Representing a non-associated constitutive law by a bipotential issued from a Fitzpatrick sequence

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WE ANALYZE THE RELATION between Géry de Saxcé's bipotentials representing non-associated constitutive laws and Fitzpatrick's functions representing maximal monotone multifunctions. We illustrate by two examples (one linear and monotone, the other non-linear and non-monotone) the fact that Fitzpatrick's representation coming from convex analysis provides a constructive method to discover the "best" bipotential modelling of a given Implicit Standard Material.

Key words: Generalized Standard Materials, non-associated material laws, Implicit Standard Materials, constitutive laws, point-to-set functions, multifunctions, maximal monotone operators, Fitzpatrick functions, Fitzpatrick sequences, bipotentials.

1. Introduction

STANDARD MATERIALS ARE MODELLED by differentiable potentials. Mainly to capture the multivalued constitutive laws, for example plastic flow rules [6, 21], they were extended to the so-called "Generalized Standard Materials" (GSM) modelled by lower-semi-continuous (lsc) convex potentials [9].

But this extension failed to describe Coulomb's dry friction law. In 1991, considering such an implicit constitutive law, GÉRY DE SAXCÉ and Z. Q. FENG [24, 25] proposed a new generalization, which they called "Implicit Standard Material" (ISM). This new class of materials is modelled by a point-to-point function which they called a bipotential. In the particular case of a GSM, the bipotential is reduced to the sum of the potential and its conjugate potential.

For a given GSM, a theorem due to R. T. ROCKAFELLAR [23] and J. J. MOREAU [22] provides a constructive method to retrieve the potential from the data of its subdifferential. A similar, crucial question for a given ISM is: how to retrieve the bipotential from the implicit constitutive law?

Independently, in 1988, Simon FITZPATRICK [7], in order to simplify the study of monotone operators, made a proposal to replace these multifunctions by point-to-point functions, nowadays called Fitzpatrick's functions.

Is there any relation between Géry de Saxcé's bipotentials representing ISM constitutive laws and Fitzpatrick's functions representing maximal monotone multifunctions? Can this last representation coming from convex analysis [3, 4, 14, 22] provide a constructive method to discover the "best" bipotential modelling of a given ISM? The aim of our paper is to give some answers to these two questions.

2. Standard materials

A constitutive law relating
- a strain-like variable x belonging to a real Banach space X (with norm $\|\cdot\|$) and
- a stress-like variable y belonging to the continuous dual space $Y = X^*$ (with duality product $\langle\cdot,\cdot\rangle$)

is defined as a subset of the set $X \times Y$.

This subset can be regarded as the graph of a multifunction $T : X \longrightarrow 2^Y$

$$G(T) = \{(x,y) \in X \times Y \mid y \in Tx\}.$$

In finite dimensions, when this subset is a maximal Lagrangian submanifold of the linear space $X \times Y$ (made symplectic by the canonical Darboux 2-form), then there exists a differentiable function ϕ, called "potential", such that

$$y = D\phi(x).$$

If additionally this potential is convex, the inverse constitutive law reads

$$x = D\phi^*(y)$$

with ϕ^* the Legendre transform of the potential ϕ (called "conjugate potential"). A material whose behavior can be described by a differentiable potential is referred to as a "Standard Material" (SM).

3. Generalized Standard Materials

For many materials, the relation between x and y is a multifunction. Dropping the differentiability of the potential ϕ, but keeping its convexity and its lower semi-continuity, a large class of materials, called "Generalized Standard Materials" (GSM), can be described by one of the following three equivalent constitutive laws:

(i) $y \in \partial\phi(x)$,

(ii) $x \in \partial\phi^*(y)$,

(iii) $\phi(x) + \phi^*(y) = \langle x, y \rangle$.

REMARK. The convexity of the potential ϕ allows to express the conjugate potential ϕ^* as a supremum [22]:

$$\phi^*(y) = \sup_{x \in X} \left[\langle x, y \rangle - \phi(x) \right].$$

REMARK. The subdifferentials

$$\partial\phi(x) = \{ y \in Y \mid \forall \xi \in X, \phi(\xi) \geq \phi(x) + \langle \xi - x, y \rangle \},$$
$$\partial\phi^*(y) = \{ x \in X \mid \forall \eta \in Y, \phi^*(\eta) \geq \phi^*(y) + \langle x, \eta - y \rangle \},$$

of the potentials ϕ and ϕ^* generalize [22] the differentials $D\phi$ and $D\phi^*$; usually they are not reduced to a unique gradient.

4. Implicit Standard Materials

4.1. Bipotentials

The equality (iii) can be regarded as an extremal case of Fenchel's inequality

$$\phi(x) + \phi^*(y) \geq \langle x, y \rangle.$$

To model the dry friction phenomena or the behavior of materials such as clays [24–26], GÉRY DE SAXCÉ noticed that it was useful to weaken Fenchel's inequality to

$$b(x, y) \geq \langle x, y \rangle.$$

Dropping the decomposition as a sum of two potentials

$$b(x, y) = \phi(x) + \phi^*(y),$$

he called the function $b(x, y)$ a bipotential.

The bipotentials $b(x, y)$ are assumed to be

(i) convex and lsc in x,

(ii) convex and lsc in y,

(iii) bounded from below by the duality product: $b(x, y) \geq \langle x, y \rangle$.

4.2. Implicit Standard Materials

A material whose behavior can be described equivalently by one of the following three constitutive laws:

(iv) y belongs to the subdifferential of $b(x, y)$ with respect to x,

(v) x belongs to the subdifferential of $b(x, y)$ with respect to y,

(vi) $b(x, y) = \langle x, y \rangle$

is referred to as an "Implicit Standard Material" (ISM).

4.3. Examples of Implicit Standard Materials

The Implicit Standard Material model has shown to be relevant for the description of many non-associated phenomena ([26] and references contained therein):

- unilateral contact with Coulomb dry friction [24],
- generalized Drucker–Prager plasticity [26],
- modified Cam–Clay model [26],
- non-linear kinematical hardening rule for cyclic plasticity of metals [1, 16],
- Lemaître's plastic-ductile damage law [15].

5. Maximal monotone constitutive laws

5.1. Basic facts

DEFINITION. A constitutive law associated with a multifunction T is monotone if [3, 4, 22, 33]

$$y_1 \in Tx_1 \text{ and } y_2 \in Tx_2 \implies \langle x_2 - x_1, y_2 - y_1 \rangle \geq 0.$$

DEFINITION. A monotone multifunction T is maximal if no proper enlargement of T is monotone [3, 4, 22, 33].

As mentioned in Sec. 2, multifunctions T are used to model constitutive laws. Due to their implicit nature, they are quite difficult to handle – this is the reason to seek equivalent representations. The following lemma provides such an equivalent representation for maximal monotone multifunctions. Its proof is straightforward (for example, by contradiction) and can be found in [33].

LEMMA 1. *If T is a maximal monotone multifunction, then*

$$\forall x \in X, \forall y \in Y, \inf_{y_1 \in Tx_1} \langle x - x_1, y - y_1 \rangle \leq 0$$

and the value 0 is attained if and only if $y \in Tx$.

5.2. Fitzpatrick's function

DEFINITION. Let T be a maximal monotone multifunction; the associated Fitzpatrick function $F_{T,2}$ is defined by [7]

$$F_{T,2}(x, y) = \langle x, y \rangle - \inf_{y_1 \in Tx_1} \langle x - x_1, y - y_1 \rangle.$$

Applying Lemma 1, we obtain the following proposition.

PROPOSITION 1. *Fitzpatrick's function is bounded from below by the duality product*

$$F_{T,2}(x,y) \geq \langle x, y \rangle$$

and the equality is attained if and only if $y \in Tx$.

Finally, we are able to represent a maximal monotone multifunction by its associated Fitzpatrick function (for the proof, see [2]).

THEOREM 1 (Representation theorem). *The Fitzpatrick function $F_{T,2}$ represents the maximal monotone multifunction T:*

$$G(T) = \{(x,y) \in X \times Y \mid F_{T,2}(x,y) = \langle x, y \rangle\}.$$

5.3. Fitzpatrick's sequence

DEFINITION. For $n \geq 2$ and $(x,y) \in X \times Y$, let (x_i, y_i) be $n-1$ elements of $G(T)$ indexed from $i = 1$ to $i = n-1$. Insert $(x_n, y_n) = (x, y)$ and close the loop by taking $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Then Fitzpatrick's sequence [2] is defined by

$$F_{T,n}(x,y) = \langle x, y \rangle + \sup_{y_i \in Tx_i} \sum_{\lambda=1}^{n} \langle x_{\lambda+1} - x_\lambda, y_\lambda \rangle$$

REMARK. For $n = 2$, we recover the function $F_{T,2}$ originally proposed by Fitzpatrick.

The following proposition [2] states a basic property of Fitzpatrick's sequence.

PROPOSITION 2. *Fitzpatrick's sequence is increasing:*

$$\text{for } n \geq 3, \quad F_{T,n}(x,y) \geq F_{T,n-1}(x,y)$$

and admits as a pointwise limit

$$F_{T,\infty}(x,y) = \sup_{n \geq 2} F_{T,n}(x,y).$$

P r o o f. In the sequence $x_1, \ldots, x_{n-1}$ choose the last term as $x_{n-1} = x_{n-2}$. $\square$

We can now extend the representation Theorem 1 (for the proof, see [2]).

THEOREM 2 (Representation theorem). *Every function of Fitzpatrick's sequence represents the maximal monotone multifunction T:*

$$\forall n \geq 2, \ G(T) = \{(x,y) \in X \times Y \mid F_{T,n}(x,y) = \langle x, y \rangle\}.$$

REMARK. Although the functions $F_{T,n}$ can be defined for any multifunction T, they are equipped with strong properties only when T is maximal monotone. For a generalization of the theorem to maximal n-cyclically monotone multifunctions T, see also [2].

The above theorem shows that we can use any function in Fitzpatrick's sequence to represent the constitutive law T. Having now a whole sequence of bipotentials to choose from, the question arises: Which one is the "best"? To find the answer, we consider some examples from [2].

5.4. Fitzpatrick's sequence for a GSM

5.4.1. Recovery of the bipotential for a GSM. When the multifunction T models a GSM, then there exists a potential ϕ such that $T = \partial\phi$; thereafter T is cyclically monotone [22, 23] and

$$\forall n \geq 2, \ F_{T,n}(x,y) \text{ exists.}$$

Moreover, the pointwise limit is

$$F_{T,\infty}(x,y) = \phi(x) + \phi^*(y)$$

and we recover the typical separated GSM bipotentials.

5.4.2. Fitzpatrick's sequence for an indicator function. When the potential ϕ is the indicator function i_K of a convex set K, then

$$\forall n \geq 2, F_{T,n}(x,y) = F_{T,\infty}(x,y) = i_K(x) + i_K^*(y).$$

5.4.3. Fitzpatrick's sequence for a support function. By duality, the same coincidence holds if the potential ϕ is the support function i_K^* of a convex set K.

REMARK. For GSM, the bipotential is given by $F_{T,\infty}$. Thus, $F_{T,\infty}$ becomes a strong candidate for the "best" bipotential.

6. Linear constitutive laws

6.1. Symmetric linear constitutive laws

Let $X = Y$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

6.1.1. Potential representing a symmetric linear constitutive law. A linear symmetric law $y = Sx$ is modelling a Standard Material if and only if S is symmetric (S coincides with its adjoint S^*). The potential is therefore

$$\phi(x) = \frac{1}{2}\langle x, Sx \rangle.$$

If S is also positive definite, this potential is convex and the symmetric linear constitutive law $y = Sx$ is modelling a GSM.

PROPOSITION 3. *Fitzpatrick's sequence of the linear symmetric positive definite law $y = Sx$ is*

$$\forall n \geq 2, \quad F_{S,n}(x,y) = \langle x, y \rangle + \frac{n-1}{2n} \|S^{-1/2}y - S^{1/2}x\|^2,$$

where $S^{1/2}$ and $S^{-1/2}$ are the square roots of S and of its inverse S^{-1}. The pointwise limit of Fitzpatrick's sequence is

$$F_{S,\infty}(x,y) = \frac{1}{2}\langle x, Sx \rangle + \frac{1}{2}\langle y, S^{-1}y \rangle.$$

REMARK. If we replace in the definition of $F_{S,n}(x,y)$ each pair (x_λ, y_λ) by the pair $(S^{1/2}x_\lambda, S^{-1/2}y_\lambda)$, we observe that it is enough to give the proof when S is the linear identity mapping I. For $S = I$, the proof was given in [2].

6.2. Monotone non-symmetric linear constitutive laws

6.2.1. Monotonicity of linear mappings. A linear law $y = Ax$ can be monotone without being symmetric; it is only necessary that the symmetric part $S = \frac{1}{2}(A + A^*)$ of A should be positive.

6.2.2. Examples of coaxial constitutive laws. Let X and Y be the six-dimensional Euclidean space of real symmetric 3×3 matrices (with e as identity matrix and $\langle x, y \rangle = \operatorname{tr}(xy)$ as duality product). Regard the variables x and y as strain and stress tensors. Consider the linear constitutive law asserting that x and y admit the same eigenvectors. Such a law, referred to as coaxial, has the general form

$$y = [\operatorname{tr}(kx)]\, e + 2\mu x$$

and involves seven coefficients: the scalar μ and six independent coefficients of the symmetric matrix k.

This law is not symmetric except if the matrix k is spheric ($k = \lambda e$), in which case λ and μ are the Lamé coefficients of Hooke's elastic constitutive law

$$y = \lambda(\operatorname{tr} x)e + 2\mu x.$$

It is a well-known result that Hooke's constitutive law is positive if and only if

$$3\lambda + 2\mu \geq 0 \qquad \text{and} \qquad \mu \geq 0.$$

P r o o f. We start to analyze the positivity of Hooke's operator:

$$\langle Ax, x \rangle \geq 0 \Longleftrightarrow \langle \lambda(\operatorname{tr} x)e + 2\mu x, x \rangle \geq 0 \Longleftrightarrow$$

$$\operatorname{tr}\left((\lambda(\operatorname{tr} x)e + 2\mu x)\, x^T\right) \geq 0 \Longleftrightarrow \operatorname{tr}\left(\lambda(\operatorname{tr} x)x + 2\mu x^2\right) \geq 0 \Longleftrightarrow$$

$$\operatorname{tr}\left(\lambda(\operatorname{tr} x)x\right) + \operatorname{tr}\left(2\mu x^2\right) \geq 0 \Longleftrightarrow \lambda(\operatorname{tr} x)^2 + 2\mu \operatorname{tr}\left(x^2\right) \geq 0.$$

We decompose x as a sum of its spherical and deviatoric part:

$$x = \frac{1}{3}(\operatorname{tr} x)e + x' \quad \text{with } \operatorname{tr} x' = 0.$$

Then we have

$$\operatorname{tr}\left(x^2\right) = \operatorname{tr}\left(\frac{1}{9}(\operatorname{tr} x)^2 e + \frac{2}{3}(\operatorname{tr} x)x' + (x')^2\right) = \frac{1}{3}(\operatorname{tr} x)^2 + \operatorname{tr}((x')^2).$$

The positivity of Hooke's operator is then equivalent to

$$\lambda(\operatorname{tr} x)^2 + 2\mu \operatorname{tr}\left(x^2\right) \geq 0 \Longleftrightarrow$$

$$\lambda(\operatorname{tr} x)^2 + 2\mu \left(\frac{1}{3}(\operatorname{tr} x)^2 + \operatorname{tr}((x')^2)\right) \geq 0 \Longleftrightarrow$$

$$\left(\lambda + \frac{2}{3}\mu\right)(\operatorname{tr} x)^2 + 2\mu \operatorname{tr}((x')^2) \geq 0 \Longleftrightarrow$$

$$\lambda + \frac{2}{3}\mu \geq 0 \text{ and } \mu \geq 0.$$

Due to the fact that both $(\operatorname{tr} x)^2$ and $\operatorname{tr}((x')^2)$ are positive for x symmetric, the last equivalence holds because the subspaces of spherical and deviatoric matrices are orthogonal. By choosing x to be a spherical matrix we obtain $\lambda + \frac{2}{3}\mu \geq 0$; for deviatoric matrices x we obtain $2\mu \geq 0$. The proof is finished. ☐

If the deviatoric part h of the matrix k is not reduced to 0, we set

$$k = \lambda e + h \quad \text{with} \quad \lambda = \frac{1}{3}(\operatorname{tr} k) \quad \text{and} \quad \operatorname{tr} h = 0.$$

The symmetric part of the coaxial constitutive law is then

$$Sx = \frac{1}{2}\left[\operatorname{tr}(kx)\right]e + \frac{1}{2}(\operatorname{tr} x)k + 2\mu x$$

$$= \lambda(\operatorname{tr} x)e + 2\mu x + \frac{1}{2}(\operatorname{tr} x)h + \frac{1}{2}\left[\operatorname{tr}(hx)\right]e.$$

It is positive if and only if

$$\begin{cases} \mu \geq 0, \\ 3\lambda + 2\mu \geq 0, \\ \operatorname{tr}\left(h^2\right) \leq \frac{8}{3}\mu(3\lambda + 2\mu). \end{cases}$$

P r o o f. We start by expressing the positiveness of S in equivalent forms:

$$\langle Sx, x \rangle \geq 0 \Longleftrightarrow \text{tr} \left(\frac{1}{2} \left(\text{tr}(kx) \right) x + \frac{1}{2} (\text{tr } x)kx + 2\mu x^2 \right) \geq 0 \Longleftrightarrow$$

$$\frac{1}{2} \left(\text{tr}(kx) \right) (\text{tr } x) + \frac{1}{2} (\text{tr } x) \left(\text{tr}(kx) \right) + 2\mu \, \text{tr}(x^2) \geq 0 \Longleftrightarrow$$

$$\text{tr}(kx)(\text{tr } x) + 2\mu \, \text{tr}(x^2) \geq 0 \Longleftrightarrow$$

$$\text{tr} \left((\lambda e + h)x \right) (\text{tr } x) + 2\mu \, \text{tr}(x^2) \geq 0 \Longleftrightarrow$$

$$\text{tr}(\lambda x + hx)(\text{tr } x) + 2\mu \, \text{tr}(x^2) \geq 0 \Longleftrightarrow$$

$$\lambda (\text{tr } x)^2 + \text{tr}(hx)(\text{tr } x) + 2\mu \, \text{tr}(x^2) \geq 0.$$

Using the decomposition of x in its spherical and deviatoric part:

$$(6.1) \qquad\qquad x = \frac{1}{3} (\text{tr } x)e + x' \quad \text{with tr } x' = 0,$$

we can continue our chain of equivalences:

$$\lambda (\text{tr } x)^2 + \text{tr}(hx)(\text{tr } x) + 2\mu \, \text{tr} \left(x^2 \right) \geq 0 \Longleftrightarrow$$

$$\lambda (\text{tr } x)^2 \text{tr} \left(\frac{1}{3} (\text{tr } x)h + hx' \right) (\text{tr } x) +$$

$$+ 2\mu \, \text{tr} \left(\frac{1}{9} (\text{tr } x)^2 e + + \frac{2}{3} (\text{tr } x)x' + (x')^2 \right) \geq 0 \Longleftrightarrow$$

$$\lambda (\text{tr } x)^2 + \text{tr}(hx')(\text{tr } x) + \frac{2}{3}\mu (\text{tr } x)^2 + 2\mu \, \text{tr}((x')^2) \geq 0 \Longleftrightarrow$$

$$\left(\lambda + \frac{2}{3}\mu \right) (\text{tr } x)^2 + \text{tr}(hx')(\text{tr } x) + 2\mu \, \text{tr}((x')^2) \geq 0.$$

For the case when x' is in the deviatoric subspace orthogonal to h, we have $\text{tr}(hx') = 0$, the middle term vanishes and the inequality is reduced to the one already analyzed for the symmetric Hooke's law. It leads, like before, to the inequalities $\lambda + \frac{2}{3}\mu \geq 0$ and $\mu \geq 0$.

For the case that x' is parallel to h, we can write $x' = \frac{1}{\alpha}h$. The inequality becomes

$$\left(\lambda + \frac{2}{3}\mu \right) (\text{tr } x)^2 + \frac{1}{\alpha} \text{tr} \left(h^2 \right) (\text{tr } x) + 2\mu \frac{1}{\alpha^2} \text{tr} \left(h^2 \right) \geq 0$$

or, after multiplying by α^2,

$$\left(\lambda + \frac{2}{3}\mu \right) (\alpha \, \text{tr } x)^2 + \text{tr} \left(h^2 \right) (\alpha \, \text{tr } x) + 2\mu \, \text{tr} \left(h^2 \right) \geq 0.$$

This is a quadratic inequality in $\alpha \operatorname{tr} x$ and thus it is always satisfied if and only if its discriminant Δ is negative:

$$\Delta \leq 0 \Longleftrightarrow \left(\operatorname{tr}\left(h^2\right)\right)^2 - 8\left(\lambda + \frac{2}{3}\mu\right)\mu \operatorname{tr}\left(h^2\right) \leq 0 \Longleftrightarrow$$

$$\operatorname{tr}\left(h^2\right)\left[\operatorname{tr}\left(h^2\right) - 8\left(\lambda + \frac{2}{3}\mu\right)\mu\right] \leq 0.$$

Because of $h = \alpha x'$, h is symmetric and thus $\operatorname{tr}\left(h^2\right) \geq 0$; the above inequality is equivalent to

$$\operatorname{tr}\left(h^2\right) \leq 8\mu\left(\lambda + \frac{2}{3}\mu\right).$$

We can interpret this condition as a bounding condition: the deviatoric part h of the matrix k should not be too large. $\qquad\square$

6.2.3. Fitzpatrick's function of monotone linear mappings. If $y_1 = Ax_1$, the infimum of

$$\langle x - x_1, y - y_1 \rangle = \langle x - x_1, y - Ax \rangle + \langle x - x_1, S\left(x - x_1\right) \rangle$$

is attained for

$$2S\left(x - x_1\right) = Ax - y.$$

Therefore the first element of Fitzpatrick's sequence is

$$F_{A,2}(x, y) = \langle x, y \rangle + \frac{1}{4}\langle y - Ax, S^{-1}(y - Ax) \rangle.$$

This is a bipotential representing the non-associated law $y = Ax$.

7. Non-monotone constitutive laws

As a pioneering work to establish a bipotential modelling Coulomb's dry friction, let us consider the following constitutive law: x and y have the same orientation. This constitutive law is not monotone, Fitzpatrick's method cannot be directly applied for finding a point-to-point function to represent it. The questions are: does this constitutive law model an IMS? Can Fitzpatrick's sequence be generalized to exhibit a representation by a bipotential?

Let us ask only for a local supremum in the definition of the first Fitzpatrick function. By "local supremum" we mean a supremum in a subset of variables,

while the remaining variables are given by the optimality condition. In our example, let u be the unit vector in the common direction of x_1 and y_1. To obtain the first Fitzpatrick function, we have to optimize

$$\langle x_1, y \rangle + \langle x, y_1 \rangle - \langle x_1, y_1 \rangle = \|x_1\| \langle u, y \rangle + \|y_1\| \langle x, u \rangle - \|x_1\| \|y_1\|$$
$$= -\left(\|x_1\| - \langle x, u \rangle\right)\left(\|y_1\| - \langle u, y \rangle\right) + \langle x, u \rangle \langle u, y \rangle.$$

The optimality condition gives a saddle point for $(\|x_1\|, \|y_1\|) = (\langle x, u \rangle, \langle u, y \rangle)$, where the first term vanishes. The second term is depending only on u and admits a supremum (this justifies the notion of "local supremum").

$$\langle x, u \rangle \langle u, y \rangle = \frac{1}{2}\left(\langle \langle u, y \rangle x, u \rangle + \langle u, \langle x, u \rangle y \rangle\right)$$
$$= \frac{1}{2}\langle u, (xy^* + yx^*)\, u \rangle$$

with respect to u, we are led from Rayleigh's quotient argument to the largest eigenvalue $\frac{1}{2}\lambda_1(xy^* + yx^*)$ of $\frac{1}{2}(xy^* + yx^*)$. This largest eigenvalue being $\frac{1}{2}(\langle x, y \rangle + \|x\|\|y\|)$, we can claim that the constitutive law asserting that two vectors have the same orientation models an IMS, and we can propose the bipotential

$$b(x, y) = \frac{1}{2}\langle x, y \rangle + \frac{1}{2}\|x\|\|y\|.$$

This function is a bipotential thanks to the Cauchy–Schwarz–Buniakovsky inequality.

REMARK. If θ is an angle (chosen between 0 and π) such that

$$\langle x, y \rangle = \|x\|\|y\| \cos \theta,$$

then

$$b(x, y) = \|x\|\|y\| \cos^2\left(\frac{\theta}{2}\right).$$

8. Conclusions

8.1. Selection rules

Backed by the examples presented above, we define the "best" bipotential to be the largest element of Fitzpatrick's sequence (including the limit $F_{T,\infty}$, if it exists).

To select the "best" bipotential modelling an Implicit Standard Material, we propose the following rules:

(1) Define $F_{T,n}(x, y)$ as a local supremum (by considering only the subset of variables for which a supremum exists, the other variables being given by the optimality condition).

(2) Choose the last existing Fitzpatrick function in this generalized Fitzpatrick sequence.

(3) Choose the pointwise limit if all generalized Fitzpatrick functions are defined.

8.2. Conjectures

8.2.1. Cauchy–Schwarz–Buniakovsky bipotential. Concerning the constitutive law asserting that the vectors x and y have the same orientation, we conjecture that the best bipotential will be [28]

$$b(x, y) = \|x\|\|y\|$$

as the pointwise limit of the generalized Fitzpatrick sequence

$$F_n(x, y) = \|x\|\|y\| \cos^n\left(\frac{\theta}{n}\right),$$

with

$$\theta = \arccos\left(\frac{\langle x, y \rangle}{\|x\|\|y\|}\right).$$

We will call it the "Cauchy–Schwarz–Buniakovsky bipotential".

8.2.2. Hill's bipotential. Let X and Y be the $\frac{1}{2}d(d+1)$-dimensional Euclidean space of real symmetric $d \times d$ matrices (with duality product $\langle x, y \rangle = \mathrm{tr}(xy)$). Let us consider the constitutive law asserting that real symmetric matrices x and y admit the same ordered spectral decomposition [13, 14, 17]. Let

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x) \quad \text{and} \quad \lambda_1(y) \geq \lambda_2(y) \geq \cdots \geq \lambda_d(y)$$

denote the ordered eigenvalues of x and y. We conjecture [28] that the best bipotential will be

$$b(x, y) = \sum_{i=1}^{d} \lambda_i(x)\lambda_i(y),$$

as the pointwise limit of a generalized Fitzpatrick sequence. We will call it "Hill's bipotential" [28].

8.3. Byproducts

8.3.1. Algorithm for determining the eigenvectors associated with the largest eigen-value of a real symmetric matrix. In linear vibration studies of mechanical or acoustical systems, the eigenvectors associated with the largest eigenvalue $\lambda_1(x)$ of a real symmetric matrix x are of main interest. The spaces X and Y are the $\frac{1}{2}d(d+1)$-dimensional Euclidean spaces of real symmetric $d \times d$ matrices (with duality product $\langle x, y \rangle = \mathrm{tr}(xy)$). This largest eigenvalue is a lsc convex function of the matrix [31]. We can regard it as the potential of a GSM, with the indicator function $i_K(y)$ of the convex part

$$K = \{y \in Y \mid y \geq 0 \text{ and } \mathrm{tr}\, y = 1\}$$

as the conjugate potential. The subdifferential of the potential λ_1 at x is consti-tuted of the elements y of the convex K, which are projections on the eigensub-spaces associated with the largest eigenvalue. According to Moreau's "proximal mappings method" [19, 20]:

$$y \in \partial \lambda_1(x) \iff y = P(y + x),$$

where P is the projection on the convex K. The fixed point method leads us to propose the iterative algorithm [31]

$$y_{i+1} = P\left(y_i + x\right).$$

Algorithms concerning the projection onto the cone of positive semidefinite matrices can be found in [8, 11, 12], and [32].

8.3.2. Algorithm for determining the subdifferential of Tresca's yield criterion. Let X and Y be the six-dimensional Euclidean space of real symmetric 3×3 matrices (with e as identity matrix and duality product $\langle x, y \rangle = \mathrm{tr}(xy)$). Let the variables x and y be strain-rate and stress tensors. Consider the problem of finding x in the subdifferential at y of Tresca's yield criterion [10]:

$$x \in \partial \frac{\lambda_1 - \lambda_3}{2}(y).$$

The solutions can be obtained by applying the fixed point method [31]

$$x = P(x + y),$$

where P is the projection on the convex

$$K = \left\{x \in X \;\middle|\; -\frac{1}{2}e \leq x \leq \frac{1}{2}e \text{ and } \mathrm{tr}\, x = 0\right\}.$$

8.4. Open problems

Determine the generalized Fitzpatrick sequence for:
- non-symmetric linear coaxial constitutive laws,
- Coulomb's dry friction law,
- generalized Drucker–Prager plasticity,
- modified Cam-Clay model,
- non-linear kinematical hardening rule for cyclic plasticity of metals,
- Lemaître's plastic-ductile damage law.

8.5. Outlook

We think that the method of the generalized Fitzpatrick sequence will prove to be very helpful to produce bipotentials for:
- shakedown analysis of non-standard elastoplastic materials,
- constitutive laws of wet clays,
- damage kinetic constitutive equation,
- granular materials,
- linear non-symmetric constitutive laws of non-standard materials.

Acknowledgment

Starting points

As starting points we recommend [2] and [26].

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