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On the deformation of transversely isotropic porous elastic circular cylinder

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In this paper we study the deformation of right circular cylinders filled by a linear transversely isotropic porous material. We construct a solution of the relaxed Saint–Venant's problem using the results established for anisotropic porous cylinders. Firstly, we decompose the relaxed Saint–Venant's problem into two problems: extension–bending–torsion problem and flexure problem. Then, for each of them we give the exact expressions of the solutions.

Key words: transversely isotropic porous materials; relaxed Saint–Venant's problem; circular cylinder.
Mathematics Subject Classifications (2000): 74E10, 74G05, 74G75, 74G99.

1. Introduction

The theory of elastic materials with voids is a special case of that for materials with microstructure. In this paper we consider the theory of materials with voids introduced by Cowin and Nunziato [1]. The basic idea of this theory is to suppose that there is a continuous distribution of voids throughout the elastic body. In this theory, the bulk density is written as the product of two fields: the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom and it was employed previously by Goodman and Cowin [2] to develop a continuum theory of granular materials. The first investigations in the theory of thermoelastic materials with voids are due to Nunziato and Cowin [3] and Ieşan [4]. The intended applications of the theory are geological materials and manufactured porous materials. A presentation of this theory can be found in [5, 6].

Saint–Venant's problem consists of determining the equilibrium of an elastic cylinder loaded by surface forces distributed over its plane ends. In the relaxed Saint–Venant's problem the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Using the method introduced by Toupin [7] in classical elasticity, Batra and

Yang [8] have proved that these changes of the ends conditions produce negligible errors, except possibly near the ends.

In this paper we consider the relaxed Saint–Venant's problem for right circular cylinders made of a transversely isotropic homogeneous elastic material with voids. An elastic material is a transversely isotropic material [9] if at each point there is a principal direction and an infinite number of principal directions in the plane normal to the first direction. The case of transversely isotropic materials is an important branch of applied mathematics and engineering science. In [10] Ding *et al.* have presented the methods to study different types of problems which arise in the theory of transversely isotropic elastic materials. Besides the well-known applications of this type of material in the mechanics of rocks [11–16], the transversely isotropic materials are very useful in many branches of biology [17–21]. The recent studies of fiber-reinforced composites [22, 23] and the modern technologies also encourage the study of transversely isotropic materials.

For the treatment of the deformation of a right circular cylinder filled with a transversely isotropic porous material, we use the results established by Ghiba [24]. These results are established using the method described by Ieşan in the books [25, 26]. This method gives a possibility to reduce the Saint–Venant's problem to some generalized plane strain problems. In fact, in the paper [24], two classes of semi-inverse solutions were described in the set of solutions of Saint–Venant's problem that may be expressed in terms of solutions of some generalized plane strain problems. We use these classes obtained in the anisotropic case to solve the extension, bending, torsion and flexure problems of transversely isotropic porous elastic circular cylinders.

We outline that a study of Saint–Venant's problem for homogeneous and isotropic porous elastic cylinders has been presented by Dell'Isola and Batra [27]. The semi-inverse method used in the present paper has been employed to study the Saint–Venant's problem for different types of materials in the papers [28–33].

2. Formulation of the problem

We consider a right circular cylinder of length L and radius a, occupied by an homogeneous porous, transversely isotropic elastic material. We denote by B the interior of cylinder, by ∂B the boundary of B and by $D \subset \mathbb{R}^2$ the interior of the bounded cross-section. As shown in Fig. 1, we choose a rectangular Cartesian system $Ox_1x_2x_3$ so that the Ox_3-axis is parallel to the generator of the cylinder and O is the center of one of its ends. The lateral boundary of the cylinder is $\Pi = \partial D \times (0, L)$ and D_0 and D_L are, respectively, the cross-sections located at $x_3 = 0$ and $x_3 = L$.

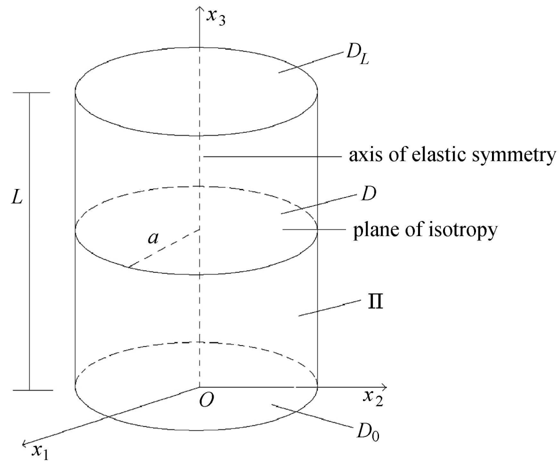

FIG. 1. Schema of the problem studied.

The Latin subscripts and superscripts are understood to range over the integers $1, 2, 3$, unless we specify else, whereas Greek subscripts and superscripts are confined to the range $1, 2$; summation over repeated subscripts is implied and comma followed by a subscript denotes partial derivative with respect to the corresponding Cartesian coordinate; where no confusion may occur, we suppress the dependence upon the spatial variables.

Let $\mathbf{u}$ be the displacement field over B and φ the volume distribution function [2]. We denote by $\mathbf{U}$ the four-dimensional vector $(u_i,\ \varphi)$. The linear strain measure e_{ij} is given by

$$e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

The components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force for anisotropic porous material [1] are

$$\begin{aligned} t_{ij}(\mathbf{U}) &= C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ijr}\varphi_{,r}, \\ h_i(\mathbf{U}) &= D_{rsi}e_{rs} + d_i\varphi + A_{ij}\varphi_{,j}, \\ g(\mathbf{U}) &= -B_{ij}e_{ij} - \xi\varphi - d_i\varphi_{,i}, \end{aligned} \tag{2.2}$$

where C_{ijrs}, B_{ij}, A_{ij}, D_{ijk}, d_i and ξ are the constitutive coefficients which satisfy the symmetry relations

$$C_{ijrs} = C_{rsij} = C_{jirs}, \qquad A_{ij} = A_{ji}, \qquad B_{ij} = B_{ji}, \qquad D_{ijk} = D_{jik}. \tag{2.3}$$

We suppose that the axis Ox_3 is an axis of elastic symmetry and the planes normal to this axis are planes of isotropy. In the case of transverse isotropy, the mechanical response of the body remains unaffected due to arbitrary rotations from the direction of Ox_3 and due to reflections from the planes perpendicular to this direction. Thus, the symmetry group is [34]

$$\mathcal{G} = \left\{ \begin{bmatrix} \pm\cos\theta & -\sin\theta & 0 \\ \pm\sin\theta & \cos\theta & 0 \\ 0, & 0 & \pm 1 \end{bmatrix};\ \theta \in [0, 2\pi) \right\}. \tag{2.4}$$

For this class of materials we have only ten non-zero independent constitutive coefficients

$$\begin{gathered} c_{ij} \equiv C_{iijj}, \quad i,j \in \{1,2,3\} \text{ (not summed)}, \quad c_{11} = c_{22}, \quad c_{13} = c_{23}, \\ c_{44} \equiv C_{2323} = C_{1313}, \quad b_1 \equiv B_{11} = B_{22}, \quad b_3 \equiv B_{33}, \\ a_1 \equiv A_{11} = A_{22}, \quad a_3 \equiv A_{33} \text{ and } \xi. \end{gathered} \tag{2.5}$$

We note that in the case of isotropic materials with voids, the number of independent constitutive coefficients is five [1].

The constitutive equations (2.2), in the case of transversely isotropic materials are reduced to

$$\begin{aligned} t_{11}(\mathbf{U}) &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + b_1\varphi, \\ t_{22}(\mathbf{U}) &= c_{12}e_{11} + c_{22}e_{22} + c_{13}e_{33} + b_1\varphi, \\ t_{33}(\mathbf{U}) &= c_{13}e_{11} + c_{13}e_{22} + c_{33}e_{33} + b_3\varphi, \\ t_{12}(\mathbf{U}) &= (c_{11} - c_{12})e_{12}, \\ t_{13}(\mathbf{U}) &= 2c_{44}e_{13}, \\ t_{23}(\mathbf{U}) &= 2c_{44}e_{23}, \\ h_1(\mathbf{U}) &= a_1\varphi_{,1}, \\ h_2(\mathbf{U}) &= a_1\varphi_{,2}, \\ h_3(\mathbf{U}) &= a_3\varphi_{,3}, \\ g(\mathbf{U}) &= -b_1(e_{11} + e_{22}) - b_3e_{33} + \xi\varphi. \end{aligned} \tag{2.6}$$

The surface force and the equilibrated stress at a regular point of ∂B, are given by

$$t_i(\mathbf{U}) = t_{ij}(\mathbf{U})n_j, \qquad h(\mathbf{U}) = h_j(\mathbf{U})n_j, \tag{2.7}$$

respectively, where n_j are the components of the outward unit normal to ∂B.

The equilibrium equations, in the absence of the body force and the extrinsic equilibrated body force, are

$$t_{ji,j} = 0, \qquad h_{i,i} + g = 0 \quad \text{in } B. \tag{2.8}$$

Throughout this paper we assume that the internal energy density

$$\begin{aligned}\mathcal{W}(\mathbf{U}) = {} & \frac{1}{2}c_{11}e_{11}^2 + c_{12}e_{11}e_{22} + c_{13}e_{11}e_{33} + \frac{1}{2}c_{11}e_{22}^2 + c_{13}e_{22}e_{33} \\ & + \frac{1}{2}c_{33}e_{33}^2 + 2c_{44}e_{23}^2 + 2c_{44}e_{13}^2 + (c_{11} - c_{12})e_{12}^2 + b_1e_{11}\varphi + b_1e_{22}\varphi \\ & + b_3e_{33}\varphi + \frac{1}{2}\xi\varphi^2 + \frac{1}{2}a_1\varphi_{,1}\varphi_{,1} + \frac{1}{2}a_1\varphi_{,2}\varphi_{,2} + \frac{1}{2}a_3\varphi_{,3}\varphi_{,3}\end{aligned} \tag{2.9}$$

is positive definite quadratic in terms of e_{ij}, φ and $\varphi_{,i}$. This is true if and only if

$$\begin{gathered} c_{11} > 0, \qquad c_{11} > c_{12} > -c_{11}, \qquad (c_{11} + c_{12})c_{33} > 2c_{13}^2, \\ c_{44} > 0, \qquad a_1 > 0, \qquad a_3 > 0, \\ \xi[-2c_{13}^2 + (c_{11} + c_{12})c_{33}] > b_3^2(c_{11} + c_{12}) - 4b_1b_3c_{13} + 2b_1^2c_{33}. \end{gathered} \tag{2.10}$$

The cylinder is assumed to be free from lateral loading, so that the conditions on the lateral surface are

$$t_i = 0, \qquad h = 0 \quad \text{on } \Pi. \tag{2.11}$$

We consider the loading at the end D_0 to be statically equivalent to the given force $\mathbf{R}$ and the given moment $\mathbf{M}$. Then, for $x_3 = 0$, we have the conditions

$$\int_D t_{3i}(\mathbf{U})da = -R_i, \qquad \int_D \varepsilon_{ijk}x_jt_{3k}(\mathbf{U})da = -M_i. \tag{2.12}$$

From the existence results [5], for the equilibrium, we must have similar conditions at the end D_L.

The relaxed Saint–Venant's problem for B consists in determination of the displacement field $\mathbf{u}$ and the volume distribution function φ on B, solution of the equilibrium equations (2.8), which satisfy the requirements (2.11) and (2.12).

We decompose the relaxed Saint–Venant's problem $(\mathcal{P})$, into the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ characterized by

$(\mathcal{P}_1)$ (extension–bending–torsion): $\quad R_\alpha = 0,$

$(\mathcal{P}_2)$ (flexure): $\quad R_3 = M_i = 0.$

In this paper we study the Saint–Venant's problem reducing the above problems to some generalized plane problems.

By the state of generalized plane strain for the interior of the cross-section domain, $D \subset \mathbb{R}^2$, of the considered cylinder, we mean the state in which the displacement field $\mathbf{w}$ and the volume distribution ψ depend only on x_1 and x_2:

$$w_i = w_i(x_1, x_2), \qquad \psi = \psi(x_1, x_2), \qquad (x_1, x_2) \in D. \tag{2.13}$$

In this case, the components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force are functions of x_1 and x_2.

For a state of generalized plane strain $\mathbf{W} = (w_i(x_1, x_2), \psi(x_1, x_2))$, $(x_1, x_2) \in D$, we define the operators

$$\begin{aligned}
\mathcal{S}_1(\mathbf{W}) &= c_{11} w_{1,11} + c_{12} w_{2,21} + \frac{1}{2}(c_{11} - c_{12})(w_{1,22} + w_{2,12}) + b_1 \psi_{,1},\\
\mathcal{S}_2(\mathbf{W}) &= c_{11} w_{2,22} + c_{12} w_{1,12} + \frac{1}{2}(c_{11} - c_{12})(w_{2,11} + w_{1,21}) + b_1 \psi_{,2},\\
\mathcal{S}_3(\mathbf{W}) &= w_{3,\alpha\alpha},\\
\mathcal{C}(\mathbf{W}) &= a_1 \psi_{,\alpha\alpha} - b_1 w_{\alpha,\alpha} - \xi \psi,\\
\mathcal{H}_1(\mathbf{W}) &= (c_{11} w_{1,1} + c_{12} w_{2,2} + b_1 \psi) n_1 + \frac{1}{2}(c_{11} - c_{12})(w_{1,2} + w_{2,1}) n_2,\\
\mathcal{H}_2(\mathbf{W}) &= \frac{1}{2}(c_{11} - c_{12})(w_{2,1} + w_{1,2}) n_1 + (c_{11} w_{2,2} + c_{12} w_{1,1} + b_1 \psi) n_2,\\
\mathcal{H}_3(\mathbf{W}) &= w_{3,\alpha} n_\alpha,\\
\mathcal{D}(\mathbf{W}) &= a_1 \psi_{,\alpha} n_\alpha.
\end{aligned} \tag{2.14}$$

In what follows, we construct a solution of the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ using the semi-inverse method [25, 26].

3. Construction of solution of the problem $(\mathcal{P}_1)$

Let us consider $\mathbf{W}^{(s)} = (\mathbf{w}^{(s)}, \psi^{(s)})$, $s = 1, 2, 3$ solutions of the problems characterized by the equations

$$\mathcal{S}_i(\mathbf{W}^{(s)}) + f_i^{(s)} = 0, \quad \mathcal{C}(\mathbf{W}^{(s)}) + \ell^{(s)} = 0 \quad \text{in } D, \tag{3.1}$$

and the boundary conditions

$$\mathcal{H}_i(\mathbf{W}^{(s)}) = \widetilde{T}_i^{(s)}, \quad \mathcal{D}(\mathbf{W}^{(s)}) = \widetilde{H}^{(s)} \quad \text{on } \partial D, \tag{3.2}$$

where

$$
\begin{aligned}
f_\alpha^{(\gamma)} &= c_{13}\delta_{\alpha\gamma}, & f_3^{(\beta)} &= 0, & f_i^{(3)} &= 0,\\
\ell^{(\gamma)} &= -b_3 x_\gamma, & \ell^{(3)} &= -b_3,\\
\widetilde{T}_\alpha^{(\gamma)} &= -c_{13}x_\gamma n_\alpha, & \widetilde{T}_3^{(i)} &= 0,\\
\widetilde{T}_\alpha^{(3)} &= -c_{13}n_\alpha, & \widetilde{H}^{(i)} &= 0.
\end{aligned}
\tag{3.3}
$$

According to the existence results presented in [5], these generalized plane strain problems have solutions.

In view of the results established in [24], we find a solution of the problem $(\mathcal{P}_1)$ to be

$$
\mathbf{U}^I = \sum_{s=1}^{4} a_s \mathbf{U}^{(s)},
\tag{3.4}
$$

where the vectors $\mathbf{U}^{(s)} = (\mathbf{u}^{(s)}, \psi^{(s)})$, $s = 1, 2, 3, 4$ are defined by

$$
\begin{aligned}
u_\alpha^{(\beta)} &= -\frac{1}{2}x_3^2\delta_{\alpha\beta} + w_\alpha^{(\beta)}(x_1, x_2),\\
u_3^{(\beta)} &= x_\beta x_3 + w_3^{(\beta)}(x_1, x_2),\\
u_\alpha^{(3)} &= w_\alpha^{(3)}(x_1, x_2), \qquad u_3^{(3)} = x_3 + w_3^{(3)}(x_1, x_2),\\
u_\alpha^{(4)} &= \varepsilon_{3\beta\alpha}x_\beta x_3, \qquad u_3^{(4)} = 0,\\
\varphi^{(i)} &= \psi^{(i)}, \quad i = 1, 2, 3, \qquad \varphi^{(4)} = 0
\end{aligned}
\tag{3.5}
$$

and the unknown constants a_s, $s = 1, 2, 3, 4$, are solutions of the following algebraic system:

$$
\begin{gathered}
\sum_{s=1}^{4} a_s D_{\alpha s} = \varepsilon_{3\alpha\beta}M_\beta, \qquad \sum_{s=1}^{4} a_s D_{3s} = -R_3,\\
\sum_{s=1}^{4} a_s D_{4s} = -M_3.
\end{gathered}
\tag{3.6}
$$

with

$$
\begin{aligned}
D_{3s} &= \int_D t_{33}(\mathbf{U}^{(s)})\, da,\\
D_{\beta s} &= \int_D x_\beta t_{33}(\mathbf{U}^{(s)})\, da,\\
D_{4s} &= \int_D \varepsilon_{3\alpha\beta}x_\alpha t_{3\beta}(\mathbf{U}^{(s)}) da.
\end{aligned}
\tag{3.7}
$$

Because the internal energy is positive definite, we can prove [24] that

$$\det(D_{rs}) \neq 0, \tag{3.8}$$

so that the system (3.6) uniquely determines the constants a_s, $s = 1, 2, 3, 4$.

In what follows, we solve the three problems defined by relations (3.1)–(3.3). First, it is easy to see that $\mathbf{W}^{(3)}$ defined by

$$w_1^{(3)} = -\nu_1 x_1, \qquad w_2^{(3)} = -\nu_1 x_2, \qquad \psi^{(3)} = -\nu_2, \qquad w_3^{(3)} = 0, \tag{3.9}$$

with

$$\nu_1 = \frac{c_{13}\xi - b_1 b_3}{(c_{11} + c_{12})\xi - 2b_1}, \qquad \nu_2 = \frac{(c_{11} + c_{12})b_3 - 2c_{13}b_1}{(c_{11} + c_{12})\xi - 2b_1^2}, \tag{3.10}$$

is a solution of the third problem.

Next, we search a solution $\mathbf{W}^{(1)}$ of the first problem in the form

$$\begin{aligned} w_1^{(1)} &= v_1^{(1)} - \frac{1}{2}\nu_1(x_1^2 - x_2^2), \\ w_2^{(1)} &= v_2^{(1)} - \nu_1 x_1 x_2, \\ w_3^{(1)} &= 0, \\ \psi^{(1)} &= \phi^{(1)} - \nu_2 x_1, \end{aligned} \tag{3.11}$$

where $\mathbf{V}^{(1)} = (v_1^{(1)}, v_2^{(1)}, \phi^{(1)})$ is a solution of the problem defined by the equations

$$\mathcal{S}_\alpha(\mathbf{V}^{(1)}) = 0, \ \mathcal{C}(\mathbf{V}^{(1)}) = 0 \text{ in } D, \tag{3.12}$$

and the boundary conditions

$$\mathcal{H}_\alpha(\mathbf{V}^{(1)}) = 0, \ \mathcal{D}(\mathbf{V}^{(1)}) = a_1\nu_2 n_1 \text{ on } \partial D. \tag{3.13}$$

To solve the above problem we use the method presented by IEŞAN and NAPPA in [29]. Thus, we rewrite this problem in the polar coordinates (r, θ). We denote by u and v the components of the vector $\mathbf{v}^{(1)} = (v_1^{(1)}, v_2^{(1)})$ in polar coordinates. Because the properties of materials are invariants of rotations about the axis Ox_3, the constitutive equations in polar coordinates are

$$
\begin{aligned}
t_{rr}(\mathbf{V}^{(1)}) &= c_{11}e_{rr}(\mathbf{v}^{(1)}) + c_{12}e_{\theta\theta}(\mathbf{v}^{(1)}) + b_1\phi^{(1)},\\
t_{\theta\theta}(\mathbf{V}^{(1)}) &= c_{12}e_{rr}(\mathbf{v}^{(1)}) + c_{11}e_{\theta\theta}(\mathbf{v}^{(1)}) + b_1\phi^{(1)},\\
t_{r\theta}(\mathbf{V}^{(1)}) &= (c_{11} - c_{12})e_{r\theta}(\mathbf{v}^{(1)}),\\
h_r &= a_1\frac{\partial\phi^{(1)}}{\partial r},\\
h_\theta &= a_1\frac{1}{r}\frac{\partial\phi^{(1)}}{\partial\theta},\\
g &= -b_1\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial\theta}\right] - \xi\phi^{(1)},
\end{aligned}
\tag{3.14}
$$

where

$$
\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v}{\partial\theta} + u\right), \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{1}{r}v\right).
\tag{3.15}
$$

The equilibrium equations become

$$
\begin{aligned}
\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) &= 0,\\
\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial\theta} + \frac{2}{r}t_{r\theta} &= 0,\\
\frac{1}{r}\frac{\partial}{\partial r}(rh_r) + \frac{1}{r}\frac{\partial h_\theta}{\partial\theta} + g &= 0.
\end{aligned}
\tag{3.16}
$$

The boundary conditions (3.2) become

$$
t_{rr} = 0, \qquad t_{r\theta} = 0, \qquad h_r = \frac{1}{a}a_1\nu_2\cos\theta.
\tag{3.17}
$$

Let us introduce the quantities

$$
c_1 = \frac{1}{2}\left(1 - \frac{c_{12}}{c_{11}}\right), \qquad c_2 = \frac{b_1}{c_{11}}.
\tag{3.18}
$$

As in [29], we search a solution of the above problem in the following form:

$$
u(r,\theta) = U^{(1)}(r)\cos\theta, \quad v(r,\theta) = V^{(1)}(r)\sin\theta, \quad \phi(r,\theta) = \Psi^{(1)}(r)\cos\theta,
\tag{3.19}
$$

where $U^{(1)}, V^{(1)}$ and $\Psi^{(1)}$ are solutions of the following system of differential equation:

$$
\begin{aligned}
& r^2\frac{d^2U^{(1)}}{dr^2}+r\frac{dU^{(1)}}{dr}-(1+c_1)U^{(1)}+r(1-c_1)\frac{dV^{(1)}}{dr} \\
& \qquad\qquad -(1+c_1)V^{(1)}+c_2r^2\frac{d\Psi^{(1)}}{dr}=0, \\
& c_1\left(r^2\frac{d^2V^{(1)}}{dr^2}+r\frac{dV^{(1)}}{dr}\right)-(1+c_1)V^{(1)}-r(1-c_1)\frac{dU^{(1)}}{dr} \\
& \qquad\qquad -(1+c_1)U^{(1)}-c_2r\Psi^{(1)}=0, \\
& a_1\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi^{(1)}}{dr}\right)-\frac{1}{r^2}\Psi^{(1)}\right]-b_1\left[\frac{1}{r}\frac{d}{dr}(rU^{(1)})+\frac{1}{r}V^{(1)}\right]-\xi\Psi^{(1)}=0.
\end{aligned}
\tag{3.20}
$$

It is easy to see that, in view of (2.10), we have

$$
\xi-\frac{2b_1^2}{c_{11}+c_{12}}>\frac{(b_3(c_{11}+c_{12})-2b_1c_{13})^2}{(c_{11}+c_{12})(-2c_{13}^2+(c_{11}+c_{12})c_{33})}>0. \tag{3.21}
$$

For this type of system, IEŞAN and NAPPA [29] give the following solution:

$$
\begin{aligned}
U^{(1)} &= A_1+Q_1A_2r^2-\frac{c_2}{2p}A_3[I_0(pr)+I_2(pr)], \\
V^{(1)} &= -A_1-Q_2A_2r^2+\frac{c_2}{2p}A_3[I_0(pr)-I_2(pr)], \\
\Psi^{(1)} &= A_3I_1(pr)+\frac{8b_1c_1}{\alpha(1-3c_1)p^2}A_2r,
\end{aligned}
\tag{3.22}
$$

where I_n is the modified Bessel functions of order n and

$$
\begin{aligned}
p^2 &= \frac{\xi}{a_1}-\frac{b_1^2}{c_{11}a_1}, \\
Q_1 &= \frac{1}{1-3c_1}\left(1-3c_1-\frac{3c_1c_2b_1}{a_1p^2}\right), \\
Q_2 &= \frac{1}{1-3c_1}\left(3-c_1-\frac{c_1c_2b_1}{a_1p^2}\right).
\end{aligned}
\tag{3.23}
$$

We note that in view of relation (3.21) and (2.10), it follows that the real number p is well-defined.

From the boundary conditions (3.17), the unknown constants A_2 and A_3 are

$$
\begin{aligned}
A_2 &= -\frac{2c_1b_1I_2(pa)\nu_2p^2a_1(1-3c_1)}{\Gamma a^2p^4(1-3c_1)a_1I_1'(pa)-16c_1^2b_1^2aI_2(pa)}, \\
A_3 &= \frac{1}{apI_1'(pa)}\left(\nu_2-\frac{8b_1c_1a}{a_1p^2(1-3c_1)}A_2\right),
\end{aligned}
\tag{3.24}
$$

where

$$\Gamma = (2c_{11} + c_{12})Q_1 - c_{12}Q_2 + \frac{8b_1^2 c_1}{a_1(1-3c_1)p^2}. \tag{3.25}$$

From the above discussion we can conclude that

$$\begin{aligned}
w_1^{(1)} &= \left(U^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\cos^2\theta - \left(V^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\sin^2\theta,\\
w_2^{(1)} &= (U^{(1)}(r) + V^{(1)}(r) - r^2)\sin\theta\cos\theta,\\
w_3^{(1)} &= 0,\\
\psi^{(1)} &= (\Psi^{(1)}(r) - r)\cos\theta
\end{aligned} \tag{3.26}$$

is a solution of the first problem defined by (3.1)–(3.3).

Similarly, we can find the solution of the second problem to be

$$\begin{aligned}
w_1^{(2)} &= (U^{(1)}(r) + V^{(1)}(r) - r^2)\sin\theta\cos\theta,\\
w_2^{(2)} &= \left(U^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\sin^2\theta - \left(V^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\cos^2\theta,\\
w_3^{(2)} &= 0,\\
\psi^{(2)} &= (\Psi^{(1)}(r) - r)\sin\theta.
\end{aligned} \tag{3.27}$$

From relations (3.5), (3.7), (3.9), (3.26) and (3.27) we obtain the components of the matrix $(D_{ij})_{4\times 4}$

$$\begin{gathered}
D_{11} = D_{22} = J, \qquad D_{33} = E\pi a^2, \qquad D_{44} = \frac{c_{13}\pi a^4}{2},\\
D_{12} = D_{21} = D_{\beta 3} = D_{3\beta} = D_{\beta 4} = D_{4\beta} = D_{43} = D_{34} = 0,
\end{gathered} \tag{3.28}$$

where

$$\begin{aligned}
J &= \frac{\pi}{4}a^4 Q + \frac{\pi a}{p^2}(b_3 - c_{13}c_2)A_3\left(apI_0(pa) - 2I_1(pa)\right),\\
Q &= E + c_{13}(3Q_1 - Q_2)A_2 + \frac{8b_1 b_3 c_1}{a_1 p^2(1-3c_1)}A_2,\\
E &= -2\nu_1 c_{13} - b_3\nu_2 + c_{33}.
\end{aligned} \tag{3.29}$$

From the algebraic systems (3.6) we find the unknown constants a_s to be

$$a_1 = \frac{M_2}{J}, \quad a_2 = -\frac{M_1}{J}, \quad a_3 = -\frac{R_3}{E\pi a^2}, \quad a_4 = -\frac{M_3}{c_{13}\pi a^4}. \tag{3.30}$$

With these, we have a complete expression of the solution of the problem $(\mathcal{P}_1)$. In view of the constitutive equations (2.6) we can observe that the solution constructed in this section corresponds to the null-equilibrated stress vector's values on the ends of the cylinder.

4. Solution of the problem $(\mathcal{P}_2)$

In this section we construct a solution of the problem $(\mathcal{P}_2)$. The form proposed for this solution is suggested by the results presented in [24]. The general type of solution proposed in [24] for anisotropic case and for arbitrary cross-section of the cylinder have a complex form. Using the results established in the previous Section, we propose the following simplified expression for a solution, $\mathbf{U}^{II} = (\mathbf{u}^{II}, \varphi^{II})$, of the problem $(\mathcal{P}_2)$:

$$\begin{aligned} u_\alpha^{II} &= -\frac{1}{6} b_\alpha x_3^3 + \sum_{\beta=1,2} x_3 b_\beta w_\alpha^{(\beta)}, \\ u_3^{II} &= \frac{1}{2} b_\rho x_\rho x_3^2 + w_3^*, \\ \varphi^{II} &= \sum_{\beta=1,2} x_3 b_\beta \varphi^{(\beta)}, \end{aligned} \tag{4.1}$$

where $\mathbf{W}^{(s)} = (\mathbf{w}^{(s)}, \psi^{(s)})$ are solutions of the generalized plane strain problem defined in the previous section, b_β are unknown constants which will be determined, while the function w_3^* is a solution of the following problem:

$$\begin{aligned} \Delta w_3^* &= -\sum_{\beta=1,2} b_\beta t_{33}(\mathbf{U}^{(\beta)}) - c_{44} \sum_{\beta=1,2} b_\beta w_{\alpha,\alpha}^{(\beta)}, \\ w_{3,\alpha}^* n_\alpha &= -c_{44} \sum_{\beta=1,2} b_\beta w_\alpha^{(\beta)} n_\alpha, \end{aligned} \tag{4.2}$$

where $\Delta = \dfrac{\partial}{\partial x_1^2} + \dfrac{\partial}{\partial x_2^2}$ is the Laplace operator in two dimensions.

We can observe that, in view of (3.7), (3.28), the necessary condition for the existence of solution of the above Neumann-type problem holds.

With the help of relations (3.4), (3.26) we can rewrite this problem in the polar coordinates (r, θ)

$$\begin{aligned} \Delta w_3^* &= (Mr + N I_1(pr))(b_1 \cos\theta + b_2 \sin\theta), \\ \frac{\partial w_3^*}{\partial r} &= -(U^{(1)}(a) - \frac{1}{2}\nu_1 a^2)(b_1 \cos\theta + b_2 \sin\theta), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} M &= -\Bigg\{ \left(1 + \frac{c_{13}}{c_{44}}\right)[(3Q_1 - Q_2)A_2 - 2\nu_1] \\ &\qquad + \frac{8 b_1 c_1}{a_1(1 - 3c_1)c_{44} p^2} A_2 + \frac{c_{13}}{c_{44}} c_{33} + \frac{b_3}{c_{44}} \Bigg\}, \\ N &= \left[\left(1 + \frac{c_{13}}{c_{44}}\right) c_3 - \frac{b_3}{c_{44}} \right] A_3. \end{aligned} \tag{4.4}$$

In [24], GHIBA gives the expression of solution of this type of problem in the form

$$w_3^* = W(r)(b_1 \cos\theta + b_2 \sin\theta), \tag{4.5}$$

with

$$W(r) = M\frac{r^3}{8} + N\frac{I_1(pr)}{p^2} - \left[\frac{3}{4}Ma^2 + 2N\frac{I_1'(pa)}{p^2} + 2(U^{(1)}(a) - \frac{1}{2}\nu_1 a^2)\right]\frac{r}{2}. \tag{4.6}$$

On the other hand, in view of the end conditions characteristic for the flexure problem (see also Remark 4.2 from the paper [24]), the unknown constants b_α must satisfy the equations

$$\sum_{\beta=1,2} b_\beta D_{\alpha\beta} = -R_\alpha. \tag{4.7}$$

Thus, we obtain

$$b_1 = -\frac{R_1}{J}, \qquad b_2 = -\frac{R_2}{J}. \tag{4.8}$$

The solution of the problem $(\mathcal{P}_2)$ corresponds to the following equilibrated stress on the ends of cylinder:

$$\begin{aligned} h &= \frac{a_1}{J}(\phi - \nu_2 r)(R_1\cos\theta + R_2\sin\theta) \text{ on } D_0,\\ h &= -\frac{a_1}{J}(\phi - \nu_2 r)(R_1\cos\theta + R_2\sin\theta) \text{ on } D_L \end{aligned} \tag{4.9}$$

and we can observe that the resultant flux of porosity vanishes on the ends of cylinder.

5. Conclusion

In the present paper we study the relaxed Saint–Venant's problem for circular cylinders filled with a transversely isotropic elastic porous material. The solution of the extension–bending–torsion problem is given by the relations (3.4), (3.9), (3.26), (3.27), (3.30) and the solution of the flexure problem is given by the relations (4.1) and (4.8). From the linearity of the problem, we note that the relaxed Saint–Venant's problem has a solution of the form

$$\mathbf{U} = \mathbf{U}^I + \mathbf{U}^{II}.$$

Thus, the relaxed Saint–Venant's problem for an elastic elastic transversely isotropic porous circular cylinder is completely solved. As a particular case we can retrieve the solution obtained for isotropic porous materials [24].

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