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Uniqueness in thermoelasticity of porous media with microtemperatures

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The problem determined by thermoelastic deformations when the internal energy is not positive definite, becomes ill-posed. We recall that this kind of situation happens in the study of prestressed thermoelastic solids. Thus, it will be of interest to obtain qualitative properties of solutions in this case. In this note we prove the uniqueness of solutions for the linear thermo-poro-elasticity with microtemperatures theory, when the internal energy is not to be assumed to be positive definite. We use the energy arguments combined with the Lagrange identities.

Key words: thermo-poro-elasticity, uniqueness of solutions, pre-stressed solids, Lagrange identities.

1. Introduction

The theory of elastic solids with voids is the simplest generalization of the classical theory of elasticity; however, it is worth recalling that porous materials have applications in many fields of engineering such as petroleum industry, material science, biology, etc. When elastic solids with voids are considered, as in this paper, one should look into the theory of porous elastic materials. Here we deal with the theory established by Cowin and Nunziato (see [3, 4, 14]). In their approach, the bulk density is the product of two scalar fields: the matrix material density and the volume fraction field. This was deeply discussed in the book by Ieşan [6].

The study of the basic qualitative properties of solutions is the fundamental work to develop for different thermomechanical situations. In this note we deal with the system of equations which governs the thermo-poro-elasticity with microtemperatures. This system has attracted much attention in last years [19, 20]. We can recall that it was proposed by Ieşan [8] (see also [10]) and recently it has been extended to a more general case, when the material points admit micropolar structure [9]. In these contributions, several existence and uniqueness

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results were obtained, however they were based on the assumption of positivity of the internal energy. Further studies, such as exponential decay of solutions or impossibility of localization of the solutions, have been also obtained recently [2, 13, 17]. It is also worth recalling that in the reference [11], the equations of the thermo-poro-elasticity with microtemperatures are also proposed, but in the context of the thermomechanical theories proposed by Green and Naghdi. It is worth noting that contributions concerning qualitative properties, for the theory we deal, always assume that the internal energy is positive. However, it is known that this assumption is not needed to prove the uniqueness of solutions in several thermomechanical situations [18]. Moreover, there are many thermomechanical situations where the internal energy is not positive definite. This happens in the study of problems concerning solids with initial pre-stress [6, 7]. However, we do not know any uniqueness result concerning thermomechanical theories with microtemperatures, when internal energy is not assumed to be positive definite. The loss of symmetry of this system introduces a new difficulty in the studies of this kind of problems (from the mathematical point of view). Thus, it is necessary to introduce an alternative argument to avoid this difficulty.

We recall that in absence of the supply terms, the evolution equations which govern the problem of the thermo-poro-elasticity with microtemperatures are (see [9]):

$$\rho\ddot{u}_i = S_{ji,j}, \qquad J\ddot{\phi} = h_{i,i} + g, \qquad \rho T_0\dot{\zeta} = q_{j,j}, \qquad \rho\dot{\eta}_i = q_{ji,j} + q_i - Q_i,$$

where S_{ij} is the stress tensor, h_i is the equilibrated stress vector, g is the intrinsic equilibrated body force, ζ is the entropy, q_i is the heat flux vector, η_i is the first moment energy vector, Q_i is the microheat flux average, q_{ji} is the first heat flux moment tensor, ρ is the mass density and J is the product of the mass density and the equilibrated inertia. As usual, u_i is the displacement, T_0 is the temperature at the reference configuration and ϕ is the volume fraction.

In this paper we restrict our attention to the case when the materials have a center of symmetry. Then, the constitutive tensors of odd order must vanish and the constitutive equations take the form

$$\begin{aligned}
S_{ji} &= C_{ijkl}e_{kl} + B_{ij}\phi - a_{ij}\theta, & & \\
h_i &= A_{ij}\phi_{,j} - N_{ij}T_j, & g &= -B_{ij}e_{ij} - \eta\phi + f\theta, \\
\rho\eta_i &= -N_{ji}\phi_{,j} - M_{ij}T_j, & q_{ij} &= -P_{ijrs}T_{s,r}, \\
\rho\zeta &= a_{ij}e_{ij} + f\phi + a\theta, & Q_i &= (H_{ij} - \Lambda_{ij})T_j + (k_{ij} - K_{ij})\theta_{,j}, \\
q_i &= k_{ij}\theta_{,j} + H_{ij}T_j. & &
\end{aligned}$$

It is worth recalling that C_{ijkl} is the elasticity tensor, A_{ij}, B_{ij}, η are tensorial functions which are typical in porous media theories and Λ_{ij}, $M_{ij}, N_{ij}, H_{ij}, K_{ij}$

and P_{ijrs} are tensors which are usual in the theories with microtemperatures, k_{ij} is the thermal conductivity tensor, a_{ij} is the thermal dilatation tensor and a is the heat capacity. As usual, θ means the relative temperature and T_i are the microtemperatures.

If we introduce the constitutive equations into the evolution equations, we obtain the system of field equations for the thermo-poro-elasticity with microtemperatures:

$$\rho\ddot{u}_i = (C_{ijkl}u_{k,l} + B_{ij}\phi + a_{ij}\theta)_{,j}, \tag{1.1}$$

$$J\ddot{\phi} = (A_{ij}\phi_{,i} - N_{ij}T_i)_{,j} - B_{ij}u_{i,j} - \eta\phi + f\theta, \tag{1.2}$$

$$a\dot{\theta} = -f\dot{\phi} - a_{ij}\dot{e}_{ij} + \frac{1}{T_0}(k_{ij}\theta_{,i} + H_{ij}T_i)_{,j}, \tag{1.3}$$

$$M_{ij}\dot{T}_j = (P_{jirs}T_{s,r})_{,j} - \Lambda_{ij}T_j - N_{ji}\dot{\phi}_{,j} - K_{ij}\theta_{,j}. \tag{1.4}$$

In this situation it is usual to assume that the inequality:

$$k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0K_{ij})\theta_{,j}T_i + T_0\Lambda_{ij}T_iT_j + T_0P_{jirs}T_{i,j}T_{s,r} \geq 0 \tag{1.5}$$

is satisfied. The last inequality is a consequence of the Clausius-Duhem inequality in the context of the theories with microtemperatures (see [9]).

In the case of isotropic and homogeneous materials, the constitutive equations become

$$\begin{aligned}
S_{ji} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + b\phi\delta_{ij} - \beta\theta\delta_{ij}, \\
h_i &= \gamma\phi_{,i} - dT_i, & g &= -be_{rr} - \eta\phi + f\theta, \\
\rho\eta_i &= d\phi_{,i} - \alpha T_i, & q_{ij} &= -\kappa_4T_{r,r}\delta_{ij} - \kappa_5T_{i,j} - \kappa_6T_{j,i}, \\
\rho\zeta &= \beta e_{rr} + f\phi + a\theta, & Q_i &= (\kappa_1 - \kappa_2)T_i + (k - \kappa_3)\theta_{,i}, \\
q_i &= k\theta_{,i} + \kappa_1T_i,
\end{aligned}$$

where $\lambda, \mu, b, \beta, \gamma, \zeta, f, a, \alpha, d, k, \kappa_i$, $(i = 1, \ldots, 6)$ are given constants. For the three-dimensional case, condition (1.5) becomes (see [5]):

$$k \geq 0,\ 3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \quad \kappa_5 + \kappa_6 \geq 0,\ \kappa_6 - \kappa_5 \geq 0, \quad (\kappa_1 + T_0\kappa_3)^2 \leq 4T_0k\kappa_2.$$

Substituting our constitutive equations into the evolution equations, we obtain the system of the field equations in the case of isotropic and homogeneous materials:

$$\begin{aligned}
\rho\ddot{u}_i &= \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + b\phi_{,i} - \beta\theta_{,i}, \\
J\ddot{\phi} &= \gamma\phi_{,jj} - bu_{i,i} - \eta\phi - dT_{i,i} + f\theta - \tau\dot{\phi}, \\
a\dot{\theta} &= k\theta_{,jj} - \beta T_0\dot{u}_{i,i} - fT_0\dot{\phi} + \kappa_1T_{i,i}, \\
\alpha\dot{T}_i &= \kappa_6T_{i,jj} + (\kappa_4 + \kappa_5)T_{j,ji} - d\dot{\phi}_{,i} - \kappa_3\theta_{,i} - \kappa_2T_i.
\end{aligned}$$

It is worth noting that after a non-dimensionalizing procedure, we can assume that the tensor and variables used here are non-dimensional.

Our approach is strongly based on the Lagrange identities and on energy arguments. However, it is worth noting that in the case of thermoelasticity with microtemperatures, the existence of the tensors H_{ij} and K_{ij} introduces new mathematical difficulties. Thus, our approach must be different from the one used in classical thermoelasticity or type III thermoelasticity [18]. In fact, we need to work with an inequality of the type (4.11) which relates a measure (on the solutions) at the moment "t" to the integral of this measure until the moment "$2t$". Nevertheless, this kind of inequality can be treated as in the case of the usual thermoelasticity for exterior domains [16].

The plan of this note is the following: in section two, we present the assumptions concerning the constitutive tensors for the thermo-poro-elasticity with microtemperatures we are going to work with. In Section 3 we obtain two equalities which will be used in Sec. 4. One of them is the conservation law of the energy and the second one is a Lagrange type identity. In Section 4 we state our main theorem, and in Sec. 5 we recall the main conclusion of this short note.

2. Assumptions and restrictions

The aim of this section is to determine the assumptions on the constitutive tensors for the thermomechanical theory we are going to work with. To clarify the situation, we present the assumptions concerning the constitutive tensors. First, we recall that the tensors C_{ijkl} and A_{ij} are symmetric. That is,

$$C_{ijkl} = C_{klij}, \qquad A_{ij} = A_{ji}. \tag{2.1}$$

In general, the tensors P_{ijrs}, M_{ij} and $\varLambda_{ij}$ are not symmetric. However, we assume that we can substitute them for symmetric tensors P^*_{ijrs}, M^*_{ij} and $\varLambda^*_{ij}$ in the field equations (of course, this is not possible in treatment of the constitutive equations). This assumption is not odd, in the sense that it happens at least in two relevant and different cases: for one-dimensional materials and in the case of isotropic and homogeneous materials, as it can be seen in the corresponding system of equations. In this case we can write:

$$M^*_{ij} = \alpha\delta_{ij}, \varLambda^*_{ij} = \kappa_2\delta_{ij}, P^*_{ijrs} = k_6\delta_{is}\delta_{jr} + (k_4 + k_5)\delta_{ij}\delta_{rs},$$

where δ_{ij} is the Kronecker delta.

Thus, from now on, we omit the stars and assume that

$$P_{ijkl} = P_{klij}, \qquad M_{ij} = M_{ji}, \qquad \varLambda_{ij} = \varLambda_{ji}. \tag{2.2}$$

We impose conditions (2.2) because of technical arguments. To be precise, we note that these conditions are needed to obtain the relation (3.8) which plays a relevant role in our proof.

In addition to the assumptions that the constitutive coefficients are bounded from above, we also need to impose the positivity of several functions and tensors. So we also assume that:

(i) $\rho(\mathbf{X}) \geq \rho_0 > 0$, $a(\mathbf{X}) \geq J_0 > 0$, $a(\mathbf{X}) \geq a_0 > 0$, $M_{ij}\xi_i\xi_j \geq m_0\xi_i\xi_i$, where m_0 is a positive constant.

(ii) $k_{ij}\xi_i\xi_j + (H_{ij} + T_0K_{ij})\xi_j\zeta_i + T_0\Lambda_{ij}\zeta_i\zeta_j \geq C_0(\xi_i\xi_i + \zeta_i\zeta_i)$, where C_0 is a positive constant, for every (ξ_j) and (ζ_i).

(iii) $P_{jirs}\xi_{ij}\xi_{sr} \geq C_1\xi_{ij}\xi_{ij}$, where C_1 is a positive constant, for every (ξ_{ij}).

The above assumptions are in agreement with the physical experience. The thermomechanical interpretation of conditions (i) is obvious. Conditions (ii) and (iii) follow from the Clausius–Duhem conditions recalled in Eq. (1.5). However, we note that the assumptions were imposed previously on the symmetrized tensors, which can be different from the ones given by (1.5). For this reason, we believe that it is necessary to clarify what happens in the case of isotropic and homogeneous materials. In this sense we note that condition (ii) holds because in this case the tensors M_{ij}, Λ_{ij} are symmetric. The condition (iii) is not the same. This condition is satisfied at least when $\kappa_6 > 0$ and $-\kappa_6 < \kappa_4 + \kappa_5$. We note that this is more suitable than the corresponding inequalities proposed by Clausius–Duhem condition.

3. Some basic relations

The aim of this section is to set down two relations which will be useful to prove our main theorem. Since we want to prove the uniqueness of solutions, we must show that the only solution of the problem determined by null initial conditions and null boundary conditions, is the null solution. We assume that we study our system in a bounded domain B with a boundary ∂B, which is smooth enough to apply the divergence theorem.

We assume the null initial conditions:

$$(3.1)\qquad u_i(\mathbf{X},0) = \dot{u}_i(\mathbf{X},0) = \phi(\mathbf{X},0) = \dot{\phi}(\mathbf{X},0) = \theta(\mathbf{X},0) = T_i(\mathbf{X},0) = 0, \quad \mathbf{X} \in B$$

and the null boundary conditions:

$$(3.2)\qquad u_i(\mathbf{X},t) = \phi(\mathbf{X},t) = \theta(\mathbf{X},t) = T_i(\mathbf{X},t) = 0, \quad \mathbf{X} \in \partial B,\ t \geq 0.$$

Now, we determine the basic relations. The first one is the energy relation. After integration and use of the boundary conditions and the initial conditions, we obtain:

$$\frac{1}{2}\int_B (\rho\dot{u}_i\dot{u}_i + J|\dot{\phi}|^2 + C_{ijkl}u_{i,j}u_{k,l} + 2B_{ij}u_{i,j}\phi + \eta\phi^2 + A_{ij}\phi_{,i}\phi_{,j} + a\theta^2 + M_{ij}T_iT_j)dV = -\int_0^t\int_B \Big(\frac{1}{T_0}(k_{ij}\theta_{,i}\theta_{,j} + (H_{ij}+T_0K_{ij})\theta_{,j}T_i + T_0\Lambda_{ij}T_iT_j + T_0P_{jirs}T_{i,j}T_{s,r})\Big)dV\,ds. \tag{3.3}$$

The second identity we need follows from the Lagrange identities method and it can be derived as in [1, 12, 15, 18]. For a fixed $t \in (0,T)$, we use the identities:

$$\frac{\partial}{\partial s}[\rho\dot{u}_i(s)\dot{u}_i(2t-s)] = \rho\ddot{u}_i(s)\dot{u}_i(2t-s) - \rho\dot{u}_i(s)\ddot{u}_i(2t-s), \tag{3.4}$$

$$\frac{\partial}{\partial s}[J\dot{\phi}(s)\dot{\phi}(2t-s)] = J\ddot{\phi}(s)\dot{\phi}(2t-s) - J\dot{\phi}(s)\ddot{\phi}(2t-s), \tag{3.5}$$

$$\frac{\partial}{\partial s}[a\theta(s)\theta(2t-s)] = a\dot{\theta}(s)\theta(2t-s) - a\theta(s)\dot{\theta}(2t-s), \tag{3.6}$$

$$\frac{\partial}{\partial s}[M_{ij}T_i(s)T_j(2t-s)] = M_{ij}\dot{T}_i(s)T_j(2t-s) - M_{ij}T_i(s)\dot{T}_j(2t-s), \tag{3.7}$$

the basic equations (1.1) to (1.4), the initial conditions (3.1) and the boundary conditions (3.2), to obtain

$$\int_B (\rho\dot{u}_i\dot{u}_i + J|\dot{\phi}|^2 - a\theta^2 - M_{ij}T_iT_j)dV = \int_B (C_{ijkl}u_{k,l}u_{i,j} + 2B_{ij}u_{i,j}\phi + \eta\phi^2 + A_{ij}\phi_{,i}\phi_{,j})dV + \Omega(t), \tag{3.8}$$

where

$$\Omega(t) = \frac{1}{T_0}\int_0^t\int_B \Big((H_{ij}+T_0K_{ij})(\theta_{,j}(s)T_i(2t-s)) + ((H_{ij}+T_0K_{ij})T_i(s))_{,j}\theta(2t-s)\Big)dV\,ds. \tag{3.9}$$

From (3.3) and (3.8) we obtain

$$(3.10)\qquad E_1(t) = \int\limits_B (\rho\dot{u}_i\dot{u}_i + J|\dot{\phi}|^2)dV$$
$$= -\int\limits_0^t\int\limits_B\Big(\frac{1}{T_0}(k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0K_{ij})\theta_{,j}T_i$$
$$+ T_0\Lambda_{ij}T_iT_j + T_0P_{jirs}T_{i,j}T_{s,r})\Big)dV\,ds + \frac{\Omega(t)}{2}.$$

This equality will be used in the next section.

4. The main result

In this section we prove the uniqueness of solutions in the case of thermo-poro-elasticity with microtemperatures. It is worth noting that in this case, the proof presents several similarities to the case of poro-elasticity with the usual heat conduction, however the tensors H_{ij} and K_{ij} need a special treatment which enables us to obtain an inequality of type (4.11) which is different of Gronwall's-type inequality. This is the main point of the proof.

LEMMA 1. *Let us assume that the conditions* (i), (ii), (iii), (2.1) *and* (2.2) *hold. Let* (u_i, ϕ, θ, T_i) *be a solution of the problem determined by the system* (1.1)–(1.4), *the initial conditions* (3.1) *and the boundary conditions* (3.2). *Then* $u_i = \phi = \theta = T_i = 0$.

P r o o f. If we multiply (1.3) by θ and (1.4) by T_i, after integration and using the divergence theorem and the boundary conditions, we obtain the equality:

$$(4.1)\qquad E_2(t) = \frac{1}{2}\int\limits_B (a\theta^2 + M_{ij}T_iT_j)dV$$
$$= -\int\limits_0^t\int\limits_B\Big(\frac{1}{T_0}(k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0K_{ij})\theta_{,j}T_i$$
$$+ T_0\Lambda_{ij}T_iT_j + T_0P_{jirs}T_{i,j}T_{s,r})\Big)dV\,ds$$
$$-\int\limits_0^t\int\limits_B (N_{ji}\dot{\phi}_{,j}T_i + a_{ij}\dot{u}_{i,j}\theta + f\theta\dot{\phi})dV\,ds.$$

Let ϵ be a small, but positive constant. Let us consider $E(t) = E_1(t) + \epsilon E_2(t)$. We note that the function:

$$E(t) = \int_B \left(\rho \dot{u}_i \dot{u}_i + J|\dot{\phi}|^2 + \frac{a\epsilon}{2}\theta^2 + \frac{\epsilon}{2} M_{ij} T_i T_j \right) dV \tag{4.2}$$

is a positive function that defines a measure on the solutions. We have

$$\begin{aligned} E(t) = -(1+\epsilon) \int_0^t \int_B \frac{1}{T_0} (k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0 K_{ij})\theta_{,j} T_i \\ + T_0 \Lambda_{ij} T_i T_j + T_0 P_{jirs} T_{i,j} T_{s,r}) dV ds \\ - \epsilon \int_0^t \int_B \left(N_{ji} \dot{\phi}_{,j} T_i + a_{ij} \dot{u}_{i,j} \theta + f\theta\dot{\phi} \right) dV ds + \frac{\Omega(t)}{2}. \end{aligned} \tag{4.3}$$

Application of the A-G inequality and the divergence theorem enables us to obtain estimates of the type

$$\left| \int_0^t \int_B a_{ij} \dot{u}_{i,j} \theta dV ds \right| \le M_1 \int_0^t \int_B \rho \dot{u}_i \dot{u}_i dV ds + \epsilon_1 \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dV ds, \tag{4.4}$$

$$\left| \int_0^t \int_B N_{ji} \dot{\phi}_{,j} T_i dV ds \right| \le M_1^* \int_0^t \int_B J|\dot{\phi}|^2 dV ds + \epsilon_1^* \int_0^t \int_B P_{jirs} T_{i,j} T_{s,r} dV ds, \tag{4.5}$$

$$\left| \int_0^t \int_B f\dot{\phi}\theta dV ds \right| \le M_1^{**} \int_0^t \int_B J|\dot{\phi}|^2 dV ds + \epsilon_1^{**} \int_0^t \int_B a\theta^2 dV ds, \tag{4.6}$$

where ϵ_1, ϵ_1^* and ϵ_1^{**} are positive constants which can be selected as small as we want and M_1, M_1^* and M_1^{**} are positive constants. They can be expressed in terms of the constitutive parameters, ϵ_1, ϵ_1^* and ϵ_1^{**}.

It will be suitable to pay attention to $\Omega(t)$. We can obtain

$$\begin{aligned} \frac{|\Omega(t)|}{2} \le M_2 \int_0^{2t} \int_B M_{ij} T_i(s) T_j(s) dV ds + \epsilon_2 \int_0^t \int_B k_{ij} \theta_{,i}(s) \theta_{,j}(s) dV ds \\ + M_3 \int_0^{2t} \int_B a\theta^2(s) dV ds + \epsilon_3 \int_0^t \int_B (P_{jirs} T_{i,j}(s) T_{s,r}(s) \\ + M_{ij} T_i(s) T_j(s)) dV ds, \end{aligned}$$

where ϵ_2 and ϵ_3 are positive, but as small as we want. Due to relation (3.3), we have

$$E(t) \le \epsilon(M_1 \int_0^t \int_B \rho \dot{u}_i \dot{u}_i dV ds + (M_1^* + M_1^{**}) \int_0^t \int_B J|\dot{\phi}|^2 dV ds \tag{4.7}$$
$$+ \epsilon_1^{**} \int_0^t \int_B a\theta^2 dV ds) + (\epsilon\epsilon_1 + \epsilon_2) \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dV ds$$
$$+ (\epsilon\epsilon_1^* + \epsilon_3) \int_0^t \int_B (P_{jirs}T_{i,j}T_{s,r} + M_{ij}T_i(s)T_j(s)) dV ds$$
$$+ M_2 \int_0^{2t} \int_B M_{ij}T_i(s)T_j(s) dV ds + M_3 \int_0^{2t} \int_B a\theta^2(s) dV ds$$
$$- (1+\epsilon) \int_0^t \int_B \frac{1}{T_0}(k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0 K_{ij})\theta_{,j}T_i$$
$$+ T_0 \Lambda_{ij} T_i T_j + T_0 P_{jirs} T_{i,j} T_{s,r}) dV ds.$$

In view of condition (ii) we can always select $\epsilon_1, \epsilon_1^*, \epsilon_2$ and ϵ_3 in such a way that the inequality

$$- (1+\epsilon) \int_0^t \int_B \frac{1}{T_0}(k_{ij}\theta_{,i}\theta_{,j} + (H_{ij} + T_0 K_{ij})\theta_{,j}T_i \tag{4.8}$$
$$+ T_0 \Lambda_{ij} T_i T_j + T_0 P_{jirs} T_{i,j} T_{s,r}) dV ds + (\epsilon\epsilon_1 + \epsilon_2) \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dV ds$$
$$+ (\epsilon\epsilon_1^* + \epsilon_3) \int_0^t \int_B (P_{jirs}T_{i,j}T_{s,r} + M_{ij}T_i(s)T_j(s)) dV ds \le 0,$$

is satisfied. It then follows that the inequality

$$E(t) \le \epsilon \left(M_1 \int_0^t \int_B \rho \dot{u}_i \dot{u}_i dV ds \right. \tag{4.9}$$
$$\left. + (M_1^* + M_1^{**}) \int_0^t \int_B J|\dot{\phi}|^2 dV ds + \epsilon_1^{**} \int_0^t \int_B a\theta^2 dV ds \right)$$
$$+ M_2 \int_0^{2t} \int_B M_{ij}T_i(s)T_j(s) dV ds + M_3 \int_0^{2t} \int_B a\theta^2(s) dV ds,$$

holds true. Thus we can obtain a positive constant C such that

$$E(t) \leq C \int_0^{2t} \int_B (\rho \dot{u}_i \dot{u}_i + J|\dot{\phi}|^2 + a\theta^2 + M_{ij} T_i T_j) dV ds. \tag{4.10}$$

In view of the definition of the function $E(t)$ in (4.2), we can find a positive constant C^* such that the estimate:

$$E(t) \leq C^* \int_0^{2t} E(s) ds \tag{4.11}$$

is satisfied for every $t \geq 0$. After integration we obtain that

$$\begin{aligned} \int_0^{2t} E(s) ds &\leq C^* \int_0^{2t} \int_0^{2s} E(\eta) d\eta ds = C^* \int_0^{2t} 2(t-s) E(s) ds \\ &\leq 2C^* t \int_0^{2t} E(s) ds. \end{aligned} \tag{4.12}$$

It then follows that

$$(1 - 2C^* t) \int_0^{2t} E(s) ds \leq 0. \tag{4.13}$$

If we assume $t_0 = (2C^*)^{-1}$, we obtain that $E(t)$ vanishes in the interval $(0, t_0)$. If we take into account the definition of $E(t)$, it follows that $\theta \equiv 0$, $\dot{\phi} \equiv 0$, $T_i \equiv 0$ and $\dot{u}_i \equiv 0$ for every $t \leq t_0$. Thus, we have proved that the problem determined by our system of equations with the boundary conditions (3.2) and the initial condition (3.1), has only one solution in the interval $[0, t_0]$. If we apply the same approach to the problem determined by our system, the boundary conditions (3.1) and initial conditions

$$\begin{aligned} u_i(\mathbf{X}, t_0) = \dot{u}_i(\mathbf{X}, t_0) = \phi(\mathbf{X}, t_0) = \dot{\phi}(\mathbf{X}, t_0) \\ = \theta(\mathbf{X}, t_0) = T_i(\mathbf{X}, t_0) = 0, \qquad \mathbf{X} \in B, \end{aligned} \tag{4.14}$$

we can conclude that $\theta \equiv 0$, $\phi \equiv 0$, $T_i \equiv 0$ and $u_i \equiv 0$ for every $t \leq 2t_0$. This approach can be repeated successively and we will conclude that for arbitrary positive time, the solution vanishes. It follows that the only solution to our problem is the null solution and the lemma is proved.

Thus, if we consider the problem determined by the system (1.1)–(1.4) with arbitrary initial and Dirichlet boundary conditions, we have proved:

THEOREM 1. *Let us assume that the conditions* (i), (ii), (iii), (2.1) *and* (2.2) *hold. Let* $(u_i^1, \phi^1, \theta^1, T_i^1)$ *and* $(u_i^2, \phi^2, \theta^2, T_i^2)$ *be two solutions of the problem determined by the system* (1.1)–(1.4), *with the same initial and Dirichlet boundary conditions. Then the two solutions agree, that is* $u_i^1 = u_i^2$, $\phi^1 = \phi^2$, $\theta^1 = \theta^2$, $T_i^1 = T_i^2$.

REMARK. Our arguments could be adapted to prove the Holder continuous dependence results if we assume *a priori* bounds on the solutions. Also, it is possible to adapt the arguments to the case of unbounded domains, and in the general case when polar effects are also considered [9].

5. Conclusions

The aim of this note was to study the uniqueness of solutions for the linear thermo-poro-elasticity with microtemperatures when internal energy is not positive definite. This situation happens in case of prestressed thermoelastic bodies. We have established the uniqueness of solutions. We have used the energy arguments combined with the Lagrange identities.

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