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Calculation of the force on solid body in the unsteady flow of incompressible fluid

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In the paper, a special method to compute the aerodynamic force is presented. This method is especially addressed to the calculation while both the velocity and vorticity fields are found as a result of the vortex method application.

In the case of vortex method, the vorticity field is shown as a sum of contributions given by a large number of the vorticity carriers. These carriers of vorticity move and change, but the vorticity distribution given by each of them is known. It means that both the vorticity and induced velocity field connected with them are easy to determine. The velocity field may also contain any potential component. This component assures the fulfillment of the asymptotic condition, and cancels the normal component of the velocity on the rigid body surface [15].

As it is known, the aerodynamic force may be calculated by using the basic definition, but in this case the boundary values of pressure and vorticity or derivatives of velocity field have to be found beforehand. These values are difficult to determine and their properties can be inconvenient. Quartapelle and Napolitano [12] introduced a special method of aerodynamic force calculation. This method does not require any surface integrals. Instead, the areas integrations are held. The integrands consist of vorticity and velocity fields only. The pressure field is excluded by special harmonic projection. The numerical experiment shows that the method of Quartapelle and Napolitano requires improvement in case of complicated, rapidly changing velocity and vorticity fields and the approximation of these fields in the neighborhood of the body not being perfect. However, if the concept of Quartapelle and Napolitano is applied to the area located outside the big sphere surrounding the body and containing the sources of vorticity, where velocity and vorticity fields have suitable properties (which permits to perform analytical calculations), we will get a simple formula for the aerodynamic force. This formula is not limited by additional properties of the pressure and velocity and vorticity fields. The numerical results are in relatively good agreement with the experimental data.

Key words: force, vortex method, viscous flows.

1. Introduction

The aerodynamic force acting on the finite solid body immersed in an incompressible fluid may be calculated using the basic definition, where v_κ are the Cartesian components of the velocity and p is the pressure field, one writes as follows:

$$A_j = -\oint_\Gamma \left[n_j\, p + \mu\, n_k \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right) \right] dS. \tag{1.1}$$

The incompressibility restriction ensures the vanishing of the integral containing the integrand $n_k \dfrac{\partial v_k}{\partial x_j}$. Using this property we obtain the following expression:

$$\mathbf{A} = -\oint_\Gamma (\mathbf{n}p + \mu\, \mathbf{n} \times \boldsymbol{\omega})\, dS \tag{1.2}$$

where $\mathbf{n}$ denotes a unit vector normal to the body surface Γ and $\boldsymbol{\omega}$ is the vorticity vector of the following form:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{1.3}$$

The direct usage of the formula (1.2) needs both boundary values p and $\boldsymbol{\omega}$. These values have to be defined before integration. The weak assumptions of the viscous liquid flow theory [4, 17] lead to the considerations of the velocity field of vector space $\mathbb{W}_2^{(1)}$ and the pressure field of $W_2^{(1)}$ space. The integrability of the squares of the first derivatives of any function does not ensure the existence of integrable boundary values of derivatives of that function [6]. The additional assumptions for a smooth approximation in the numerical realization allow to get the integrable boundary values of p and $\boldsymbol{\omega}$. Some problems may occur when the vortex method is applied. In this method, a numerous set of fictional vortex particles is considered. Each particle carries a definite charge of vorticity. The decrease of the particle's size is required to improve approximation, which results in difficulties in the evaluation of integral (1.2). Moreover, the pressure does not appear in the vorticity formulation. The fundamental equation of the vortex method, i.e. Helmholtz's equation, contains the velocity and vorticity fields only. These two fields – *ex post* – define the pressure. The gradient of pressure is included in the equation of motion. Quartapelle and Napolitano [11, 12] have introduced a skilful method of the pressure term of integral (1.2) calculation. This method does not require the boundary value of pressure and the derivatives of vorticity. This idea is based on "harmonic projection" formed by using a harmonic function with special, Neuman-type boundary condition. When we multiply the gradient of pressure by the gradient of projector, integrate over the area of motion and make some further transformations, we obtain the pressure part of force expressed by the integrals over the area. The numerical experience [10] shows that in the case of complicated velocity/vorticity fields, the obtained results are not satisfactory, at least in our calculations.

Another approach is based on the equations of motion. The integration of these equations in the area between the body and large sphere, containing the body and the centroids of vorticity carriers, admits expression of the aerodynamic force via the surface integrals taken on the far surface and the surface of the body. The integrals taken on the large sphere are relatively simple and can be determined easily. This evaluation is possible, when we simplify the equation of motion in the exterior of the sphere. NOCCA [7, 8] and PLOUMHANS [9] applied a similar idea, however they assumed fast asymptotic decay of the pressure variations and vorticity at infinity. Also, they used two special integral identities[1)]. The application of these identities is possible only in the case of sufficiently rapid diminishing of the fields mentioned above. Note that the products of $\mathbf{v}$ and gradient of p or velocity field must be, at least, integrated on the surface Γ_∞. This requirement is not obvious.

The detailed analysis of the vorticity field and the motion equations allows to find important asymptotic properties of both the considered fields. These properties do not need to fulfill assumptions provided the usage of inequalities mentioned above. Fortunately, the problem can be solved another way. As it will be shown, the concept given by Quartapelle–Napolitano applies to the area located outside the large sphere and brings us a simple formula for the aerodynamic force. This formula is not restricted by additional properties for fields of the vorticity/velocity and the pressure, which cannot be assured *a priori*.

2. The velocity and vorticity fields

The rigid body is located so that one point of it does not move and marks the origin of the inertial coordinates system. The body surface is denoted by Γ and the exterior region of the space by Ω.

[1)]Let the surface Γ_∞ be a boundary of the area denoted by Ω_0. There are two identities:

$$\oint\limits_{\Gamma_\infty} \mathbf{n} p \, dS = \oint\limits_{\Gamma_\infty} [\mathbf{n}(\mathbf{r}\cdot\nabla p) - (\mathbf{n}\cdot\mathbf{r})\nabla p] \, dS,$$

$$\int\limits_{\Omega_0} \mathbf{v} \, d\Omega = \frac{1}{2}\int\limits_{\Omega_0} \mathbf{r}\times\boldsymbol{\omega} \, d\Omega - \frac{1}{2}\oint\limits_{\Gamma_\infty} \mathbf{r}\times(\mathbf{n}\times\mathbf{v}) \, dS$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. The first identity follows $(\nabla p\cdot\nabla)\,\mathbf{r} = \nabla p$ and $\nabla p = \nabla(\mathbf{r}\cdot\nabla p) - (\mathbf{r}\cdot\nabla)\nabla p$. The second one results from $\mathbf{v} = (\mathbf{v}\cdot\nabla)\mathbf{r}$ and $2(\mathbf{G}\cdot\nabla)\,\mathbf{F} = \nabla\times(\mathbf{F}\times\mathbf{G}) + \nabla(\mathbf{F}\cdot\mathbf{G}) + \mathbf{G}(\nabla\cdot\mathbf{F}) - \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{F}\times(\nabla\times\mathbf{G}) - \mathbf{G}\times(\nabla\times\mathbf{F})$ for $\mathbf{F} = \mathbf{r}$ and $\mathbf{G} = \mathbf{v}$. Previously, an expression for $\nabla(\mathbf{F}\cdot\mathbf{G})$ where $\mathbf{F} = \mathbf{r}$ and $\mathbf{G} = \nabla p$ was employed.

The viscous liquid moves with the homogeneous velocity $\mathbf{U}_\infty$ at infinity. Also the pressure field at infinity is uniform. Both these quantities may depend on time. The non-slip condition on the body surface is assumed. It means that

$$\mathbf{v}|_\Gamma = \mathbf{v}_\Gamma. \tag{2.1}$$

The velocity of the body surface is, at least, the result of rotation which gives us

$$\mathbf{v}|_\Gamma = \boldsymbol{\gamma} \times \mathbf{R}, \tag{2.2}$$

where $\mathbf{R}$ is the radius – vector describing the surface and $\boldsymbol{\gamma}$ denotes the angular velocity of the body. The velocity field of the liquid is a sum of homogeneous field $\mathbf{U}_\infty$, vorticity-free term $\nabla\varphi$ and non-divergent term $\mathbf{v}_\omega$:

$$\mathbf{v} = \mathbf{U}_\infty + \nabla\varphi + \mathbf{v}_\omega. \tag{2.3}$$

When we apply the boundary condition (2.1), we obtain the following:

$$\frac{\partial\varphi}{\partial n}|_\Gamma = (\mathbf{n}\cdot\nabla\varphi)|_\Gamma = \mathbf{n}\cdot(\mathbf{v}_\Gamma - \mathbf{U}_\infty - \mathbf{v}_\omega). \tag{2.4}$$

It can be observed that the potential term fulfills the continuity equation and φ is a harmonic function. It vanishes at infinity. Also, the right-hand term of Eq. (2.4) guarantees that

$$\oint_\Gamma \frac{\partial\varphi}{\partial n}\, dS = 0. \tag{2.5}$$

This fact is an indispensable condition for existence of the solution of Neuman problem (2.4) for the harmonic function φ. This harmonic function is to be expressed as a sum of two potentials formed by the Green's formula

$$\varphi = \frac{1}{4\pi}\oint_\Gamma \left[\varphi(\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}\frac{1}{|\mathbf{r}-\boldsymbol{\xi}|} - \frac{1}{|\mathbf{r}-\boldsymbol{\xi}|}\frac{\partial\varphi(\boldsymbol{\xi})}{\partial n_\xi}\right] dS_\xi. \tag{2.6}$$

The unit normal is directed to the flow area Ω being outside of the surface Γ.

If the distance between the surface and the arbitrary point, where the function φ is calculated, is large, then the integral (2.6) can be expanded into a series of $1/r$. The restriction (2.5) eliminates the first term of this series and so the following estimation can be made:

$$\varphi|_{r\to\infty} = 0(1/r^2). \tag{2.7}$$

It entails a fast vanishing at infinity of the potential term constituent of velocity, i.e.

$$|\nabla \varphi|_{r\to\infty} = 0\,(1/r^3). \tag{2.8}$$

The last term of the sum in the Eq. (2.3) is the vortex term. It may be expressed by the Biot–Savart integral

$$\mathbf{v}_\omega = \frac{1}{4\pi} \int\limits_{\operatorname{supp}\boldsymbol{\omega}} \frac{\boldsymbol{\omega}(\boldsymbol{\xi}) \times (\mathbf{r}-\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|^3}\, d_3\xi = -\frac{1}{4\pi}\nabla \times \int\limits_{\operatorname{supp}\boldsymbol{\omega}} \frac{\boldsymbol{\omega}(\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|}\, d_3\xi. \tag{2.9}$$

This formula defines the temporary velocity field dependent on the temporary vorticity. If vorticity is given as a sum of many separated subsets[2)], the integral (2.9) will be equal to a sum of many "subintegrals". The "subintegral" may be expanded into a series of $1/r$. But first, a location of any subset has to be introduced. This location describes, for example, a central point of subset. The radius vector of it is $\mathbf{r}_k$. In this subset $\boldsymbol{\xi} = \mathbf{r}_k + \boldsymbol{\eta}$ in local coordinates. The radius vector in these local coordinates is denoted by $\boldsymbol{\eta}$. Thus, for any subset we have:

$$\int \frac{\boldsymbol{\omega}(\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|}\, d_3\xi = \frac{1}{r}\,\boldsymbol{\Omega} + \frac{(\mathbf{r}\cdot\mathbf{r}_k)}{r^3}\,\boldsymbol{\Omega} + \frac{1}{4\pi}\frac{\int \boldsymbol{\omega}\,(\boldsymbol{\eta})(\mathbf{r}\cdot\boldsymbol{\eta})\, d_3\eta}{r^3} \tag{2.10}$$

where $\boldsymbol{\Omega}$ denotes a value proportional to the vorticity charge in the subset. Assuming that the diameter of the subset is small enough, for instance of the order ϵ, we get

$$\left| \int \boldsymbol{\omega}\,(\mathbf{r}\cdot\boldsymbol{\eta})\, d_3\,\eta \right| \le |\boldsymbol{\Omega}|\cdot|\mathbf{r}|\operatorname{diam}(\text{subset}) = |\boldsymbol{\Omega}|\cdot|\mathbf{r}|\,\epsilon, \tag{2.11}$$

which guarantees that the last term is smaller than the second one, so that ϵ is less than the diameter of the body surface Γ (because $\min r_k \ge \operatorname{diam}\Gamma$ and $\operatorname{diam}\Gamma \gg \epsilon$).

Omitting the last term[3)], we obtain the expansion for the vortex term of the velocity:

$$\mathbf{v}_\omega = \frac{(\sum \boldsymbol{\Omega}_k)\times\mathbf{r}}{r^3} - \frac{\sum \boldsymbol{\Omega}_k \times \mathbf{r}_k}{r^3} + 3\,\frac{\sum \boldsymbol{\Omega}_k \times \mathbf{r}\,(\mathbf{r}\cdot\mathbf{r}_k)}{r^5} + \dots \tag{2.12}$$

The total charge of vorticity

$$\boldsymbol{\Omega} = \sum \boldsymbol{\Omega}_k \tag{2.13}$$

[2)]Each of them may be interpreted as a small vortex particle.

[3)]Considering the vortex particle, we put charge of vorticity in the central point of the subset and assume the symmetry. Thus $\int \omega_k \eta_i d_3\,\eta$ is a zero tensor.

is described by the ordinary differential equation which depends on the properties of motion on the surface Γ. Taking into account the Helmholtz's equation, we obtain:

$$\int\limits_{\operatorname{supp}\omega} \frac{d\boldsymbol{\omega}}{dt} d_3 r = -\nu \oint \mathbf{n} \times \operatorname{rot} \boldsymbol{\omega}\, dS + \oint (\mathbf{n} \cdot \boldsymbol{\omega})\, \mathbf{v}\, dS. \tag{2.14}$$

Having the equation of motion in the close vicinity of the surface Γ, we take the vector product of the normal $\mathbf{n}$ and this equation, and then integrate, transform and finally obtain the following relation:

$$\begin{aligned} \oint \mathbf{n} \times \frac{\partial \mathbf{v}}{\partial t}\, dS + \oint \mathbf{n} \times \nabla \left(\frac{v^2}{2} + p \right) dS \\ = -\nu \oint \mathbf{n} \times \operatorname{rot} \boldsymbol{\omega}\, dS + \oint \mathbf{n} \times (\mathbf{v} \times \boldsymbol{\omega})\, dS. \end{aligned} \tag{2.15}$$

The second integral on the left-hand part of the above equation is equal to zero as a result of Green's theorem application. Thus, one subtracts (2.14) and (2.15) and obtains an important result[4)]:

$$\frac{d}{dt} \int\limits_{\operatorname{supp}\omega} \boldsymbol{\omega}\, d_3\, r = \oint \mathbf{n} \times \frac{\partial \mathbf{v}}{\partial t}\, dS + \oint (\mathbf{n} \cdot \mathbf{v})\, \boldsymbol{\omega}\, dS. \tag{2.16}$$

If the angular velocity of the body is not considered, then total vorticity charge equals zero due to the fact that it is zero at initial time when $\mathbf{v} = 0$. Gathering the estimations of the velocity terms, we conclude that for a large value of r, the following asymptotic expression is to be used:

$$\mathbf{v} = \mathbf{U}_\infty + \nabla \varphi - \frac{\sum (\boldsymbol{\Omega}_k \times \mathbf{r}_k)}{r^3} + 3 \frac{\sum (\boldsymbol{\Omega}_k \times \mathbf{r}(\mathbf{r} \cdot \mathbf{r}_k))}{r^5} + O(1/r^4), \tag{2.17}$$

where the elementary charges of vorticity $\boldsymbol{\Omega}_k$ are restricted by

$$\sum \boldsymbol{\Omega}_k = 0. \tag{2.18}$$

The expression (2.12) may be applied for $r > R_\infty$ where R_∞ is the radius of a large sphere surrounding the body and the centroids of all vortex particles.

3. The aerodynamic force

Let us assume that the rigid body is encircled by the large sphere mentioned above. This sphere Γ_∞ has a large radius R_∞ and, together with the body surface Γ, defines the region Ω_0 between them.

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4) The formula $\frac{d}{dt} \int\limits_{\operatorname{supp} F} F\, d\Omega = \frac{d}{dt} \int\limits_{\operatorname{supp} F} (\frac{dF}{dt} + F \operatorname{div} \mathbf{v})\, d\Omega$ has been used.

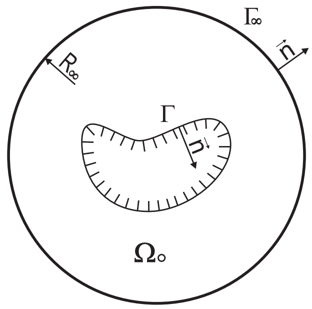


FIG. 1. The area Ω_0 covered between the surface of rigid body Γ and the big sphere of radius R_∞.

Integrating the equation of motion in the region Ω_0 one gets:

$$-\mathbf{A} = \int\limits_{\Omega_0} \frac{\partial \mathbf{v}}{\partial t}\, d\Omega + \oint\limits_{\Gamma_\infty} (\mathbf{n}\cdot\mathbf{v})\,\mathbf{v}\, dS + \oint\limits_{\Gamma_\infty} \mathbf{n}\, p\, dS - 2\mu \oint\limits_{\Gamma_\infty} \dot{\mathbb{D}}\,\mathbf{n}\, dS, \tag{3.1}$$

where $\mathbf{A}$ denotes the aerodynamic force acting on the body, i.e.

$$-\mathbf{A} = \oint\limits_{\Gamma} \mathbf{n}\cdot\mathbb{T}\, dS = \oint\limits_{\Gamma} [-\mathbf{n}\, p + 2\mu\, \mathbf{n}\cdot\dot{\mathbb{D}}\, dS]. \tag{3.2}$$

In these formulas

$$[\dot{\mathbb{D}}]_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right)$$

and μ is the viscosity. Taking into account the expression (2.17) and the estimation (2.8), we write

$$\left|\oint\limits_{\Gamma_\infty} \dot{\mathbb{D}}\,\mathbf{n}\, dS\right| \le \text{const}' \oint\limits_{\Gamma_\infty} \frac{1}{R_\infty^4}\, dS(R_\infty) = 4\,\pi \text{const}' \frac{1}{R_\infty^2}.$$

It means that the last term in the Eq. (3.1) is not significant and can be neglected. Also, the momentum flux term is not important. In the written form, we obtain an estimation:

$$\left|\oint\limits_{\Gamma_\infty} (\mathbf{n}\cdot\mathbf{v})\mathbf{v}\, dS\right| \le \left|\mathbf{U}_\infty \oint\limits_{\Gamma_\infty} \mathbf{n}\cdot\mathbf{U}_\infty\, dS\right| + |\mathbf{U}_\infty| \oint\limits_{\Gamma_\infty} O\left(\frac{1}{R_\infty^3}\right) dS(R_\infty) = 4\pi\, \text{const} \frac{1}{R_\infty}$$

which shows that the term can be omitted, too.

The remaining part of the formula for the force is

$$-\mathbf{A} = \oint\limits_{\Gamma_\infty} \mathbf{n}\, p\, dS + \int\limits_{\Omega_0} \frac{\partial \mathbf{v}}{\partial t}\, d\Omega. \tag{3.3}$$

Noca [7] for the simple case of 2-D motion and Ploumhans [9] for the 3-D, used two identities written previously. To apply those formulas for evaluation of integrals in (3.3), the equation of motion must be simplified first. The reduced form of this equation used by these authors is

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla p. \tag{3.4}$$

In our opinion, the following procedure is not sufficiently accurate. First, the velocity is formed also by a vortex component which cannot be performed as a gradient. Second, the velocity field is going to the uniform field like $O(1/r^3)$. It means that the pressure (in non-stationary motion) is going to a constant value like $O(1/r^2)$. This is an improper asymptotic behavior for the application of the integral identities mentioned above. The conclusion is following: both integrals forming the formula (3.3) must be considered with special attention. It will be done using the Quartapelle–Napolitano idea based on the harmonic projection. To do so we will use the equation of motion in the following form:

$$\nabla P = \nabla \left(p + \frac{\partial \tilde{\varphi}}{\partial t} \right) = \mathbf{B} = -\frac{\partial \mathbf{v}_\omega}{\partial t} - \frac{\partial}{\partial x_i}(v_i\, \mathbf{v}) + \mu \Delta \mathbf{v}. \tag{3.5}$$

In this equation $\tilde{\varphi}$ is the full potential

$$\tilde{\varphi} = \mathbf{U}_\infty \cdot \mathbf{r} + \varphi.$$

In the vortex methods, the velocity and vorticity fields are to be found independently of the pressure field. It means that the right part of Eq. (3.5), denoted as $\mathbf{B}$, is known. Multiplying the gradient of P by gradient of harmonic function H:

$$H = \frac{R_\infty^3}{2} \frac{\mathbf{e}_x \cdot \mathbf{r}}{r^3} = \frac{R_\infty^3}{2} \frac{x}{r^3} \tag{3.6}$$

and integrating in the outside of Γ_∞ we get, after some elementary transformations, another formula

$$\oint\limits_{\Gamma_\infty} n_x\, P\, dS = -\oint\limits_{\Gamma_\infty} (\mathbf{n} \cdot \nabla P) H\, dS - \int\limits_{r > R_\infty} H \Delta P\, d\Omega. \tag{3.7}$$

Laplacian of P is given by the divergence of gradient P which is

$$\Delta P = -\frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_i}. \tag{3.8}$$

For $r > R_\infty$ this value vanishes very fast. The following estimation holds

$$\left| \int\limits_{r>R_\infty} H\,\Delta P\,d\Omega \right| \le 4\pi R_\infty^3 \int\limits_{R_\infty}^{\infty} \frac{\text{const}}{r^8}\,dr, \tag{3.9}$$

which guarantees that the last expression vanishes for $R_\infty \to \infty$. Generalization of the Eqs. (3.6) and (3.7) gives

$$\oint\limits_{\Gamma_\infty} \mathbf{n}\,p\,dS = -\oint\limits_{\Gamma_\infty} \mathbf{n}\cdot\frac{\partial\tilde{\varphi}}{\partial t} - \frac{1}{2}\oint\limits_{\Gamma_\infty} (\mathbf{n}\cdot\nabla P)\,\mathbf{r}\,dS. \tag{3.10}$$

The last integral can be evaluated as

$$\begin{aligned} &-\oint\limits_{\Gamma_\infty} (\mathbf{n}\cdot\nabla P)\,\mathbf{r}\,dS \\ &= \oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS + \oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial}{\partial x_i}(v_i\mathbf{v})\right)\mathbf{r}\,dS - \mu\oint\limits_{\Gamma_\infty} (\mathbf{n}\cdot\Delta\mathbf{v})\,\mathbf{r}\,dS. \end{aligned} \tag{3.11}$$

It can be shown (as a result of a simple estimation) that the last integral is of $1/R_\infty^2$ order. The momentum flux integral also vanishes like $1/R_\infty$, which means that only one term is to be considered. Thus, the force can be given by the formula

$$-\mathbf{A} = \frac{1}{2}\oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS - \oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\tilde{\varphi}}{\partial t}\right)dS + \int\limits_{\Omega_0} \frac{\partial\mathbf{v}}{\partial t}\,d\Omega. \tag{3.12}$$

The last term contains the integrand of the following contribution $\dfrac{\partial}{\partial t}\nabla\tilde{\varphi} = \nabla\dfrac{\partial\tilde{\varphi}}{\partial t}$. Applying Green's formula we obtain:

$$\begin{aligned} -\mathbf{A} &= \frac{1}{2}\oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS - \oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\tilde{\varphi}}{\partial t}\right)dS \\ &\quad + \frac{d}{dt}\oint\limits_{\Gamma} [\mathbf{n}\tilde{\varphi} + \mathbf{n}\times\mathbf{\Psi}]\,dS + \frac{d}{dt}\oint\limits_{\Gamma_\infty} [\mathbf{n}\tilde{\varphi} + \mathbf{n}\times\mathbf{\Psi}]\,dS. \end{aligned} \tag{3.13}$$

Obviously, the vortex part of the velocity is a curl of vector potential $\mathbf{\Psi}$. We can simplify the above equation to:

$$-\mathbf{A} = \frac{d}{dt}\oint\limits_{\Gamma} [\mathbf{n}\,\tilde{\varphi} + \mathbf{n}\times\mathbf{\Psi}]\,dS + \frac{d}{dt}\oint\limits_{\Gamma_\infty} \mathbf{n}\times\mathbf{\Psi}\,dS + \frac{1}{2}\oint\limits_{\Gamma_\infty} \left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS. \tag{3.14}$$

It is interesting that both surface integrals taken on the sphere Γ_∞ may be evaluated easily. The expansion (2.12) is recalled. The appropriate vector potential $\mathbf{\Psi}$ for $r \geq R_\infty$ is

$$\mathbf{\Psi} = \frac{\sum \mathbf{\Omega}_k(\mathbf{r}\cdot\mathbf{r}_k)}{r^3} + O(1/r^3). \tag{3.15}$$

The scalar product $\mathbf{n}\cdot\mathbf{v}_\omega$ cuts off the second term of expansion (3.15) and so

$$\frac{1}{2}\oint\limits_{\Gamma_\infty}\left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS = -\frac{1}{2}\frac{d}{dt}\oint\limits_{\Gamma_\infty}\frac{\mathbf{r}\cdot\mathbf{M}}{R_\infty^4}\mathbf{r}\,dS = -\frac{1}{2}\frac{d}{dt}\oint\limits_{S(1)}(\boldsymbol{\xi}\cdot\mathbf{M})\,\boldsymbol{\xi}\,dS, \tag{3.16}$$

where $\mathbf{M}$ is the moment of the vorticity

$$\mathbf{M} = \sum \mathbf{\Omega}_k \times \mathbf{r}_k. \tag{3.17}$$

The unit vector $\boldsymbol{\xi} = \mathbf{r}/R_\infty$ is involved here. After this transformation, the integral is calculated on a unit sphere $S(1)$. Due to spherical symmetry we obtain

$$\oint\limits_{S(1)}\xi_\alpha\xi_\beta\,dS = \begin{cases} 0 & \text{if } \alpha\neq\beta, \\ \dfrac{1}{3}\oint\limits_{S(1)}\boldsymbol{\xi}\cdot\boldsymbol{\xi}\,dS = \dfrac{4\pi}{3} & \text{otherwise.}\end{cases}$$

Thus we get

$$\frac{1}{2}\oint\limits_{\Gamma_\infty}\left(\mathbf{n}\cdot\frac{\partial\mathbf{v}_\omega}{\partial t}\right)\mathbf{r}\,dS = -\frac{2\pi}{3}\frac{d\mathbf{M}}{dt}. \tag{3.18}$$

Similarly, the integral containing the vector potential $\mathbf{n}\times\mathbf{\Psi}$ is evaluated in the same way. We write

$$\oint\limits_{\Gamma_\infty}\mathbf{n}\times\mathbf{\Psi} = \mathbf{e}_\alpha\epsilon_{\alpha\ \beta\gamma}(\Omega_k)_\gamma\,(x_k)_i\oint\limits_{\Gamma_\infty}\frac{x_\beta x_i}{R_\infty^4}\,dS = -\frac{4\pi}{3}\mathbf{M}. \tag{3.19}$$

Gathering all the terms, i.e. (3.19), (3.18) and (3.14), we form the final formula for the force:

$$-\mathbf{A} = 2\pi\frac{d}{dt}\mathbf{M} + \frac{d}{dt}\oint\limits_{\Gamma}[\mathbf{n}\,\tilde{\varphi} + \mathbf{n}\times\mathbf{\Psi}]\,dS. \tag{3.20}$$

It is possible that only one surface integral forms this value. Knowing the set of vortex particles (and their properties), the moment $\mathbf{M}$ can be calculated in the algebraic way. Note, that both potentials $\tilde{\varphi}$ and $\mathbf{\Psi}$ are much more regular functions on the surface Γ than the vorticity $\boldsymbol{\omega}$ and preassure p at Γ involved in the basic formula (1.2).

4. Numerical implementation

The method described above was applied to the calculation of the aerodynamic force acting on a sphere immersed in the unsteady flow of the incompressible fluid. The flow past the sphere had been calculated earlier by the vortex method [15, 16].

The following data are necessary to calculate the force:

- the elementary charge of vorticity $\mathbf{\Omega}_k$ for each particle,
- the position of the vortex particles centres, which are described by the vectors $\mathbf{r}_k(t)$,
- the potential $\tilde{\varphi} = \mathbf{U}_\infty \cdot \mathbf{r} + \varphi$,
- the vector potential $\mathbf{\Psi}$.

The charges of the particles $\mathbf{\Omega}_k$ change when the fluid is moving. Their values are obtained from the system of differential equations, which is solved in each step of time. The procedure is described in [16].

The positions of the vortex particles $\mathbf{r}_k(t)$ are the stochastic processes. Each of them results from Ito equations

$$d\mathbf{r}_k = \mathbf{v}(t, \mathbf{r})|_{\mathbf{r}_k}\, dt + \sqrt{2\,\nu}\, d\mathbf{W}, \qquad d\mathbf{r}_k|_0 = d\mathbf{r}_{k\,0}, \tag{4.1}$$

where $d\mathbf{W}$ is an increment of the vector Wiener process and ν is the kinematic viscosity.

The potential φ is a harmonic function. In order to get this function, the Neumann problem should be resolved outside the sphere. To do so, the standard code of the harmonic analysis on the sphere was used [1].

The vector potential $\mathbf{\Psi}$ is expressed by the formula

$$\mathbf{\Psi} = \sum \mathbf{\Omega}_k \phi(|\mathbf{r} - \mathbf{r}_k|), \tag{4.2}$$

where

$$\phi(x) = \int_0^x \frac{F(\xi)}{\xi^2}\, d\xi \tag{4.3}$$

and the function F, which defines the structure of a particle, is given by

$$F(\rho) = \int_0^\rho \xi^2 f(\xi)\, d\xi. \tag{4.4}$$

The function f specifies a distribution of the vorticity which is brought by the particle. It can be chosen freely, however it should fulfill some conditions:

- $f(\rho \geq \delta) = 0$ (where δ is the conventional radius of the particle),
- f is limited for $\rho > 0$.

The details of calculations and general investigations have been given in [16]. When the potential $\tilde{\varphi}$ and the potential vector $\mathbf{\Psi}$ are known, it is easy to calculate the integral $\oint\limits_{\Gamma} \mathbf{n}\,\tilde{\varphi} + \mathbf{n} \times \mathbf{\Psi}\, dS$ using for instance the generalized trapezoid formula for two dimensions.

When $\mathbf{\Omega}_k$ and $\mathbf{r}_k$ are known, the moment $\mathbf{M}$ can be calculated from (3.17). But to get the force, the function

$$\mathbf{G}(t) = 2\pi\,\mathbf{M} + \oint\limits_{\Gamma} [\mathbf{n}\,\tilde{\varphi} + \mathbf{n} \times \mathbf{\Psi}]\, dS \tag{4.5}$$

has to be differentiated with respect to time. The differentiation includes vector $\mathbf{r}_k$ – the random quantity, which is not differentiable. Fortunately, the derivative of $\sum \mathbf{\Omega}_k \times \mathbf{r}_k$ has to be calculated so we can expect the average of random quantities. Furthermore, the function $\mathbf{G}(t)$ is approximated by the smooth function (local polynomials in this case), so the difference

$$|\mathbf{G}(t) - \mathbf{W}(t)| \tag{4.6}$$

is minimized. Then the function $\mathbf{W}(t)$ can be differentiated. The instance of the local approximation is shown in the Fig 1.

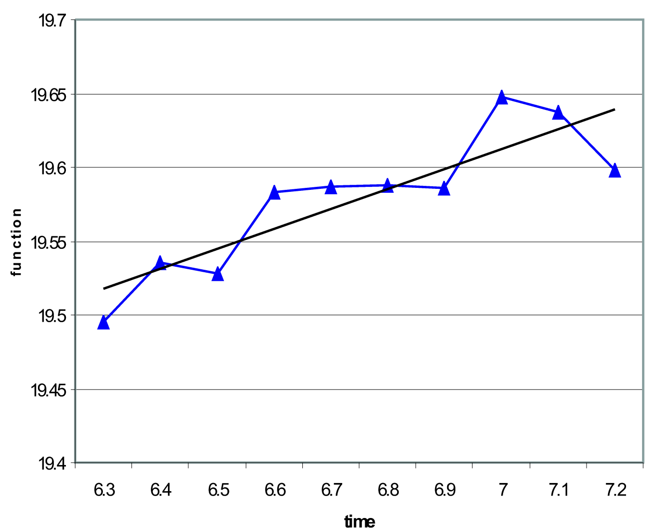


FIG. 2. Local approximation of function $\mathbf{G}(t)$ by the function $f_v = c_0 + c_1 t$.

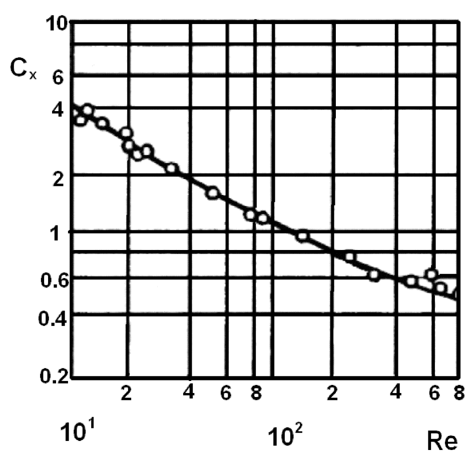
The aerodynamic coefficients C_x, C_y, C_z as functions of time for flows of Reynolds numbers $\mathrm{Re} = 100, 200, 400$ are presented in Fig. 5. These coefficients are calculated basing on temporary forces values, for which the following definitions are used:

$$
\begin{aligned}
C_x &= \frac{A_x}{\dfrac{\varrho U_\infty^2}{2} \cdot \dfrac{\pi d^2}{4}}, \\
C_y &= \frac{A_y}{\dfrac{\varrho U_\infty^2}{2} \cdot \dfrac{\pi d^2}{4}}, \qquad (4.7) \\
C_z &= \frac{A_z}{\dfrac{\varrho U_\infty^2}{2} \cdot \dfrac{\pi d^2}{4}},
\end{aligned}
$$

where A_x, A_y, A_z – the components of aerodynamic force, $|U_\infty| = 1$ – the velocity at infinity, $\rho = 1$ – density, and $d = 2$ – diameter of the sphere.

The flow past the sphere has been frequently explored numerically and experimentally [2, 5, 18, 19]. The results which were obtained with the current method can be compared with experimental data. In this instance, the values of coefficients were taken from the book by H. SCHLICHTING and K. GERSTEN [13]. The standard, mean value of the drag coefficient C_x in function of the Reynolds number is presented in the Fig. 3.

a)



b)

Re	C_x exper.	C_x average
100	1.1	1.007
200	0.8	0.77
400	0.6	0.7

FIG. 3. a) Drag coefficient for the sphere (data from the book by Schlichting and Gersten); b) experimental data and average values obtained from calculations.

It is seen that numerical results for the Re = 100 and Re = 200 fluctuate near the experimental data. The coefficient C_x fluctuates near the value 1 for the Re = 100 and near the value 0.8 for the Re = 200. The coefficients C_y and C_z fluctuate near zero what agrees with expectation.

The coefficient C_x for the Re = 400 is a little greater than the experimentally measured value and stronger fluctuations of the C_y and C_z are observed. A possible reason is that the flow in the range of Reynolds number higher than 400 is unsteady and loses the periodicity and symmetry.

a)

b)

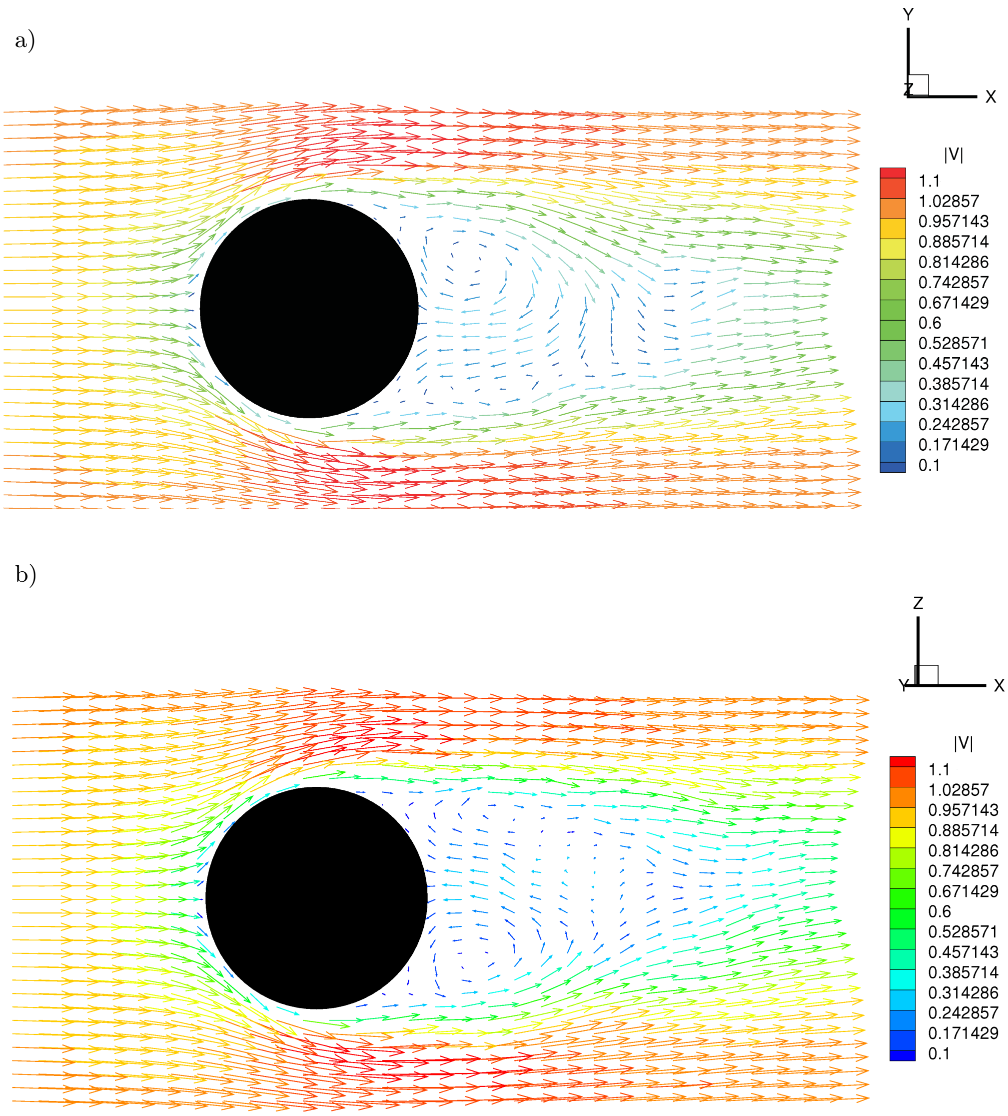


Fig. 4. Velocity fields in two orthogonal planes. The breaking of symmetry is observed.

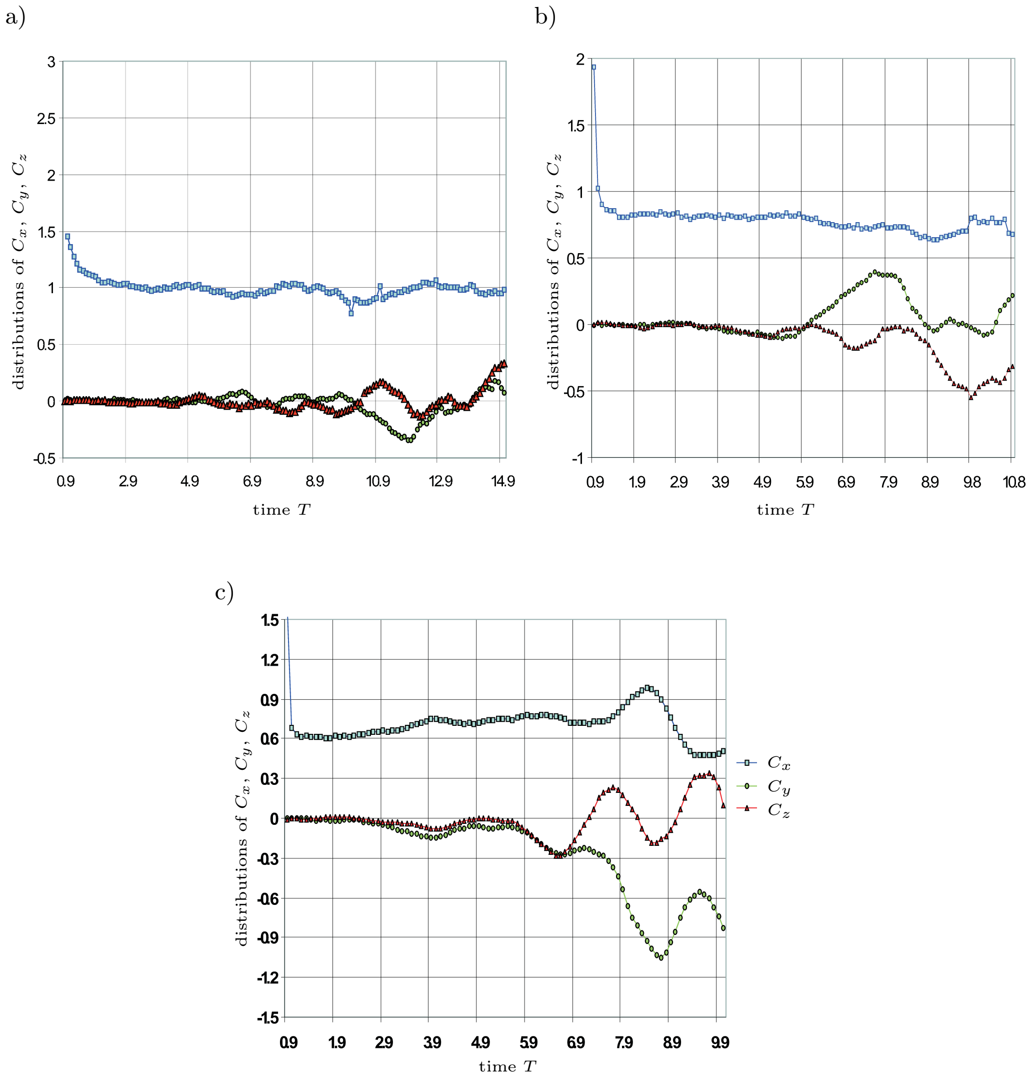


FIG. 5. Distributions of aerodynamic coefficients C_x, C_y, C_z in time for various Reynolds numbers: a) Re = 100, b) Re = 200, c) Re = 400.

5. Conclusion

The method of a field type was proposed to calculate the aerodynamic force. The well-known formula of Quartapelle–Napolitano was applied to the exterior of the large sphere, where initial velocity and vorticity fields have properties which permit to perform analytical calculations. This method does not have any drawbacks which are present in the procedure described in [7, 9]. However, the necessity of differentiation of a function of one variable can be regarded as a weak point. If the stochastic simulation of diffusion is used, the estimation of the undifferentiable stochastic process by the differentiable function is required. If the deterministic method is used, such estimation is not necessary.

As it is known, the wake of axisymmetric body may lose its symmetry [3]. This fact brings up waving of the drag coefficient and essentially stronger oscilations of the two remaining coefficients. Excluding these oscilations, we observe that the calculated results are relatively close to the empirical ones. It is known [3, 5, 14, 19] that the phenomenona such that appear while the Reynolds number are bigger than 212. The oscilations of wake break effectively the symmetry of flow and introduce non-stationary behaviours of aerodynamic force.

References

1. J.C. Adams, P.N. Swarztrauber, www.scd.ucar.edu/css/software/spherepack, 1998.
2. T.A. Johnson, V.C. Patel, *Flow Past a Sphere up to a Reynolds Number of* 300, J. Fluid Mech., **378**, 19–70, 1999.
3. D.J. Fabre, A. Franc, J. Magnaudet, *Bifurcations and symmetry breaking in the wake of axisymmetric bodies*, Physic of Fluids, **20**, 051702–4, 2008.
4. O.A. Ladhyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, GiFM, Moskva, 1963; English transl.: Gordon and Breach, N.Y. 1963.
5. R.H. Margavey, R.L. Bishop, *Vortices in Sphere Wake*, Can. J. Phys., **43**, 1649, 1965.
6. S. Mizohata, *The theory of Partial Differential Equations*, Cambridge 1973.
7. F. Noca, D. Shiels, D. Jeon, *Measuring Instantaneous Fluid Dynamic Forces on Bodies, Using Only Velocity Fields and Their Derivatives*, Journal of Fluids and Structures, **11**, 345–350, 1997.
8. F. Noca, D. Shiels, D. Jeon, *A Comparison of Methods for Evaluating Time-dependent Fluid Dynamic Forces on Bodies Using only Velocity Fields and Their Derivatives*, Journal of Fluid and Structures, **13**, 551, 1999.
9. P. Ploumhans, G.S. Winckelmans, J.K. Salmon, A. Leonard, M.S. Warren, *Vortex Methods for Direct Numerical Simulation of Three-Dimensional Bluff Body: Application to the Sphere at* Re = 300*,* 500*, and* 1000, Journal of Computational Physics, **178**, 427–463, 2002.
10. M. Poćwierz, *The Pressure Field, Reaction and Typical Structures of POD for the Unsteady Flow Past the Solid Body* [in Polish], PhD Thesis, Warsaw University of Technology, 2005.
11. L. Quartapelle, M. Napolitano, *Integral Condition for Pressure in the Computation of Incompressible Viscous Flows*, Journal of Computational Physics, **62**, 340–348, 1986.
12. L. Quartapelle, M. Napolitano, *Force and Moment in Incompressible Flows*, AIAA Journal, **21**, 911–913, 1983.
13. H. Schlichting, K. Gersten, *Boundary Layer Theory*, Springer, 2000.

14. L. SCHOUVEILER, M. PROVANSAL, *Self-sustained oscillations in the wake of a sphere*, Phisics of Fluids, **14**, 11, 2002.

15. A. STYCZEK, P. DUSZYŃSKI, M. POĆWIERZ, J. SZUMBARSKI, *Random Vortex Method for Three Dimensional Flows. Part I: Mathematical Background*, Journal of Theoretical and Applied Mechanics, **42**, 1, 3–20, 2004.

16. A. STYCZEK, P. DUSZYŃSKI, M. POĆWIERZ, J. SZUMBARSKI, *Random Vortex Method for Three Dimensional Flows. Part II: Numerical Implementation and Sample Results*, Journal of Theoretical and Applied Mechanics, **42**, 2, 223–238, 2004.

17. R. TEMAM, *Navier–Stokes Equations. Theory and Numerical Analysis*, North Holland, 1984.

18. A.G. TOMBOULIDES, S.A. ORSZAG, *Numerical Investigation of Transitional and Weak Turbulent Flow Past a Sphere*, J. Fluid Mech., **416**, 45, 2000.

19. J.S. WU, G.M. FAETH, *Sphere Wakes in Still Surroundings at Intermediate Reynolds Numbers*, AIAA J., **31**, 1448, 1993.

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