi=k\cos(\theta_0+i\tau_1)$, the contour over ξ also transforms into a hyperbola. The two hyperbolae will not cross each other if $\theta<\theta_0$. However, if the inequality is reversed, there will be a contribution from the pole which, in fact, cancels the incident wave in the shadow region [2]. The explicit expression for the unknown constant p^{*} is determined with the help of Eqs. (2.40), (2.46) and (2.47) and it comes out to be

$$p^{*}=\frac{2^{\frac{1}{4}}\sin\dfrac{\theta_0}{4}}{2\pi i}\frac{e^{ik\rho_0+\frac{i\pi}{4}}}{\sqrt{2\pi k\rho_0}}. \tag{2.54}$$

Omitting the details of calculations, using the method of steepest descent [13] the field due to a line source at a large distance from the plate for $y>0$ is given as

$$(2.55)_1 \qquad \psi(\rho,\theta) \approx \frac{i}{4\pi^2}\left\{ \sin\frac{\theta_0}{4}\sin\frac{\theta}{4} - \frac{\cos\frac{\theta}{4}\sin\frac{\theta_0}{4}\sin\theta + \sin\frac{\theta}{4}\cos\frac{\theta_0}{4}\sin\theta_0}{\cos\theta+\cos\theta_0}\right\}\frac{\sqrt{2}e^{ik(\rho+\rho_0)}}{k\sqrt{\rho\rho_0}},$$

which after some trigonometric simplification reduces to

$$(2.55)_2 \qquad \psi(\rho,\theta) \approx -\frac{e^{\frac{i\pi}{2}}}{\sqrt{2}\pi^2}\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\frac{e^{ik(\rho+\rho_0)}}{k\sqrt{\rho\rho_0}}.$$

3. The point source scattering

3.1. Mathematical formulation

For the case of point source scattering we suppose that a point source is occupying the position (x_0, y_0, z_0). Thus for harmonic time variation e^{-iwt}, we require the solution of the equation

$$(3.1) \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi_t(x,y,z) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0),$$

subject to the following boundary conditions, for $x > 0$:

$$(3.2) \qquad \Phi_t(x,0^+,z) = 0, \qquad x>0, \qquad -\infty<z<\infty,$$

$$(3.3) \qquad \frac{\partial\Phi_t(x,0^-,z)}{\partial y} = 0, \qquad x>0, \qquad -\infty<z<\infty,$$

and for $x < 0$:

$$(3.4) \qquad \Phi_t(x,0^+,z) = \Phi_t(x,0^-,z), \qquad x<0, \qquad -\infty<z<\infty,$$

$$(3.5) \qquad \frac{\partial\Phi_t(x,0^+,z)}{\partial y} = \frac{\partial\Phi_t(x,0^-,z)}{\partial y}, \qquad x<0, \qquad -\infty<z<\infty,$$

where Φ_t is the total acoustic field, defined as

$$(3.6) \qquad \Phi_t(x,y,z) = \Phi_0(x,y,z) + \Phi(x,y,z),$$

where Φ is the scattered field and Φ_0 represents the effect due to a point source.

Let us define the Fourier transform and the inverse Fourier transform with respect to the variable z as follows

$$\bar{\Phi}(x,y,\mu) = \int_{-\infty}^{\infty} \Phi(x,y,z)e^{ik\mu z}dz,$$
$$\Phi(x,y,z) = \frac{k}{2\pi}\int_{-\infty}^{\infty} \bar{\Phi}(x,y,\mu)e^{-ik\mu z}d\mu. \tag{3.7}$$

Taking the Fourier transform of the Eqs. (3.1) to (3.5), the problem with boundary conditions in the transformed domain μ takes the following form:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\eta^2\right)\bar{\Phi}_t = a\delta(x-x_0)\delta(y-y_0), \tag{3.8}$$

with $\eta = \sqrt{1-\mu^2}$, and $a = e^{ik\mu z_0}$.

The transformed boundary conditions take the form

$$\bar{\Phi}_t(x,0^+,\mu) = 0, \qquad x > 0, \tag{3.9}$$

$$\frac{\partial\bar{\Phi}_t(x,0^-,\mu)}{\partial y} = 0, \qquad x > 0, \tag{3.10}$$

$$\bar{\Phi}_t(x,0^+,\mu) = \bar{\Phi}_t(x,0^-,\mu), \qquad x < 0, \tag{3.11}$$

$$\frac{\partial\bar{\Phi}_t(x,0^+,\mu)}{\partial y} = \frac{\partial\bar{\Phi}_t(x,0^-,\mu)}{\partial y}, \qquad x < 0. \tag{3.12}$$

Thus, we see that the problem (3.8) together with the boundary conditions (3.9)–(3.12) in the transformed domain μ is the same as in the case of two dimensions, formulated in the Sec. 2 except that $k^2\eta^2$ replaces k^2 [8, 10, 11].

3.2. Solution of the problem

As mentioned before, the mathematical problem (3.8) together with the boundary conditions (3.9)–(3.12) in the transformed domain μ is the same as in the case of two dimensions, formulated in the Sec. 2 except that $k^2\eta^2$ replaces k^2. Thus by making use of the the solution obtained in Sec. 2, we calculate the diffracted field due to a point source as follows.

For $y > 0$, we have

$$\bar{\Phi}(\rho,\theta,\mu) \approx -\frac{e^{\frac{i\pi}{2}}}{\pi^2}\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\frac{e^{ik\eta(\rho+\rho_0)+ik\mu z_0}}{k\eta\sqrt{2\rho\rho_0}}. \tag{3.13}$$

The scattered field in the spatial domain can now be obtained by taking the inverse Fourier transform of Eq. (3.13). Thus, for $y > 0$, we have

$$(3.14)\qquad \Phi(\rho,\theta,z) \approx \frac{k}{2\pi}\int_{-\infty}^{\infty}\left\{-\frac{e^{\frac{i\pi}{2}}}{\pi^2}\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\right\}\frac{e^{ik\eta(\rho+\rho_0)-ik\mu(z-z_0)}}{k\eta\sqrt{2\rho\rho_0}}\,d\mu.$$

In order to solve the problem completely, we introduce the following substitutions [10, 11] in the Eq. (3.14):

$$(3.15)\qquad \mu = \cos\beta, \qquad \eta = \sqrt{1-\mu^2} = \sin\beta,$$

$$(3.16)\qquad \rho+\rho_0 = R_1\sin\nu, \qquad z-z_0 = R_1\cos\nu,$$

$$(3.17)\qquad R_1 = \sqrt{(z-z_0)^2+(\rho+\rho_0)^2}.$$

Using the method of steepest descent [13], the integral appearing in Eq. (3.14) can be evaluated asymptotically for large kR_1. The contour of integration is taken such that it passes through the point of steepest descent $\beta = \nu$. Therefore, for $kR_1 \gg 1$, omitting the details of calculations, the final form of field for $y > 0$ is given as follows.

For $y > 0$,

$$(3.18)\qquad \Phi(\rho,\theta,z) \approx \frac{1}{2\pi^2}\left[\left(\frac{\sin\frac{\theta_0}{4}\sin\frac{\theta}{4}}{\cos\theta+\cos\theta_0}\right)\left(1+\cos\frac{\theta}{2}+\cos\frac{\theta_0}{2}\right)\right]\frac{e^{-ikR_1+i\frac{\pi}{4}}}{\sqrt{\pi kR_1\rho\rho_0}}.$$

4. Graphical results and discussions

In this section we shall present some graphical results showing the effect of some dimensionless parameters such as the observer distance from the origin $k\rho$, source distance from the origin $k\rho_0$ and the observation angle θ_0 at the diffracted field ψ produced by the line source. Figures 2–4 show the variation of the parameter $k\rho_0$ by fixing $\theta_0 = \pi/2$ and $k\rho = 1, 2$ and 3, respectively. It is observed that by increasing the parameter $k\rho_0$ the magnitude of the diffracted field decreases. Figures 5–7 are plotted to note the variation of the parameter $k\rho$ on the diffracted field. It is observed that the diffracted field decreases by increasing the parameter $k\rho$ and fixing the other parameters to be $\theta_0 = \pi/2$ and $k\rho_0 = 0.01$, 0.05 and 1, respectively, but the field lines are almost very close in these figures as compared to the Fig. 2–4. Figures 8–10 depict the variation of the observation angle θ_0 on the diffracted field. It is observed that the highest curve corresponds to the normal incidence and the magnitude of the diffracted field decreases and the peaks shift toward right as the angle of incidence decreases, which is as expected.

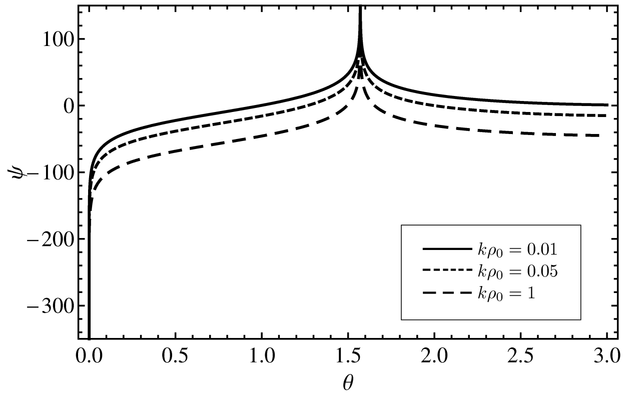

Fig. 2. Plots of ψ Vs θ for $\rho k = 1$ and $\theta_0 = \pi/2$.

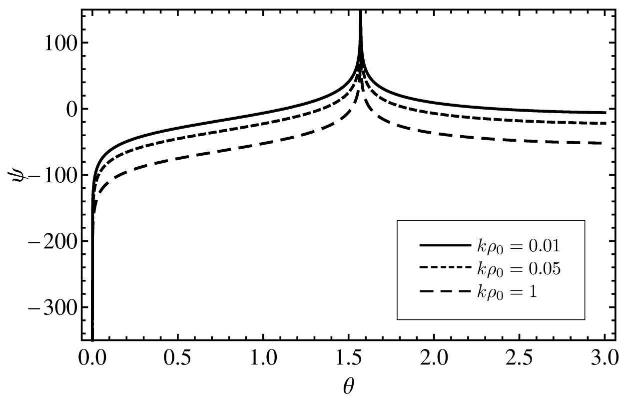

Fig. 3. Plots of ψ Vs θ for $\rho k = 2$ and $\theta_0 = \pi/2$.

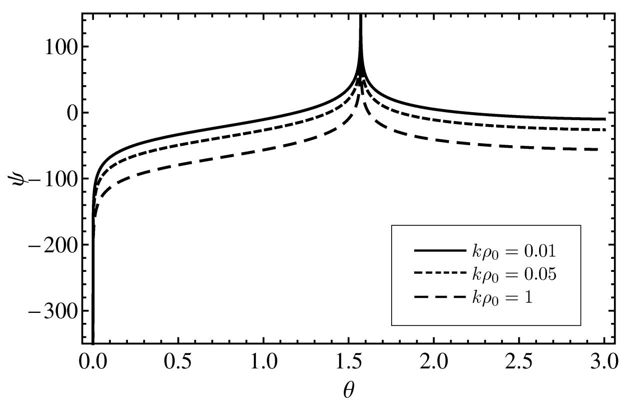

Fig. 4. Plots of ψ Vs θ for $\rho k = 3$ and $\theta_0 = \pi/2$.

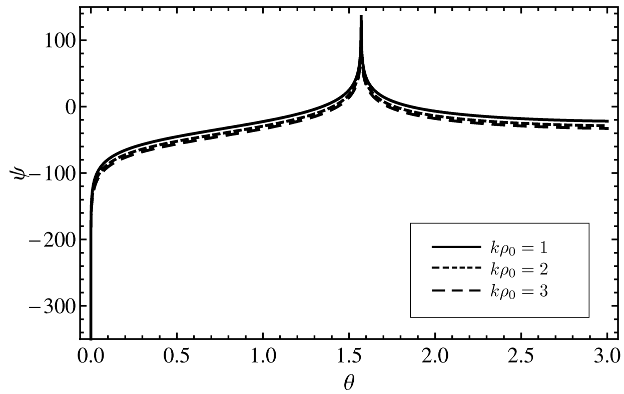

FIG. 5. Plots of ψ Vs θ for $\rho_0 k = 0.01$ and $\theta_0 = \pi/2$.

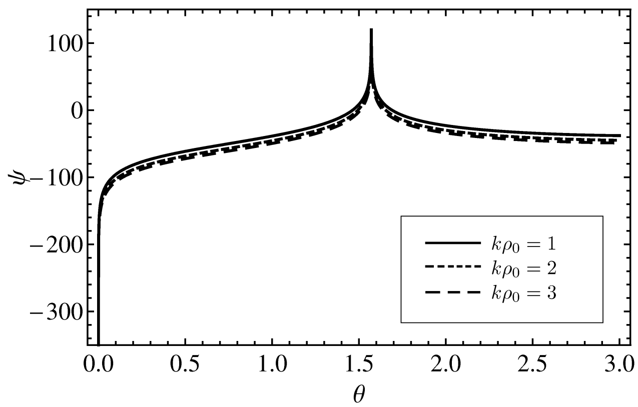

FIG. 6. Plots of ψ Vs θ for $\rho_0 k = 0.05$ and $\theta_0 = \pi/2$.

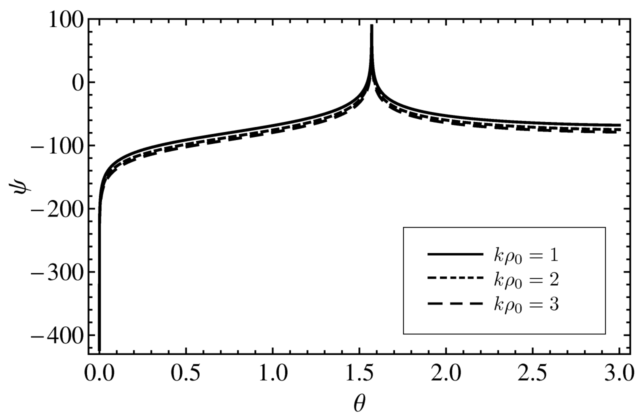

FIG. 7. Plots of ψ Vs θ for $\rho_0 k = 1$ and $\theta_0 = \pi/2$.

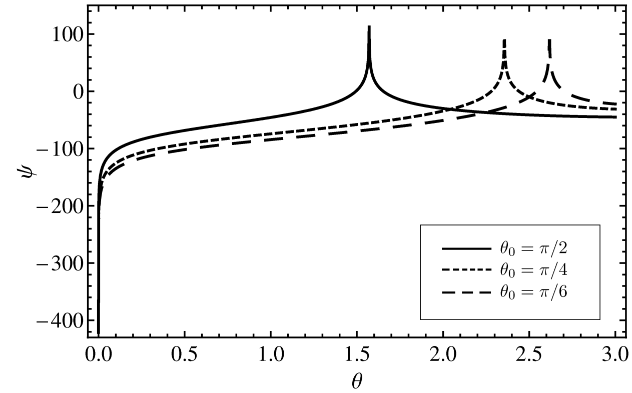

Fig. 8. Plots of ψ Vs θ for $\rho_0 k = 1$ and $\rho k = 1$.

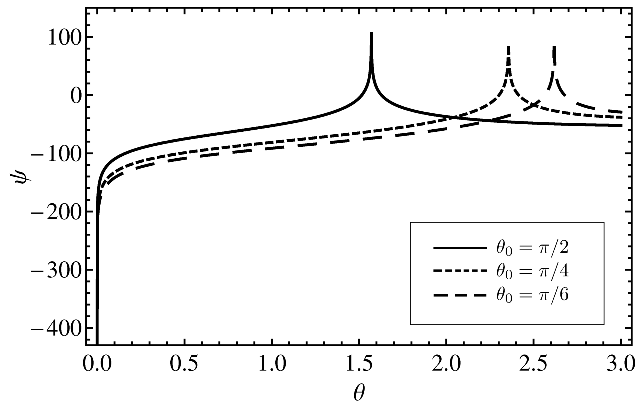

Fig. 9. Plots of ψ Vs θ for $\rho_0 k = 1$ and $\rho k = 2$.

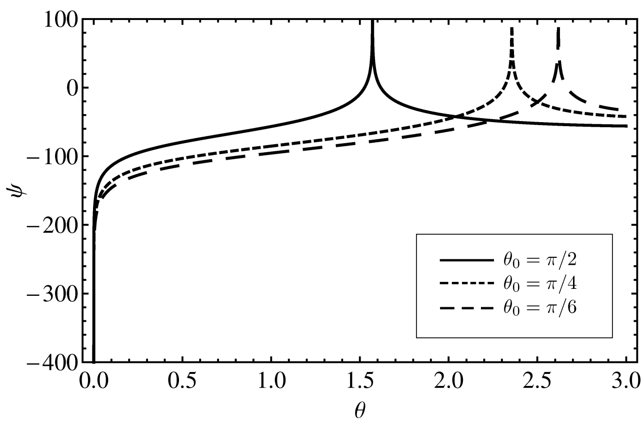

Fig. 10. Plots of ψ Vs θ for $\rho_0 k = 1$ and $\rho k = 3$.

5. Concluding remarks

In this paper, the line source and the point source scattering of acoustic waves by the soft/hard half-plane are studied. By means of the Fourier transform technique, the boundary value problem is reduced to the matrix Wiener–Hopf equation, whose solution is obtained by considering the Wiener–Hopf factorization of the kernel matrix. It is observed that our results differ due to [16] by a multiplicative factor. The result of point source excitation is also obtained. Some graphs, showing the effect of sundry parameters, for the case of line source situation are also plotted.

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