

Taxonomy of polar decompositions for singular second-order tensors in \mathbb{R}^3

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THE WELL-KNOWN POLAR DECOMPOSITION THEOREM is analyzed for the case of second-order, singular tensors on \mathbb{R}^3 . A precise analytical and geometric characterization of the split is provided for tensors of rank two and one.

Key words: polar decomposition, singular tensor.

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1. Introduction

THE POLAR DECOMPOSITION THEOREM is a basic theorem of tensor algebra which is commonly employed to understand the kinematics of a continuous medium at the differential level. In almost every book on Continuum Mechanics (e.g., [1, 2, 3]), the polar decomposition theorem is used to split the deformation gradient at each point of a body, revealing the principal stretches, the principal frame, and the local rotation of such a frame. It can be easily proven that this decomposition always exists and it is unique if the deformation gradient has a positive determinant, which is always the case due to physical reasons.

Other situations, apart from Continuum Mechanics kinematics, benefit from polar decomposition theorems. In these cases, the tensor that requires to be split, might have a negative or even zero determinant. In such circumstances it proves to be useful to identify the rotational and stretching parts of an arbitrary tensor, and to ascertain in what sense they are unique.

In Structural Mechanics, for example, there is a mapping that transforms the two directors that span a rod section from its reference to its current configuration [4, 5]. This map is of rank two, and when the cross-section changes due to the motion, the polar decomposition serves to identify which deformation actually takes place, and what is the true section rotation. In a similar fashion, when a directed shell model is employed [4, 6], the motion maps its mid-surface and

its director from their reference to their current values. In the case of a director, the polar decomposition of the mapping representing the director motion serves to clearly identify which part of this motion corresponds to a rotation and which one corresponds to a deformation. Even though in the latter one case, this can be done without the use of a polar decomposition theorem, it provides a unified treatment of the three nontrivial possible situations: a map that transforms three independent vectors (for continua), two vectors (for sections), and a single vector (for shell directors).

Other applications might be also of interest. In computer graphics, for example, given one body in two different positions related by an affine map, the polar decomposition theorem identifies the rotation and the deformation of this map, and can be employed to smoothly interpolate the two body positions [7, 8]. The deformation can be applied along a path connecting the two centers of mass, proportionally to the distance to one of the ends. In the case of rotation, a spherical interpolation (Slerp) [9] can be done also along this connecting path. The reconstruction of deformation maps at every point from the rotation and the pure deformation, result in the smoothest possible interpolation of the affine map. Using the results of this article, this approach can be employed for interpolating either the bodies, plane surfaces, or one-dimensional-type bodies. Moreover, since the approach presented herein can be done locally, the trajectory of triangulated 1D, 2D, or 3D bodies adopting two different positions related by local affine maps, can be smoothly interpolated.

The goal of this note is to study polar decompositions of second-order tensors, with emphasis on the singular ones. We identify the form of the orthogonal tensors that appear in the polar decompositions, studying their (possibly lack of) uniqueness, and providing geometric interpretations for them. All these results are provided in Section 2, making use of standard results following from differential geometry summarized in the Appendix.

2. The polar decomposition theorem for regular and singular tensors

In this section we state and prove a generalized version of the polar decomposition theorem for second-order tensors that describes in detail the form of the polar split in terms of their kernel.

We start by stating a general form of the theorem that indicates the existence of a polar decomposition for any second-order tensor. Its proof can be found, for example, in [10].

THEOREM 1. *Let \mathbf{F} be a second-order tensor. This tensor can be expressed as the product $\mathbf{F} = \mathbf{R}\mathbf{U}$ of an orthogonal tensor \mathbf{R} and a symmetric, positive*

semidefinite tensor \mathbf{U} . Moreover, \mathbf{U} is unique, defined as $\mathbf{U} = \sqrt{\mathbf{F}^T \mathbf{F}}$, hence of the same rank as \mathbf{F} .

In fact, the previous theorem defines the so-called “right” polar decomposition. In a similar fashion, there exists a “left” polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$, where \mathbf{V} is a symmetric, positive semidefinite tensor and \mathbf{R} is the orthogonal tensor mentioned in the theorem. All the results presented in this article are valid for both polar decompositions, but we will present them only for the “right” one.

In what follows, we specialize Theorem 1 to particular types of second-order tensors, providing most of the details concerning the characteristics of \mathbf{R} . The first result, which we present for completeness, is the classical polar decomposition theorem whose statement and proof can be found in most books dealing with Continuum Mechanics for the case $\det(\mathbf{F}) > 0$. References [11, 12], for example, state and prove the complete theorem.

THEOREM 2. *Let \mathbf{F} be a second-order tensor with a positive determinant. Then the polar decomposition of Theorem 1 is unique, \mathbf{U} is invertible, and \mathbf{R} is a proper orthogonal tensor. If \mathbf{F} is such that $\det(\mathbf{F}) < 0$, then the polar decomposition is still unique, \mathbf{U} is as before, and \mathbf{R} is an improper orthogonal tensor.*

The singular cases will now be discussed. First we address the situation when the rank of \mathbf{F} is two, and later – the most degenerated case when the rank is one. In each case we describe in what sense the rotation \mathbf{R} is not unique.

2.1. Rank two tensors

THEOREM 3. *Let \mathbf{F} be a singular second-order tensor with $\text{rank}(\mathbf{F}) = 2$. This tensor has exactly two polar decompositions:*

$$(2.1) \quad \mathbf{F} = \mathbf{R}^+ \mathbf{U} = \mathbf{R}^- \mathbf{U},$$

where $\mathbf{R}^+, \mathbf{R}^-$ are, respectively, a proper and an improper orthogonal tensor, and \mathbf{U} is a rank-two, symmetric, positive semidefinite tensor. Moreover, the two orthogonal tensors are related as follows:

$$(2.2) \quad \mathbf{R}^- = \mathbf{R}^+ (\mathbf{1} - 2 \mathbf{w} \otimes \mathbf{w}),$$

where \mathbf{w} is the unit vector spanning $\text{Ker}(\mathbf{F})$.

To prove Theorem 3, two preliminary results will be needed.

LEMMA 1. *Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ denote two linearly independent vectors in \mathbb{R}^3 and \mathbf{R}, \mathbf{S} – two orthogonal tensors with determinants of the same sign. If*

$$(2.3) \quad \mathbf{R} \mathbf{v}_\alpha = \mathbf{S} \mathbf{v}_\alpha$$

for $\alpha = 1, 2$, then $\mathbf{R} \equiv \mathbf{S}$.

P r o o f. In what follows, the usual notation for repeated indices will be employed unless otherwise stated. Let $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$, where “ \times ” denote the cross product in \mathbb{R}^3 . Let J denote the sign of either $\det(\mathbf{R})$ or $\det(\mathbf{S})$, assumed to be the same. Then, if (2.3) holds, it follows from the properties of orthogonal tensors that:

$$(2.4) \quad \begin{aligned} \mathbf{R}\mathbf{v}_3 &= \mathbf{R}(\mathbf{v}_1 \times \mathbf{v}_2) = J(\mathbf{R}\mathbf{v}_1) \times (\mathbf{R}\mathbf{v}_2) = J(\mathbf{S}\mathbf{v}_1) \times (\mathbf{S}\mathbf{v}_2) \\ &= J^2\mathbf{S}(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{S}\mathbf{v}_3. \end{aligned}$$

Hence, for any vector $\mathbf{a} \in \mathbb{R}^3$ expressed in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as $\mathbf{a} = a^i\mathbf{v}_i$ we have

$$(2.5) \quad \mathbf{R}\mathbf{a} = \mathbf{R}a^i\mathbf{v}_i = a^i\mathbf{R}\mathbf{v}_i = a^i\mathbf{S}\mathbf{v}_i = \mathbf{S}\mathbf{a},$$

and we conclude that $\mathbf{R} \equiv \mathbf{S}$. \square

LEMMA 2. *Let $\mathbf{v}_1, \mathbf{v}_2$, be two linearly independent vectors in \mathbb{R}^3 , and \mathbf{R} be a proper orthogonal tensor, and \mathbf{S} – an improper orthogonal tensor such that*

$$(2.6) \quad \mathbf{R}\mathbf{v}_\alpha = \mathbf{S}\mathbf{v}_\alpha$$

for $\alpha = 1, 2$. Then

$$(2.7) \quad \mathbf{S} = \mathbf{R}(\mathbf{1} - 2\mathbf{w} \otimes \mathbf{w}),$$

where $\mathbf{w} = (\mathbf{v}_1 \times \mathbf{v}_2)/|\mathbf{v}_1 \times \mathbf{v}_2|$.

P r o o f. If (2.6) holds and $\mathbf{w} = (\mathbf{v}_1 \times \mathbf{v}_2)/|\mathbf{v}_1 \times \mathbf{v}_2|$, then \mathbf{S} defined in (2.7) is an orthogonal tensor, satisfies (2.6) and is improper, because

$$(2.8) \quad \mathbf{S}\mathbf{w} = \mathbf{S}(\mathbf{v}_1 \times \mathbf{v}_2) = -(\mathbf{S}\mathbf{v}_1) \times (\mathbf{S}\mathbf{v}_2) = -(\mathbf{R}\mathbf{v}_1) \times (\mathbf{R}\mathbf{v}_2) = -\mathbf{R}\mathbf{w}.$$

Moreover, according to Lemma 1, the improper rotation \mathbf{S} must be the unique tensor that satisfies (2.6). \square

P r o o f (of Theorem 3). The existence of at least one polar decomposition of \mathbf{F} is guaranteed by Theorem 1. To prove that there are only two possible polar decompositions of the type described in the theorem, let us proceed by contradiction. Let \mathbf{R} and \mathbf{S} be two proper orthogonal tensors that satisfy:

$$(2.9) \quad \mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{S}\mathbf{U}.$$

The ranks of \mathbf{F} and \mathbf{U} are identical and thus \mathbf{U} , which is symmetric, must have two real (not necessarily distinct) nonzero eigenvalues λ_1, λ_2 with associated orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. For $\alpha = 1, 2$,

$$(2.10) \quad \mathbf{0} = (\mathbf{R}\mathbf{U} - \mathbf{S}\mathbf{U})\mathbf{v}_\alpha = \lambda_\alpha (\mathbf{R}\mathbf{v}_\alpha - \mathbf{S}\mathbf{v}_\alpha) \Leftrightarrow \mathbf{R}\mathbf{v}_\alpha = \mathbf{S}\mathbf{v}_\alpha,$$

where summation is not implied. Applying Lemma 1 to the last identity we conclude that $\mathbf{R} \equiv \mathbf{S}$.

Repeating the same argument, but with \mathbf{R} and \mathbf{S} being improper orthogonal tensors, one concludes again that they must be identical.

Finally, if \mathbf{R} and \mathbf{S} satisfy (2.9) and have determinants of opposite signs, they must still satisfy (2.10). Thus, Lemma 2 applies and \mathbf{R}, \mathbf{S} must be related as indicated in (2.7). \square

A possible geometric interpretation of Theorem 3 is as follows. If \mathbf{F} is a rank-two tensor with a given polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, then the two orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of \mathbf{U} associated with nonzero eigenvalues λ_1, λ_2 span the orthogonal complement to $\text{Ker}(\mathbf{F})$. By applying \mathbf{F} to each of them, two orthogonal vectors $\mathbf{w}_1, \mathbf{w}_2$ are obtained whose lengths are $|\mathbf{w}_1| = \lambda_1$, $|\mathbf{w}_2| = \lambda_2$.

The contents of the polar decomposition theorem is that these transformations can be decomposed in two steps. First, a stretch of the vectors \mathbf{v}_α by a factor λ_α , and then an isometry taking $\lambda_\alpha \mathbf{v}_\alpha$ (no sum) to \mathbf{w}_α . This step can be achieved by means of a (unique) rigid rotation \mathbf{R}^+ , taking the triad $\{\lambda_1 \mathbf{v}_1, \lambda_2 \mathbf{v}_2, \lambda_1 \lambda_2 \mathbf{v}_1 \times \mathbf{v}_2\}$ onto the triad $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_1 \times \mathbf{w}_2\}$. The same isometry, however, could be achieved if, before applying of \mathbf{R}^+ , a reflection from the plane spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$ is performed. The combined effect of this reflection plus the rigid rotation \mathbf{R}^+ is what corresponds to the improper rotation \mathbf{R}^- defined in the theorem, and the underlying reason for non-uniqueness of the decomposition.

There is a third way of obtaining the previous isometry. If the rigid rotation \mathbf{R}^+ is applied first, and then a reflection from the plane spanned by $\{\mathbf{w}_1, \mathbf{w}_2\}$ is performed, the same result is obtained. It turns out, however, that the composition of \mathbf{R}^+ and this second type of reflection gives the same orthogonal tensor \mathbf{R}^- as before.

2.2. Rank-one tensors

We conclude with a study of the most degenerate (nontrivial) situation. A rank one second-order tensor will be shown to have infinitely many polar decompositions. But, among all the possible orthogonal tensors, the following theorem will identify exactly one *proper* orthogonal tensor that in some sense will be the smallest one.

THEOREM 4. *Let \mathbf{F} be a singular second-order tensor with $\text{rank}(\mathbf{F}) = 1$, and let \mathbf{v} be orthogonal to $\text{Ker}(\mathbf{F})$. This tensor has infinite polar decompositions of the type described in Theorem 1, where now \mathbf{U} is of rank one. However, there is only one polar decomposition*

$$(2.11) \quad \mathbf{F} = \mathbf{R}^o \mathbf{U},$$

where \mathbf{R}^o is a proper orthogonal that rotates \mathbf{v} without drill.

P r o o f. Existence is guaranteed by Theorem 1. If λ is the unique positive eigenvalue of \mathbf{U} , the tensor \mathbf{F} maps \mathbf{v} onto $\lambda \mathbf{w}$, where \mathbf{w} is also a unit vector. Then

$$(2.12) \quad \mathbf{F}\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{v} = \lambda \mathbf{R}\mathbf{v},$$

and thus the rotation \mathbf{R} is any rotation that maps \mathbf{v} onto \mathbf{w} . Since these two vectors are both in S^2 , according to (A.6), we can find a particular rotation

$$(2.13) \quad \mathbf{R}^o = \exp[\widehat{\mathbf{v} \times \mathbf{w}}]$$

that performs this transformation without drill.

To verify that indeed there are infinite rotations that satisfy the polar decomposition of \mathbf{F} , it suffices to consider rotations of the form

$$(2.14) \quad \mathbf{R}^\alpha = \mathbf{R}^o \exp[\alpha \hat{\mathbf{v}}],$$

for any $\alpha \in \mathbb{R}$. Using the property of the exponential map

$$(2.15) \quad \exp[\hat{\boldsymbol{\theta}}]\boldsymbol{\theta} = \boldsymbol{\theta},$$

for any $\boldsymbol{\theta} \in \mathbb{R}^3$, it is easy to observe that all rotations of the form (2.14) satisfy the polar decomposition. Moreover, if \mathbf{S} is a reflection that leaves \mathbf{v} unchanged, any orthogonal tensor of the form

$$(2.16) \quad \mathbf{S}^\alpha = \mathbf{R}^\alpha \mathbf{S},$$

is also an orthogonal tensor that satisfies $\mathbf{S}^\alpha \mathbf{v} = \mathbf{w}$, and thus it is valid for the polar decomposition of \mathbf{F} . \square

3. Summary

In this note, we have reviewed the application of the polar decomposition theorem to second-order tensors, and identified (coordinate-free) expressions for the split appearing in the decomposition of regular and singular tensors. Since the regular case is well-known, we have concentrated on the rank-two and rank-one problems, providing explicit expressions of the non-unique orthogonal tensors appearing in the decompositions in terms of the kernel of the tensor, or its orthogonal complement. The results obtained are useful for the study of deformations in structural members and in computer graphics.

Appendix A. Properties of $SO(3)$ and S^2

The set of proper orthogonal tensors, known as $SO(3)$, is a Lie group with algebra $so(3)$, the set of skew-symmetric tensors. Every $\hat{\boldsymbol{\theta}} \in so(3)$ has an eigenvector $\boldsymbol{\theta} \in \mathbb{R}^3$ that lies in its kernel. This vector is called the axial vector of $\hat{\boldsymbol{\theta}}$. The exponential map $\exp : so(3) \rightarrow SO(3)$ is given in a closed form by Rodrigues' formula:

$$(A.1) \quad \exp[\hat{\boldsymbol{\theta}}] = \mathbf{1} + \frac{\sin(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|} \hat{\boldsymbol{\theta}} + \frac{1}{2} \left(\frac{\sin(|\boldsymbol{\theta}|/2)}{|\boldsymbol{\theta}|/2} \right)^2 \hat{\boldsymbol{\theta}}^2,$$

where $\boldsymbol{\theta}$ is the axial vector of $\hat{\boldsymbol{\theta}}$. See, for example, [13]. For nonzero vectors $\boldsymbol{\theta}$, the tensor $\exp[\hat{\boldsymbol{\theta}}]$ corresponds to a rotation about the axis $\boldsymbol{\theta}/|\boldsymbol{\theta}|$ of magnitude $|\boldsymbol{\theta}|$. It is then straightforward to verify that $\exp[\hat{\boldsymbol{\theta}}]\boldsymbol{\theta} = \boldsymbol{\theta}$.

We also recall the definition of the unit sphere S^2

$$(A.2) \quad S^2 = \{\mathbf{v} \in \mathbb{R}^3, \mathbf{v} \cdot \mathbf{v} = 1\}.$$

This set is a two-dimensional Lie group, and the tangent space at a point $\mathbf{d} \in S^2$ is the linear space

$$(A.3) \quad T_{\mathbf{d}}S^2 = \{\mathbf{w} \in \mathbb{R}^3, \mathbf{w} \cdot \mathbf{d} = 0\}.$$

The exponential map: $\exp_{S^2} : TS^2 \rightarrow S^2$ at a point $\mathbf{d} \in S^2$ is the function mapping the tangent space $T_{\mathbf{d}}S^2$ to the geodesic on the unit sphere passing through \mathbf{d} . The explicit expression of this mapping is, for any $\mathbf{w} \in T_{\mathbf{d}}S^2$,

$$(A.4) \quad \exp_{S^2}[\mathbf{d}, \mathbf{w}] = \cos(|\mathbf{w}|)\mathbf{d} + \frac{\sin(|\mathbf{w}|)}{|\mathbf{w}|}\mathbf{w}.$$

The function $G : [0, 1] \times S^2 \times TS^2 \rightarrow S^2$ given by

$$(A.5) \quad G(t, \mathbf{d}, \mathbf{w}) = \exp_{S^2}[\mathbf{d}, t\mathbf{w}],$$

defines the *unique* geodesic on S^2 connecting the pair of unit vectors \mathbf{d} and $\mathbf{e} = \exp_{S^2}[\mathbf{d}, \mathbf{w}]$.

The crucial geometric property that relates $SO(3)$ to S^2 is that, for every pair $\mathbf{d}, \mathbf{e} \in S^2$ as before, there exists a unique proper rotation \mathbf{R} which rotates \mathbf{d} onto \mathbf{e} and whose axis of rotation is perpendicular to both \mathbf{d} and \mathbf{e} ([6]). This map is said to rotate \mathbf{d} to \mathbf{e} without drill and its given by:

$$(A.6) \quad \mathbf{R} = \exp[\widehat{\mathbf{d} \times \mathbf{w}}].$$

A closed-form expression for this rotation is:

$$(A.7) \quad \mathbf{R} = \frac{1}{\sin^2 \phi} (\cos \phi (\mathbf{d} \otimes \mathbf{d} + \mathbf{w} \otimes \mathbf{w}) + (1 - 2 \cos^2 \phi) \mathbf{w} \otimes \mathbf{d} - \mathbf{d} \otimes \mathbf{w} + \mathbf{p} \otimes \mathbf{p}),$$

with $\mathbf{p} = \mathbf{d} \times \mathbf{w}$ and $\cos \phi = \mathbf{d} \cdot \mathbf{w}$.

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