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On the thermoelastic problem of uniform heat flow disturbed by a circular rigid lamellate inclusion

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A complete solution in elementary functions is given for the three-dimensional thermoelastic field in an elastic space, containing an absolutely rigid circular inclusion (anticrack) under a normally incident uniform heat flow. The inclusion is assumed to be slightly conducting, with a certain thermal resistance. The analysis is based on the potential theory method. The resulting boundary-value problems are reduced to classical mixed problems of the potential theory. The temperature, fluxes, thermal stresses and displacements in the inclusion plane are given in closed forms and interpreted from the point of view of the failure theory.

Key words: heat flow, rigid circular inclusion, thermal resistance, potential theory method, thermal stress singularity.

1. Introduction

Problems involving stress concentrations in deformable bodies containing different kinds of imperfections have long been studied by researchers from many fields, such as geomechanics, metallurgy, materials science. Cracks (with traction-free surfaces) and rigid inclusions (with displacements-free surfaces) represent the two extreme cases of inhomogeneity that lead to the formation of high stress concentrations. The safety of a structure containing these defects can be determined basing on the knowledge of the magnitude and distribution of stresses following from all types of loads, including thermal loads. Thus, it is essential to obtain the theoretical solutions of the realistic three-dimensional thermoelastic fracture problems. The main advances in this field dealing with cracks are discussed in monographs written by Kassir and Sih [1], Kit and Khay [2], Dell'erba [3]. The potential theory method proposed by Fabrikant [4, 5] has provided the basis for obtaining the analytical solutions to many thermoelastic crack problems of practical interest (see Kaczyński [6], Kaczyński and Matysiak [7], Chen *et al.* [8]). However, although 3-D isothermal problems involving rigid sheet-like inclusions in homogeneous elastic materials have been considered in numerous papers (see e.g. Collins [9], Kassir and Sih [10], Selvadurai [11, 12], Silovanyuk [13], Huang and Liu [14], Rahman [15, 16], Kachanov *et al.* [17], Chaudhuri [18], and the fundamental monographs by Mura [19], Panasyuk *et al.* [20] and Alexandrov *et al.* [21]), thermal effects have been investigated to a much smaller extent and are concerned with two-dimensional formulations (see Sekine [22] and Chao and Shen [23] and references cited therein). Only some inadequate results for transversally isotropic bodies having rigid elliptic inclusions under thermal loads can be found in Podil'chuk [24].

This contribution may be regarded as a companion paper to the publication given by Kaczyński and Kozłowski [25] in which, as an illustration of the general considerations, the complete elementary solution was presented for the rigid penny-shaped inclusion in thermoelastic space subjected to uniform perpendicular heat flow at infinity under a classical assumption that the faces of the inclusion are thermally insulated. The present work is concerned with an extension of [25] for more general thermal conditions by taking into account the conductivity of the inclusion. A two-stage method for obtaining the solution is used. The steady-state temperature field is determined first with regard to the heat transfer through a rigid inclusion. Next, the associated induced thermal stresses are sought by using the potential method previously developed by Kaczyński [6]. The resulting anti-symmetric problems are reduced to classical mixed problems of the potential theory. The results obtained by Fabrikant [4, 5] are utilized to derive complete and exact expressions in terms of elementary functions for the entire thermoelastic field. The properties and singular behaviour of the thermal stresses near the inclusion border as well as the influence of its thermal conductivity are examined.

2. Basic equations and general potential solution

Introduce a rectangular Cartesian coordinate system $OX_1X_2X_3$, and denote at the point (x_1, x_2, x_3) of the infinite thermoelastic homogeneous material, the unknown quantities: the temperature (a small change from the stress-free state) by T and the components of displacement, stress, heat flux by u_i, σ_{ij}, q_i, respectively.

Throughout the paper, the Latin subscripts run over 1, 2, 3, repeated indices imply summation and a comma denotes partial differentiation.

By neglecting the effects of both the inertia and coupling between temperature and strains, the general thermal-stress problem separates into two distinct subproblems – determination of the temperature distribution and employing it to find the induced stress field. The first thermal problem is governed by the Fourier law of heat conduction and the three-dimensional Laplace equation for the temperature field in absence of the heat sources (NOWACKI [26])

$$q_i = -kT_{,i}, \tag{2.1}$$

$$T_{,ii} = 0, \tag{2.2}$$

where k is the thermal conductivity. Knowing the temperature distribution, the resulting displacements and stresses may be found, respectively, from the generalized Lamé equations and the Duhamel–Neumann stress-displacement relations (in absence of the body forces):

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - \beta T_{,i} = 0, \tag{2.3}$$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j}+u_{j,i}) - \beta T\delta_{ij}, \tag{2.4}$$

in which λ and μ are the Lamé constants, $\beta = \alpha(3\lambda+2\mu)$ with α being the linear coefficient of thermal expansion, and δ_{ij} is Kronecker's delta.

For problems connected with the discontinuities at $x_3 = 0^{\pm}$, a suitable general solution to the above equations of thermoelastic equilibrium can be obtained by four spatial harmonic functions (potentials): ω – the thermal potential and φ_i, $i \in \{1,2,3\}$ – the mechanical potentials such that the temperature and displacements are represented in the form (KACZYŃSKI [6])

$$T = \omega_{,3}, \tag{2.5}$$

$$u_1 = \left[\varphi_1 + c\int_{x_3}^{\infty}\omega dx_3 + x_3F\right]_{,1} - \varphi_{3,2}, \tag{2.6}$$

$$u_2 = \left[\varphi_1 + c\int_{x_3}^{\infty}\omega dx_3 + x_3F\right]_{,2} + \varphi_{3,1}, \tag{2.7}$$

$$u_3 = \varphi_{1,3} - \frac{\lambda+3\mu}{\lambda+\mu}\varphi_2 + x_3F_{,3}, \tag{2.8}$$

where

$$F = \varphi_2 + c\omega, \qquad c = \frac{\beta}{2(\lambda+2\mu)}. \tag{2.9}$$

Using Eqs. (2.5) and (2.1), the components of the fluxes are given by

$$q_i = -k\omega_{,3i}. \tag{2.10}$$

Making use of the constitutive relations (2.4) and bearing Eqs. (2.6)–(2.8) in mind, the corresponding stresses are found to be

$$\sigma_{31} = 2\mu\left[\varphi_{1,3} - \frac{\mu}{\lambda+\mu}\varphi_2 + x_3 F_{,3}\right]_{,1} - \mu\varphi_{3,23}, \tag{2.11}$$

$$\sigma_{32} = 2\mu\left[\varphi_{1,3} - \frac{\mu}{\lambda+\mu}\varphi_2 + x_3 F_{,3}\right]_{,2} + \mu\varphi_{3,13}, \tag{2.12}$$

$$\sigma_{33} = 2\mu\left[\varphi_{1,33} - \frac{\lambda+2\mu}{\lambda+\mu}\varphi_{2,3} - cT + x_3 F_{,33}\right], \tag{2.13}$$

$$\sigma_{11} = 2\mu\left[\left(\varphi_1 + c\int\limits_{x_3}^{\infty}\omega\, dx_3 + x_3 F\right)_{,11} - \frac{\lambda}{\lambda+\mu}\varphi_{2,3} + \varphi_{3,12} - 2cT\right], \tag{2.14}$$

$$\sigma_{22} = 2\mu\left[\left(\varphi_1 + c\int\limits_{x_3}^{\infty}\omega\, dx_3 + x_3 F\right)_{,22} - \frac{\lambda}{\lambda+\mu}\varphi_{2,3} - \varphi_{3,12} - 2cT\right], \tag{2.15}$$

$$\sigma_{12} = 2\mu\left(\varphi_1 + c\int\limits_{x_3}^{\infty}\omega\, dx_3 + x_3 F\right)_{,12}. \tag{2.16}$$

3. Thermal rigid inclusion problem and its solution

Suppose that a homogeneous isotropic material occupies the entire space except the region

$$S = \{(x_1 = r\cos\theta, x_2 = r\sin\theta, x_3 = 0) : \\ 0 \le r = \sqrt{x_1^2 + x_2^2} \le a,\ 0 \le \theta < 2\pi\}, \tag{3.1}$$

where there is a rigid sheet-like inclusion (anticrack). The inclusion obstructs the heat flow as shown in Fig. 1.

For the present thermal problem governed by the Laplace equation (2.2), we have the boundary conditions at infinity

$$q_1 = q_2 = 0, \qquad q_3 = -kT_{,3} \to -q_0 \qquad \text{as} \quad \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \tag{3.2}$$

and some conditions involving the thermal conducting properties of the rigid inclusion.

Starting with the superposition principle, the total temperature field T can be represented in the form

$$T = \overset{0}{T} + \widetilde{T}. \tag{3.3}$$

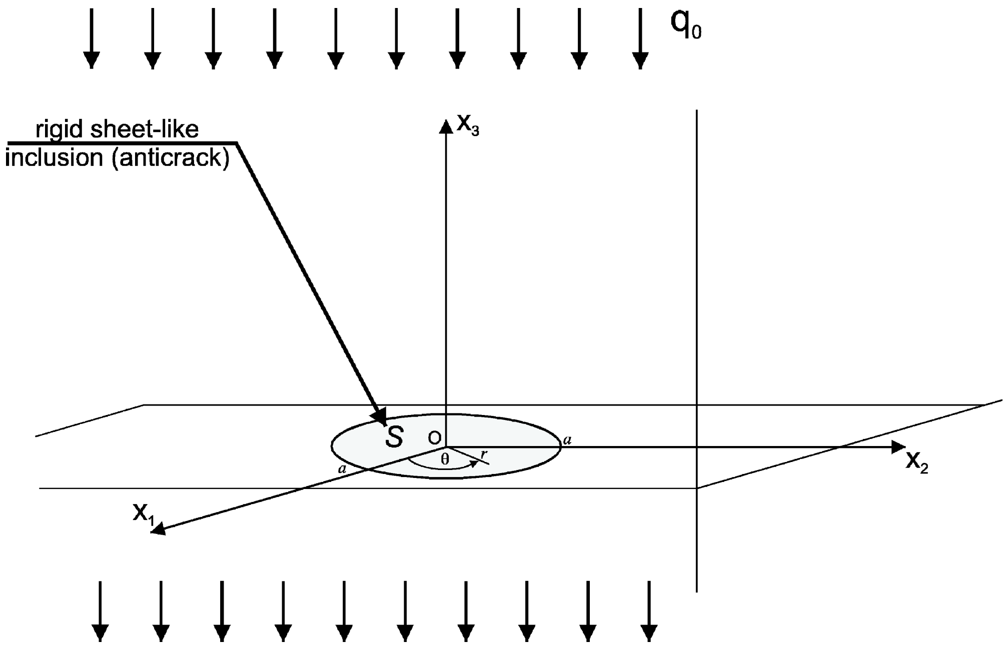

FIG. 1. A rigid penny-shaped inclusion in an elastic space under thermal flow.

Here $\overset{0}{T}$ is the temperature field corresponding to a perpendicular homogeneous flow of uniform heat at infinity with the positive constant gradient q_0 (satisfying Eqs. (3.2)) and $\widetilde{T}$ is the disturbed temperature due to the presence of a rigid inclusion.

It is readily found that

$$\overset{0}{T}(x_1, x_2, x_3) = \frac{q_0}{k} x_3. \tag{3.4}$$

It suffices to consider only the perturbed thermal problem, for which we have to formulate the appropriate boundary conditions. At this stage, the general model relations for thin heat-conducting elastic inclusions, devised by KIT and KHAI [2], are utilized:

$$\begin{aligned} &\Delta\langle\widetilde{T}\rangle + \frac{k}{h(x_1,x_2)k_{\text{in}}}\left[(\widetilde{T}_{,3})^+ - (\widetilde{T}_{,3})^-\right] = -\frac{k-k_{\text{in}}}{h(x_1,x_2)k_{\text{in}}}\left[(\overset{0}{T}_{,3})^+ - (\overset{0}{T}_{,3})^-\right], \\ &\Delta[\widetilde{T}] - \frac{3}{h^2(x_1,x_2)}[\widetilde{T}] + \frac{3k}{h(x_1,x_2)k_{\text{in}}}\left[(\widetilde{T}_{,3})^+ - (\widetilde{T}_{,3})^-\right] - \Lambda(\widetilde{T}) \\ &\qquad = -\frac{3(k-k_{\text{in}})}{h(x_1,x_2)k_{\text{in}}}\left[(\overset{0}{T}_{,3})^+ + (\overset{0}{T}_{,3})^-\right], \end{aligned} \tag{3.5}$$

in which Δ and Λ stand for the differential operators given by

$$
\begin{aligned}
&\Delta(\cdot) = (\cdot)_{,11} + (\cdot)_{,22}, \\
&\Lambda(\cdot) = \frac{2h_{,1}}{h(x_1,x_2)}(\cdot)_{,1} + \frac{2h_{,2}}{h(x_1,x_2)}(\cdot)_{,2} + \frac{h\Delta(h) + (h_{,1})^2 + (h_{,2})^2}{h^2(x_1,x_2)}
\end{aligned}
\tag{3.6}
$$

and k_{in} is the thermal conductivity of the inclusion material, $2h(x_1,x_2)$ is the thickness of the inclusion. Moreover, $(f)^{\pm}$ are the values of function f for $x_3 = \pm h(x_1,x_2)$ and $\langle\widetilde{T}\rangle = \widetilde{T}^{+} + \widetilde{T}^{-}$, $[\widetilde{T}] = \widetilde{T}^{+} - \widetilde{T}^{-}$.

For the present problem, consider a special case regarding the inclusion of negligibly small thickness and with a low heat conductivity. By using in Eqs. (3.5) the limits $k_{\text{in}} \to 0$ and $h_0 \to 0$ (where $h(x_1,x_2) = h_0\chi(x_1,x_2)$) such that $k_{\text{in}}/h_0 \to \text{const}$ (> 0), we arrive at the following boundary conditions related to the anticrack S:

$$
\widetilde{T}_{,3}(x_1,x_2,x_3 = 0^{+}) - \widetilde{T}_{,3}(x_1,x_2,x_3 = 0^{-}) = 0,
\tag{3.7}
$$

$$
\Big[\widetilde{T}(x_1,x_2,0^{+}) - \widetilde{T}(x_1,x_2,0^{-})\Big] - kR(x_1,x_2)\widetilde{T}_{,3}(x_1,x_2,0^{+}) = q_0R(x_1,x_2)
\tag{3.8}
$$

with

$$
R(x_1,x_2) \equiv \frac{2h(x_1,x_2)}{k_{\text{in}}} = \frac{2h_0\chi(x_1,x_2)}{k_{\text{in}}},
\tag{3.9}
$$

interpreted as the thermal resistance of the inclusion. In addition, the disturbed temperature at a sufficiently large distance away from the inclusion must vanish, i.e.

$$
\widetilde{T}(r,x_3) \to 0 \quad \text{as} \quad \sqrt{r^2 + x_3^2} \to \infty.
\tag{3.10}
$$

The conditions (3.7) and (3.10) suggest to assume the sought harmonic function $\widetilde{T}$ as a Newtonian potential of a double layer of intensity $\gamma(x_1,x_2)$, distributed over the inclusion region S (KELLOGG [27]). In view of Eq. (2.5), this function may be written as

$$
\widetilde{T} = \widetilde{\omega}_{,3}, \qquad \widetilde{\omega}(x_1,x_2,x_3) = -\frac{1}{2\pi}\iint\limits_S \frac{\gamma(\xi_1,\xi_2)d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}}
\tag{3.11}
$$

and the remaining condition (3.8) yields the integro-differential singular equation of Newton's potential type for the unknown density γ

$$
2\gamma(x_1,x_2) - \frac{kR(x_1,x_2)}{2\pi}\Delta\iint\limits_S \frac{\gamma(\xi_1,\xi_2)d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} = q_0R(x_1,x_2).
\tag{3.12}
$$

The solution of this equation involves serious mathematical difficulties (see Khai [28] for a full account). However, assuming next that

$$R(x_1,x_2)=R(r)=R_0\sqrt{a^2-r^2},\qquad R_0>0, \tag{3.13}$$

we obtain an analytical solution to Eq. (3.12) in the form

$$\gamma(x_1,x_2)=\gamma(r)=\frac{2q_{\text{res}}}{\pi k}\sqrt{a^2-r^2}, \tag{3.14}$$

provided

$$q_{\text{res}}=\delta_{\text{res}}q_0\le q_0,\qquad \delta_{\text{res}}=\frac{1}{1+4/\pi kR_0}\le 1, \tag{3.15}$$

remembering that [2]

$$\Delta\iint\limits_S\frac{\sqrt{a^2-\xi_1^2-\xi_2^2}\,d\xi_1d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}=-\pi^2,\qquad (x_1,x_2)\in S. \tag{3.16}$$

Inserting Eq. (3.14) into Eq. (3.11) and using the exact results expressed in terms of elementary functions for the thermal potential $\widetilde{\omega}$ and its partial derivatives obtained by Fabrikant [5], it is found that for $x_3\ge 0$

$$\begin{aligned}\widetilde{\omega}(x_1,x_2,x_3)&=\widetilde{\omega}(r,x_3)\\&=-\frac{q_{\text{res}}}{2\pi k}\left[(2a^2+2x_3^2-r^2)\sin^{-1}\frac{a}{l_2}-\frac{2a^2-3l_1^2}{a}\sqrt{l_2^2-a^2}\right],\end{aligned} \tag{3.17}$$

so the complete solution to the perturbed heat conduction problem, independent of the angular coordinate θ, takes the form:

$$\widetilde{T}(x_1,x_2,x_3)=\widetilde{\omega}_{,3}=\begin{cases}-\dfrac{2q_{\text{res}}}{\pi k}\left(x_3\sin^{-1}\dfrac{a}{l_2}-\sqrt{a^2-l_1^2}\right), & x_3\ge 0,\\[2ex] -\dfrac{2q_{\text{res}}}{\pi k}\left(x_3\sin^{-1}\dfrac{a}{l_2}+\sqrt{a^2-l_1^2}\right), & x_3<0,\end{cases} \tag{3.18}$$

and

$$\begin{aligned}\widetilde{q}_i(x_1,x_2,x_3)&=-k\widetilde{T}_{,i}=-k\widetilde{\omega}_{,3i}\\&=\begin{cases}\dfrac{2q_{\text{res}}a^2}{\pi}\dfrac{x_i\sqrt{a^2-l_1^2}}{l_2^2(l_2^2-l_1^2)}, & i=1,2,\\[2ex] \dfrac{2q_{\text{res}}}{\pi}\left(\sin^{-1}\dfrac{a}{l_2}-\dfrac{a\sqrt{l_2^2-a^2}}{l_2^2-l_1^2}\right), & i=3,\end{cases}\end{aligned} \tag{3.19}$$

with Fabrikant's notations

$$
\begin{aligned}
l_1 &\equiv l_1(a,r,x_3) = \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2} - \sqrt{(r-a)^2+x_3^2}\right],\\
l_2 &\equiv l_2(a,r,x_3) = \frac{1}{2}\left[\sqrt{(r+a)^2+x_3^2} + \sqrt{(r-a)^2+x_3^2}\right]
\end{aligned}
\tag{3.20}
$$

and their relevant properties

$$
l_1|_{x_3=0} = \min(a,r), \qquad l_2|_{x_3=0} = \max(a,r).
\tag{3.21}
$$

In particular, from the above expressions and in view of Eqs. (3.3) and (3.4), we find on the inclusion plane

$$
T(r,0^\pm) = \begin{cases} \pm\dfrac{2q_{\text{res}}}{\pi k}\sqrt{a^2-r^2}, & 0 \le r \le a,\\ 0, & r > a, \end{cases}
\tag{3.22}
$$

which is in agreement with the property of the single layer potential $\widetilde{\omega}$ [27], and

$$
\begin{aligned}
q_r(r,0^\pm) &= -k\frac{\partial T(r,0^\pm)}{\partial r} = \begin{cases} \pm\dfrac{2q_{\text{res}}}{\pi}\dfrac{r}{\sqrt{a^2-r^2}}, & 0 \le r < a,\\ 0, & r > a, \end{cases}\\
q_3(r,0^\pm) &= -kT_{,3}(r,0^\pm)\\
&= \begin{cases} q_{\text{res}} - q_0 = -(1-\delta_{\text{res}})q_0, & 0 \le r < a,\\ \dfrac{2q_{\text{res}}}{\pi}\left(\sin^{-1}\dfrac{a}{r} - \dfrac{a}{\sqrt{r^2-a^2}}\right) - q_0, & r > a. \end{cases}
\end{aligned}
\tag{3.23}
$$

A glance at these formulas shows that the rigid inclusion acts as an obstruction to the heat flow, producing thermal local disturbances such as a jump of the temperature and an infinite increase of heat flux q_3 in the vicinity of the inclusion contour. Moreover, from Eq. (3.15) it follows that the heat transfer ratio $\delta_{\text{res}} = q_{\text{res}}/q_0$ as a function of $R_0 \ge 0$ increases from 0 to 1, and by letting $R_0 \to \infty$ we get $\delta_{\text{res}} \to 1$, so $q_{\text{res}} \to q_0$, obtaining the solution corresponding to the limit case of thermally insulated rigid circular disc-inclusion (cf. [25]).

4. Thermal stress problem and its solution

Now we pass to the associated thermoelastic rigid inclusion problem. It is divided into two parts: the first one corresponding to the simple flow of heat with the temperature distribution $\overset{0}{T}$ given by Eq. (3.4) and the second, non-trivial part connected with determining of the induced state of stress and deformation resulting from the disturbed temperature $\widetilde{T}$. Consequently (see (3.3)), we can write

$$u_i = \overset{0}{u}_i + \tilde{u}_i, \qquad \sigma_{ij} = \overset{0}{\sigma}_{ij} + \tilde{\sigma}_{ij}, \tag{4.1}$$

where the components attributed to zero describe the principal state of inclusion-free space, and the components having tilde represent the perturbations due to the anticrack. The global mechanical boundary conditions resulting from the fact that the inclusion is perfectly rigid and due to the given thermal loading, may experience only a small vertical rigid-body translation ε along the X_3-axis (PODIL'CHUK [24]), are

$$u_1 = u_2 = 0, \qquad u_3 = \varepsilon, \qquad (x_1, x_2, x_3 = 0^{\pm}) \in S, \tag{4.2}$$

where the parameter ε must be determined from the equilibrium condition that the net resultant force acting on the rigid inclusion vanishes, i.e.,

$$\iint\limits_S \big[\sigma_{33}(x_1, x_2, 0^+) - \sigma_{33}(x_1, x_2, 0^-)\big] dx_1 dx_2 = 0. \tag{4.3}$$

Solution of the basic Equations (2.3)–(2.4) with the given temperature $\overset{0}{T}$ and the stress-free conditions at infinity, yields the results:

$$\begin{aligned}
&\overset{0}{u}_1(x_1, x_2, x_3) = \frac{q_0\alpha}{k} x_1 x_3, \qquad \overset{0}{u}_2(x_1, x_2, x_3) = \frac{q_0\alpha}{k} x_2 x_3,\\
&\overset{0}{u}_3(x_1, x_2, x_3) = \frac{q_0\alpha}{2k}(x_3^2 - x_1^2 - x_2^2),\\
&\overset{0}{\sigma}_{ij}(x_1, x_2, x_3) = 0.
\end{aligned} \tag{4.4}$$

Next, our attention is focused on the thermoelastic perturbed problem marked by tilde, its solution tending to zero at infinity and satisfying, in view of Eqs. (4.4), (4.1) and (4.2), the corrective displacement conditions related to the anticrack S

$$\tilde{u}_1(r, 0^{\pm}) = \tilde{u}_2(r, 0^{\pm}) = 0, \qquad \tilde{u}_3(r, 0^{\pm}) = \frac{q_0\alpha}{2k} r^2 + \varepsilon, \qquad 0 \le r \le a. \tag{4.5}$$

Proceeding as in the case of 3-D rigid inclusions problems involving external loadings (see [10, 16]), we shall seek the solution in a potential form taking into account the anti-symmetry of the temperature and of the deformation state. As a result, the problem can be posed as a mixed boundary-value problem related to the upper half-space $x_3 \ge 0$ subjected to the following mixed boundary conditions:

$$\tilde{u}_1(r,0^+) = \tilde{u}_2(r,0^+) = 0, \qquad 0 \le r \le a, \tag{4.6}$$

$$\tilde{u}_3(r,0^+) = \frac{q_0\alpha}{2k} r^2 + \varepsilon, \qquad 0 \le r \le a, \tag{4.7}$$

$$\tilde{\sigma}_{33}(r,0^+) = 0, \qquad r > a, \tag{4.8}$$

$$\tilde{u}_i = O\left(\frac{1}{\sqrt{r^2 + x_3^2}}\right) \quad \text{as} \quad \sqrt{r^2 + x_3^2} \to \infty. \tag{4.9}$$

Moreover, having obtained the distribution of the normal stress in the region S, the unknown rigid translation ε can be calculated from Eq. (4.3).

We now proceed to construct the potentials in the general solution (2.6)–(2.9) with the knowledge of the thermal potential $\tilde{\omega}$ (see Eq. (3.17)) well suited to the above boundary conditions. It is expedient to make the assumptions:

$$\varphi_1 = -c \int_{x_3}^{\infty} \tilde{\omega}\, dx_3, \qquad \varphi_2 = -f, \qquad \varphi_3 = 0, \tag{4.10}$$

which are substituted in Eqs. (2.6)–(2.16) to give the following displacement and stress expressions containing the harmonic function f:

$$\tilde{u}_1 = x_3(-f_{,1} + c\tilde{\omega}_{,1}), \qquad \tilde{u}_2 = x_3(-f_{,2} + c\,\tilde{\omega}_{,2}), \tag{4.11}$$

$$\tilde{u}_3 = \frac{\lambda + 3\mu}{\lambda + \mu} f + c\tilde{\omega} + x_3(-f_{,3} + \widetilde{T}), \tag{4.12}$$

$$\tilde{\sigma}_{31} = 2\mu \left[\frac{\mu}{\lambda + \mu} f_{,1} + c\,\tilde{\omega}_{,1} + x_3(-f_{,31} + \widetilde{T}_{,1})\right], \tag{4.13}$$

$$\tilde{\sigma}_{32} = 2\mu \left[\frac{\mu}{\lambda + \mu} f_{,2} + c\,\tilde{\omega}_{,2} + x_3(-f_{,32} + \widetilde{T}_{,2})\right], \tag{4.14}$$

$$\tilde{\sigma}_{33} = 2\mu \left[\frac{\lambda + 2\mu}{\lambda + \mu} f_{,3} + x_3(-f_{,33} + \widetilde{T}_{,3})\right], \tag{4.15}$$

$$\tilde{\sigma}_{11} = 2\mu \left[\frac{\lambda}{\lambda + \mu} f_{,3} - 2c\widetilde{T} + x_3(-f_{,11} + c\,\tilde{\omega}_{,11})\right], \tag{4.16}$$

$$\tilde{\sigma}_{22} = 2\mu \left[\frac{\lambda}{\lambda + \mu} f_{,3} - 2c\widetilde{T} + x_3(-f_{,22} + c\,\tilde{\omega}_{,22})\right], \tag{4.17}$$

$$\tilde{\sigma}_{12} = 2\mu x_3(-f_{,12} + c\,\tilde{\omega}_{,12}). \tag{4.18}$$

Note that on the boundary $x_3 = 0^+$, this representation automatically satisfies the condition (4.6). The remaining conditions (4.7) and (4.8), when expressed in terms of the potential function f by using Eqs. (4.12) and (4.15), become

$$f(r, x_3 = 0^+) = \frac{\lambda + \mu}{\lambda + 3\mu} f_0(r), \qquad 0 \le r \le a, \tag{4.19}$$

$$f_{,3}(r, x_3 = 0^+) = 0, \qquad r > a, \tag{4.20}$$

where the right-hand side in Eq. (4.19) is given by

$$f_0(r) = \frac{q_0 \alpha}{2k} r^2 + \varepsilon - c\,\tilde{\omega}(r, 0^+). \tag{4.21}$$

It is noteworthy here that only the values of $\tilde{\omega}(r, 0^+)$ from the perturbed temperature problem are required. Making use of Eqs. (3.11), (3.14), (3.15) and (3.17), it is found that

$$\tilde{\omega}(r, 0^+) = -\frac{q_{\text{res}}}{\pi^2 k} \iint\limits_S \frac{\sqrt{a^2 - \xi_1^2 - \xi_2^2}\, d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} = \frac{q_{\text{res}}}{4k}(2a^2 - r^2). \tag{4.22}$$

Thus, the inclusion-perturbed problem reduces to the classical problem of the potential theory (SNEDDON [29]) of determining the harmonic function f in the half-space $x_3 \ge 0$, which vanishes at infinity and satisfies the mixed boundary conditions:

$$\begin{aligned} f(r, x_3 = 0^+) &= \frac{\lambda + \mu}{\lambda + 3\mu} f_0(r), \quad 0 \le r \le a, \\ f_{,3}(r, x_3 = 0^+) &= 0, \qquad\qquad\quad r > a, \end{aligned} \tag{4.23}$$

with (see Eqs. (4.21) and (4.22))

$$f_0(x_1, x_2) = f_0(r) = \frac{q_0 \alpha}{2k} r^2 + \varepsilon + \frac{q_{\text{res}} \beta}{8k(\lambda + 2\mu)}(2a^2 - r^2). \tag{4.24}$$

Comparing this relation with the corresponding relation (59) in [25] obtained for the case of non-conducting penny-shaped rigid inclusion, we conclude that they differ solely by the factor q_{res}. In what follows, the results of the paper [25] will be utilized.

The solution to Eqs. (4.23) in the potential theory is represented by the Newton potential of a simple layer distributed over the region S as

$$f(x_1, x_2, x_3) = -\frac{1}{4\pi} \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \iint\limits_S \frac{\sigma(\xi_1, \xi_2) d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2}}, \tag{4.25}$$

where the unknown layer density σ can be identified as the normal stress, namely

$$\sigma(x_1, x_2) = \tilde{\sigma}_{33}(x_1, x_2, x_3 = 0^+), \qquad (x_1, x_2) \in S. \tag{4.26}$$

From the well-known properties of this potential, the second condition in Eqs. (4.23) is satisfied, and satisfaction of the first one yields the governing two-dimensional singular integral equation (similar to that arising in classical contact mechanics, see [4, 5])

$$H \iint_S \frac{\sigma(\xi_1, \xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} = -f_0(x_1, x_2), \qquad (x_1, x_2) \in S, \tag{4.27}$$

with f_0 given by Eq. (4.24), and a constant H is defined as

$$H = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}. \tag{4.28}$$

A closed-form solution to the above integral equation is obtained by using the theorems of Dyson and Galin (full details are given in RAHMAN [16]). It is of the following form:

$$\sigma(r) = \frac{c_0 - c_2 r^2}{H\pi^2\sqrt{a^2 - r^2}}, \qquad 0 \le r < a, \tag{4.29}$$

where c_0 and c_2 are the unknown constants to be determined.

Substituting Eq. (4.29) into Eq. (4.27) with the right-hand side of (4.24) and equating the terms of the left and right-hand sides, we obtain a set of linear algebraic equations solving them we obtain the following expressions for the unknown coefficients (see [25] for more details):

$$c_0 = \frac{c_2 a^2}{2} - \frac{\beta q_{\mathrm{res}} a^2}{4k(\lambda + 2\mu)} - \varepsilon, \tag{4.30}$$

$$c_2 = \frac{2q_0\alpha - q_{\mathrm{res}} c}{k} = \frac{q_0 \delta_{\mathrm{res}} \alpha}{k} \left[\frac{\lambda + 6\mu}{2(\lambda + 2\mu)} + \frac{8}{\pi k R_0} \right]. \tag{4.31}$$

Additionally, Eqs. (4.3) with (4.26) and (4.29) yield

$$c_0 = \frac{2}{3} a^2 c_2 \tag{4.32}$$

and from Eq. (4.30), the rigid vertical displacement ε is found to be

$$\varepsilon = -\frac{q_0 \alpha a^2}{3k} - \frac{q_{\mathrm{res}} \beta a^2}{6k(\lambda + 2\mu)} = -\frac{a^2(q_0\alpha + q_{\mathrm{res}} c)}{3k}. \tag{4.33}$$

The main harmonic function (4.25) giving a complete solution of the problem expressed by elementary functions is (see the appendices in FABRIKANT [4, 5] for details and his notations (3.20))

$$f(r,x_3) = -\frac{c_2(\lambda+\mu)}{3\pi(\lambda+3\mu)}\left[\left(a^2+3x_3^2-\frac{3}{2}r^2\right)\sin^{-1}\frac{a}{l_2} - \frac{3(2a^2-3l_1^2)}{2a}\sqrt{l_2^2-a^2}\right], \tag{4.34}$$

where use has been made of Eqs. (4.26), (4.29) and (4.32), together with (4.31).

The full-space stress-displacement field can then be obtained by superimposing the two parts, as given in Eq. (4.1). All the derivatives of the governing potentials $\tilde{\omega}$ and f that enter the formulas in Eqs. (4.11)–(4.18) are derived by using Fabrikant's results (see Appendix A in [25]). To save the space of the present paper, the results are omitted. In order to investigate the singular behavior of the stress field near the inclusion edge, however, the axially symmetric solution on the inclusion plane $x_3 = 0^{\pm}$ is presented below:

$$u_1(r,0^{\pm}) = u_2(r,0^{\pm}) = 0, \quad 0 \le r < \infty,$$

$$u_3(r,0^{\pm}) = \begin{cases} \varepsilon, & 0 \le r \le a, \\ \dfrac{2}{\pi}\left(\varepsilon+\dfrac{q_0\alpha}{2k}r^2\right)\sin^{-1}\dfrac{a}{r} \\ \qquad -\dfrac{q_0 a\alpha}{\pi k}\sqrt{r^2-a^2}-\dfrac{q_0\alpha}{2k}r^2, & r > a, \end{cases} \tag{4.35}$$

$$\sigma_{33}(r,0^{\pm}) = \begin{cases} \pm\dfrac{\beta_3 q_0}{\pi}\dfrac{2a^2-3r^2}{\sqrt{a^2-r^2}}, & 0 \le r < a, \\ 0, & r > a, \end{cases} \tag{4.36}$$

$$\sigma_{3r}(r,0^{\pm}) = \begin{cases} \tilde{\beta} q_0 r, & 0 \le r \le a, \\ \dfrac{2q_0}{\pi}\left(\tilde{\beta} r\sin^{-1}\dfrac{a}{r}-\dfrac{\mu}{4(\lambda+2\mu)}\dfrac{\beta_3 a^3}{r\sqrt{r^2-a^2}} \right. \\ \qquad \left. -\dfrac{\tilde{\beta} a\sqrt{r^2-a^2}}{r}\right), & r > a, \end{cases} \tag{4.37}$$

$$\sigma_{12}(r,0^{\pm}) = \sigma_{r\theta}(r,0^{\pm}) = \sigma_{3\theta}(r,0^{\pm}) = 0, \quad 0 \le r < \infty, \tag{4.38}$$

$$\begin{aligned} \sigma_{11}(r,0^{\pm}) &= \sigma_{22}(r,0^{\pm}) = \sigma_{rr}(r,0^{\pm}) = \sigma_{\theta\theta}(r,0^{\pm}) \\ &= \begin{cases} \mp\dfrac{2q_0}{\pi}\left[\dfrac{\lambda\beta_3 a^2}{2(\lambda+2\mu)}\cdot\dfrac{1}{\sqrt{a^2-r^2}}+\beta^*\sqrt{a^2-r^2}\right], & 0 \le r < a, \\ 0, & r > a, \end{cases} \end{aligned} \tag{4.39}$$

where the following notations are used:

$$\beta_3 = \frac{4\mu\alpha\delta_{\text{res}}(\lambda+2\mu)}{3k(\lambda+3\mu)}\left[\frac{\lambda+6\mu}{2(\lambda+2\mu)} + \frac{8}{\pi k R_0}\right], \tag{4.40}$$

$$\tilde{\beta} = \frac{\alpha\mu}{2k(\lambda+3\mu)}[4\mu + \delta_{\text{res}}(3\lambda+2\mu)], \tag{4.41}$$

$$\beta^* = \frac{2\alpha\mu}{k(\lambda+3\mu)}\left[\frac{3\delta_{\text{res}}(3\lambda+2\mu)}{2} - 2\lambda\right]. \tag{4.42}$$

One can readily observe from the above solution that:

1. All components of the stress tensor (excluding σ_{12}) are singular near the inclusion front $r = a$ (strictly, singularities in $\sigma_{33}, \sigma_{11}, \sigma_{22}$ occur at the points on the edge of the disc where $r = a^-$, and in σ_{3r} – at the points exterior to the disc where $r = a^+$), with the familiar inverse square-root singularity in classical fracture mechanics;
2. The normal thermal stress σ_{33} suffers a jump across the anticrack surfaces and changes the sign at $r = \sqrt{2/3}a$;
3. The results for the limiting case of thermally insulated rigid inclusion are obtained by letting $R_0 \to \infty$, $\delta_{\text{res}} \to 1$ in Eqs. (4.40)–(4.42); they are precisely the same as those obtained in the companion paper [25].

In view of the linear fracture mechanics, it indicates that there are two major mechanisms controlling the material cracking around the inclusion front:

- Mode II (edge-sliding) deformation characterized by the stress intensity factor

$$K_{\text{II}} = \lim_{r\to a^+} \sqrt{2\pi(r-a)}\sigma_{3r}(r,0) = -\frac{q_0\mu\beta_3 a\sqrt{a}}{2(\lambda+2\mu)\sqrt{\pi}}, \tag{4.43}$$

- separation of the material from the inclusion described by the stress singularity coefficients

$$S_{\text{I}}^{\pm} = \lim_{r\to a^-} \sqrt{2\pi(a-r)}\sigma_{33}(r,0^{\pm}) = \mp\frac{\beta_3 q_0 a\sqrt{a}}{\sqrt{\pi}}. \tag{4.44}$$

These parameters may be used in a suitable criterion for initiating the fractures near the edge of the rigid inclusion (see Rahman [16]).

Finally, by comparison with the thermally insulated rigid inclusion, we see that the consideration of certain conductivity leads to quantitative changes in the thermomechanical behavior characterized by the factor δ_{res} (see (3.15)).

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