

On the problems of Almansi and Michell for anisotropic Cosserat elastic shells

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THIS PAPER INVESTIGATES the equilibrium of cylindrical elastic shells under the action of prescribed body loads, external loads on the lateral edges, and resultant forces and moments on the end edges. We consider anisotropic and inhomogeneous cylindrical shells with arbitrary (open or closed) cross-sections, and we employ the linear theory of Cosserat surfaces. We present a method to construct the solutions to the problems of Almansi and Michell. Then, we apply these results to study the deformation of orthotropic and homogeneous circular cylindrical tubes and Cosserat plates.

Key words: anisotropic cylindrical shells, Cosserat surfaces, problems of Almansi and Michell.

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1. Introduction

THIS PAPER IS CONCERNED with the deformation of thin cylindrical shells made from an elastic, anisotropic and inhomogeneous material.

We employ the theory of Cosserat surfaces to describe the mechanics of thin elastic shells. The Reader is referred to the paper of NAGHDI [1] for the foundations of this theory and to the monograph of RUBIN [2] for a modern presentation of the Cosserat theories, together with several applications. A Cosserat shell is a two-dimensional continuum, endowed with a single deformable vector (called *director*) assigned to each of its points. Thus, the two-dimensional continuum can describe the deformation of the middle surface of a three-dimensional shell, while the variations of the director field give information about the three-dimensional effects that occur in the mechanics of thin shells.

The problem of Saint–Venant has been investigated in many articles within the classical theories of shells (see e.g. [3–5]). In the context of the Cosserat theory, this problem has been discussed by ERICKSEN [6]. We mention that, for isotropic and homogeneous Cosserat shells, the torsion problem has been solved previously by WENNER [7], while the solution of the relaxed Saint–Venant’s problem has been presented in [8].

In this work, we consider anisotropic and inhomogeneous Cosserat shells. The constitutive coefficients are assumed to be independent of the axial coordinate. We study the static deformation of cylindrical surfaces with arbitrary (open or closed) cross-sections, under the action of assigned body loads and external forces and couples distributed over the edges. On the end edges, the resultant forces and resultant moments are given. Along the lateral edges (for open cylindrical shells), we consider some tractions and couples, which are prescribed pointwise. We give a solution to this general problem in the case when the assigned body loads and the lateral loading are polynomials in the axial coordinate. To this aim, we employ the method established by İEŞAN [9] for the treatment of Almansi's problem in three-dimensional elasticity.

We begin this paper by presenting a summary of the basic equations for the linear theory of Cosserat shells. Then, we confine our attention to cylindrical shells and formulate the problems of Almansi and Michell. We mention that these problems have been extensively studied in the three-dimensional elasticity, for anisotropic materials (see e.g., [10, 11]). The solution of Almansi's problem is based on some results concerning Saint–Venant's problem established in [12], which are summarized in Sec. 2.3 and in the Appendix. In Sec. 3 we present a method to construct the solutions to the problems of Almansi and Michell for anisotropic Cosserat shells. These results are applicable for any specific geometry of the cross-section and for various types of material symmetry, to determine the static deformation of loaded cylindrical shells. We investigate in Sec. 4 the deformation of an orthotropic and homogeneous circular cylindrical tube subject to a hydrostatic pressure applied to its major surfaces, which depends linearly on the axial coordinate. Finally, we solve in Sec. 5 the Almansi–Michell problem for Cosserat plates. The results are in very good agreement with the exact solutions for three-dimensional plates made of orthotropic and homogeneous materials.

2. Preliminaries

2.1. Basic equations for the linear theory

In this section we summarize the basic equations of equilibrium for the linear theory of Cosserat shells. We denote by \mathcal{S} the reference configuration of a Cosserat surface and by θ^α ($\alpha = 1, 2$) – a curvilinear material coordinate system on \mathcal{S} . The static deformation of the Cosserat shell is defined by the position vector $\mathbf{r}(\theta^1, \theta^2)$ and the deformable director $\mathbf{d}(\theta^1, \theta^2)$, assigned to every point of the surface.

The reference configuration \mathcal{S} , which is assumed to coincide with the initial configuration, is characterized by the the position vector $\mathbf{R}(\theta^1, \theta^2)$ and the

director field $\mathbf{D}(\theta^1, \theta^2)$. We introduce the following fields:

$$(2.1) \quad \mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha}, \quad \mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|}, \quad A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta, \quad B_{\alpha\beta} = \mathbf{A}_3 \cdot \mathbf{A}_{\alpha,\beta},$$

which represent the covariant base vectors along the θ^α -curves, the unit normal to \mathcal{S} and the first and second fundamental forms of the surface \mathcal{S} , respectively. Throughout this paper, a subscript comma stands for partial differentiation with respect to the coordinates (θ^α) , while a subscript vertical bar denotes the covariant differentiation with respect to the metric tensor $A_{\alpha\beta}$. Also, we make use of the summation convention over repeated indices and assume that the Latin indices take the values $\{1, 2, 3\}$, while the Greek indices are confined to the range $\{1, 2\}$.

In the linear theory, we introduce the infinitesimal displacement \mathbf{u} and director displacement $\boldsymbol{\delta}$ by

$$(2.2) \quad \mathbf{u} = \mathbf{r} - \mathbf{R}, \quad \boldsymbol{\delta} = \mathbf{d} - \mathbf{D}.$$

The strain measures $e_{\alpha\beta}$, γ_i and $\rho_{i\alpha}$ are defined by

$$(2.3) \quad e_{\alpha\beta} = \frac{1}{2}(\hat{u}_{\alpha|\beta} + \hat{u}_{\beta|\alpha}) - B_{\alpha\beta}\hat{u}_3, \quad \gamma_\alpha = \hat{\delta}_\alpha + \hat{u}_{3,\alpha} + B_\alpha^\beta \hat{u}_\beta, \\ \gamma_3 = \hat{\delta}_3, \quad \rho_{\beta\alpha} = \hat{\delta}_{\beta|\alpha} - B_\alpha^\gamma \hat{u}_{\gamma|\beta} + B_\alpha^\gamma B_{\beta\gamma} \hat{u}_3, \quad \rho_{3\alpha} = \hat{\delta}_{3,\alpha},$$

where $\hat{u}_i = \mathbf{u} \cdot \mathbf{A}_i$ and $\hat{\delta}_i = \boldsymbol{\delta} \cdot \mathbf{A}_i$. We consider the case of shells with constant thickness in the reference configuration, which is characterized by the relation $\mathbf{D} = \mathbf{A}_3$ (cf. [1], page 447).

For an arbitrary curve c on \mathcal{S} (which may also be the boundary $\partial\mathcal{S}$) we denote by \mathbf{v} the (outward) unit normal to c , tangent to the surface \mathcal{S} . Let \mathbf{N} and \mathbf{M} denote the force vector and the director force vector (also called director couple) acting per unit length of c , and $\nu_\alpha = \mathbf{v} \cdot \mathbf{A}_\alpha$. Then, we have the decompositions of Cauchy type

$$(2.4) \quad \mathbf{N} = (N^{\alpha\beta} \mathbf{A}_\beta + V^\alpha \mathbf{A}_3) \nu_\alpha, \quad \mathbf{M} = (M^{\alpha i} \mathbf{A}_i) \nu_\alpha.$$

Let us introduce the surface tensor defined by $N^{\prime\alpha\beta} = N^{\alpha\beta} + B_\gamma^\beta M^{\gamma\alpha}$. For anisotropic and inhomogeneous shells, the constitutive equations have the form

$$(2.5) \quad \mathcal{W} = \frac{1}{2} C_{(1)}^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + \frac{1}{2} C_{(2)}^{i\alpha j\beta} \rho_{i\alpha} \rho_{j\beta} + C_{(3)}^{\alpha\beta i\gamma} e_{\alpha\beta} \rho_{i\gamma} + C_{(1)}^{\alpha\beta i} e_{\alpha\beta} \gamma_i \\ + C_{(2)}^{ij\alpha} \gamma_i \rho_{j\alpha} + \frac{1}{2} C_{(1)}^{ij} \gamma_i \gamma_j, \\ N^{\prime\alpha\beta} = \frac{1}{2} \left(\frac{\partial \mathcal{W}}{\partial e_{\alpha\beta}} + \frac{\partial \mathcal{W}}{\partial e_{\beta\alpha}} \right), \quad V^i = \frac{\partial \mathcal{W}}{\partial \gamma_i}, \quad M^{\alpha i} = \frac{\partial \mathcal{W}}{\partial \rho_{i\alpha}},$$

where \mathcal{W} represents the strain energy density per unit area of \mathcal{S} . The function $\mathcal{W} = \mathcal{W}(e_{\alpha\beta}, \gamma_i, \rho_{i\alpha})$ is a quadratic form of its arguments, with coefficients $C_{(n)}^{\alpha\dots}$ satisfying the following symmetry conditions:

$$\begin{aligned} C_{(1)}^{\alpha\beta\gamma\delta} &= C_{(1)}^{\beta\alpha\gamma\delta} = C_{(1)}^{\gamma\delta\alpha\beta}, & C_{(2)}^{i\alpha j\beta} &= C_{(2)}^{j\beta i\alpha}, \\ C_{(3)}^{\alpha\beta i\gamma} &= C_{(3)}^{\beta\alpha i\gamma}, & C_{(1)}^{\alpha\beta i} &= C_{(1)}^{\beta\alpha i}, & C_{(1)}^{ij} &= C_{(1)}^{ji}. \end{aligned}$$

The equations of equilibrium for Cosserat shells can be written as

$$(2.6) \quad N^{\alpha\beta}|_{\alpha} - B^{\beta}_{\alpha} V^{\alpha} + f^{\beta} = 0, \quad V^{\alpha}|_{\alpha} + B_{\alpha\beta} N^{\alpha\beta} + f^3 = 0, \quad M^{\alpha i}|_{\alpha} - V^i + l^i = 0,$$

where $\mathbf{f} = f^i \mathbf{A}_i$ and $\mathbf{l} = l^i \mathbf{A}_i$ represent the assigned force and assigned director force, respectively, measured per unit area of \mathcal{S} .

For any displacement field $v = (\mathbf{u}, \boldsymbol{\delta})$, the strain energy of the Cosserat shell is

$$(2.7) \quad U(v) = \int_{\mathcal{S}} \mathcal{W}(e_{\alpha\beta}(v), \gamma_i(v), \rho_{i\alpha}(v)) da.$$

Also, we consider the energy norm $\|\cdot\|$ and the scalar product $\langle \cdot, \cdot \rangle$ given by

$$(2.8) \quad \begin{aligned} \|v\|^2 &= 2U(v) = \langle v, v \rangle, \\ \langle v, \tilde{v} \rangle &= \int_{\mathcal{S}} [N^{\alpha\beta}(v) e_{\alpha\beta}(\tilde{v}) + V^i(v) \gamma_i(\tilde{v}) + M^{\alpha i}(v) \rho_{i\alpha}(\tilde{v})] da. \end{aligned}$$

2.2. Cylindrical shells

In this section we confine our attention to cylindrical Cosserat shells and write the relevant field equations for this particular geometry of the surface.

We assume that the reference configuration \mathcal{S} of the Cosserat shell is a cylindrical (open or closed) surface with arbitrary cross-section, with generators parallel to the axis Ox_3 of the rectangular Cartesian coordinate frame $Ox_1x_2x_3$ (see Fig. 1).

The cylindrical surface \mathcal{S} is situated between the planes $x_3 = 0$ and $x_3 = \bar{z}$, and we denote by \mathcal{C}_z the cross-section boundary curve which belongs to the plane $x_3 = z$, $z \in [0, \bar{z}]$. Let us choose the surface curvilinear coordinates $\theta^1 = s$, $\theta^2 = z$ on \mathcal{S} , where $s \in [0, \bar{s}]$ represents the arc parameter along the curves \mathcal{C}_z and $z = x_3$, $z \in [0, \bar{z}]$. We denote by \mathbf{e}_i the unit vectors along the Ox_i axes. The parametric equations of \mathcal{S} are given by

$$(2.9) \quad \mathbf{R} = \mathbf{R}(s, z) = x_{\alpha}(s) \mathbf{e}_{\alpha} + z \mathbf{e}_3, \quad \mathbf{D} = \mathbf{D}(s) = \epsilon_{\alpha\beta} x'_{\beta}(s) \mathbf{e}_{\alpha},$$

where the functions $x_{\alpha}(s)$ are assumed to be of class $C^3[0, \bar{s}]$. We denote by $(\cdot)' = \frac{d}{ds}(\cdot)$ and $\epsilon_{\alpha\beta}$ is the two-dimensional alternator ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$).

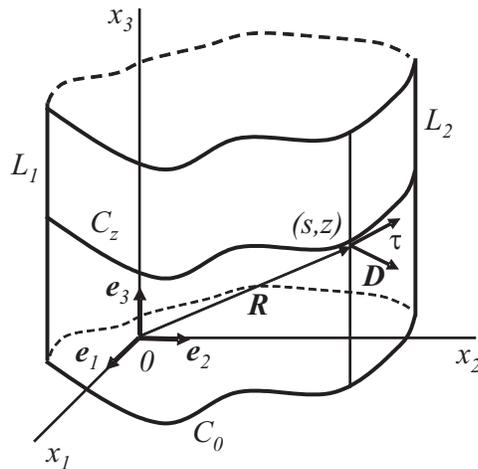


FIG. 1. Reference configuration of the cylindrical shell.

The end edge curves are C_0 and C_z . For closed shells, the functions $x_\alpha(s)$ (and their derivatives) satisfy the continuity conditions at $s = 0$ and $s = \bar{s}$. For open shells, we denote by L_1 and L_2 the lateral edges characterized by $s = 0$ and $s = \bar{s}$, respectively. We remark that, for the surface \mathcal{S} given by (2.9), we have

$$(2.10) \quad \begin{aligned} \mathbf{A}_1 = \boldsymbol{\tau}(s) &= x'_\alpha(s)\mathbf{e}_\alpha, & \mathbf{A}_2 &= \mathbf{e}_3, & \mathbf{A}_3 = \mathbf{n}(s) &= \epsilon_{\alpha\beta}x'_\beta(s)\mathbf{e}_\alpha, \\ A_{\alpha\beta} &= \delta_{\alpha\beta}, & B_{11} &= -r^{-1}, & B_{12} = B_{21} = B_{22} &= 0, \end{aligned}$$

where $\boldsymbol{\tau}$ and \mathbf{n} represent the unit tangent and normal vectors to C_z , r is the curvature radius of C_z and $\delta_{\alpha\beta}$ is the Kronecker symbol. Thus, the physical components of any tensor on the cylindrical surface coincide with the covariant and with the contravariant components of the same tensor. Taking into account that $\theta^1 = s$, $\theta^2 = z$ and $\mathbf{A}_3 = \mathbf{n}$, in what follows we shall employ the subscripts s , z and n instead of the indices 1, 2 and 3, respectively, for any tensor components. In particular, we can decompose any vector \mathbf{v} as $\mathbf{v} = v_s\boldsymbol{\tau} + v_z\mathbf{e}_3 + v_n\mathbf{n}$. Using this notation convention together with (2.10), the geometrical relations (2.3) become

$$(2.11) \quad \begin{aligned} e_{ss} &= \frac{\partial u_s}{\partial s} + \frac{u_n}{r}, & e_{sz} = e_{zs} &= \frac{1}{2} \left(\frac{\partial u_s}{\partial z} + \frac{\partial u_z}{\partial s} \right), & e_{zz} &= \frac{\partial u_z}{\partial z}, & \gamma_n &= \delta_n, \\ \gamma_s &= \delta_s - \frac{u_s}{r} + \frac{\partial u_n}{\partial s}, & \gamma_z &= \delta_z + \frac{\partial u_n}{\partial z}, & \rho_{ss} &= \frac{\partial \delta_s}{\partial s} + \frac{1}{r} \frac{\partial u_s}{\partial s} + \frac{u_n}{r^2}, \\ \rho_{zz} &= \frac{\partial \delta_z}{\partial z}, & \rho_{sz} &= \frac{\partial \delta_s}{\partial z}, & \rho_{zs} &= \frac{\partial \delta_z}{\partial s} + \frac{1}{r} \frac{\partial u_s}{\partial z}, & \rho_{ns} &= \frac{\partial \delta_n}{\partial s}, & \rho_{nz} &= \frac{\partial \delta_n}{\partial z}. \end{aligned}$$

The equations of equilibrium (2.6) can be written in the condensed form

$$(2.12) \quad \mathcal{A}(v) = -\mathcal{F},$$

where $\mathcal{F} = (f_s, f_z, f_n, l_s, l_z, l_n)$ and we have denoted by \mathcal{A} the linear operator defined on the set of displacement fields $v = (\mathbf{u}, \boldsymbol{\delta})$, by $\mathcal{A}(v) = (\mathcal{A}_1(v), \dots, \mathcal{A}_6(v))$ with

$$(2.13) \quad \begin{aligned} \mathcal{A}_1(v) &= \frac{\partial}{\partial s} N_{ss}(v) + \frac{\partial}{\partial z} N_{zs}(v) + \frac{1}{r} V_s(v), \\ \mathcal{A}_2(v) &= \frac{\partial}{\partial s} N_{sz}(v) + \frac{\partial}{\partial z} N_{zz}(v), \\ \mathcal{A}_3(v) &= \frac{\partial}{\partial s} V_s(v) + \frac{\partial}{\partial z} V_z(v) - \frac{1}{r} N_{ss}(v), \\ \mathcal{A}_4(v) &= \frac{\partial}{\partial s} M_{ss}(v) + \frac{\partial}{\partial z} M_{zs}(v) - V_s(v), \\ \mathcal{A}_5(v) &= \frac{\partial}{\partial s} M_{sz}(v) + \frac{\partial}{\partial z} M_{zz}(v) - V_z(v), \\ \mathcal{A}_6(v) &= \frac{\partial}{\partial s} M_{sn}(v) + \frac{\partial}{\partial z} M_{zn}(v) - V_n(v). \end{aligned}$$

We investigate the deformation of cylindrical shells loaded by assigned body loads \mathbf{f} and \mathbf{l} , by external forces and couples on the lateral edges (for open shells) and by resultant forces and moments acting on the end edges $\mathcal{C}_0, \mathcal{C}_{\bar{z}}$. The boundary conditions on lateral edges (in the case of open cylindrical shells) are

$$(2.14) \quad \mathbf{N} = \mathbf{N}^{(\gamma)}, \quad \mathbf{M} = \mathbf{M}^{(\gamma)} \quad \text{on } L_\gamma \quad (\gamma = 1, 2),$$

where $\mathbf{N}^{(\gamma)}$ and $\mathbf{M}^{(\gamma)}$ are prescribed vector fields.

For simplicity, we use the notation $s_1 = 0, s_2 = \bar{s}$. In the case of closed cylindrical shells, we consider the following continuity conditions for the displacement field $v = (\mathbf{u}, \boldsymbol{\delta})$ at the end points of the interval $[s_1, s_2]$:

$$(2.15) \quad v(s_1, z) = v(s_2, z), \quad \frac{\partial v}{\partial s}(s_1, z) = \frac{\partial v}{\partial s}(s_2, z), \quad \frac{\partial^2 v}{\partial s^2}(s_1, z) = \frac{\partial^2 v}{\partial s^2}(s_2, z),$$

where $z \in [0, \bar{z}]$.

Adopting the usual approach of the relaxed Saint–Venant’s problem, we assume that the resultant forces and moments acting on the end edges are prescribed. For any displacement field $v = (\mathbf{u}, \boldsymbol{\delta})$, we define the vectors

$$(2.16) \quad \mathcal{R}(v) = \int_{\mathcal{C}_0} \mathbf{N}(v) dl, \quad \mathcal{M}(v) = \int_{\mathcal{C}_0} [\mathbf{R} \times \mathbf{N}(v) + \mathbf{D} \times \mathbf{M}(v)] dl,$$

which represent the resultant force and the resultant moment about O of the external forces and director couples acting on the end edge \mathcal{C}_0 , corresponding to the displacement field v . Relations (2.16) can be expressed using the tensor components

$$(2.17) \quad \begin{aligned} \mathcal{R}_i(v) \mathbf{e}_i &= - \int_{\mathcal{C}_0} [x'_\alpha N_{zs}(v) + \epsilon_{\alpha\beta} x'_\beta V_z(v)] dl \mathbf{e}_\alpha - \int_{\mathcal{C}_0} N_{zz}(v) dl \mathbf{e}_3, \\ \mathcal{M}_i(v) \mathbf{e}_i &= \int_{\mathcal{C}_0} [\epsilon_{\beta\alpha} x_\beta N_{zz}(v) + x'_\alpha M_{zz}(v)] dl \mathbf{e}_\alpha \\ &\quad + \int_{\mathcal{C}_0} [\epsilon_{\alpha\beta} x'_\alpha x_\beta N_{zs}(v) + x_\alpha x'_\alpha V_z(v) - M_{zs}(v)] dl \mathbf{e}_3. \end{aligned}$$

The boundary conditions on the end edge \mathcal{C}_0 are

$$(2.18) \quad \mathcal{R}(v) = \mathcal{R}^0, \quad \mathcal{M}(v) = \mathcal{M}^0.$$

From the conditions of equilibrium for the shell and the relations (2.14) and (2.18), we can readily deduce the boundary conditions on the end edge $\mathcal{C}_{\bar{z}}$.

To resume, our problem consists in determining the equilibrium of a cylindrical shell subject to the assigned body loads \mathbf{f} , \mathbf{l} , the external loads on the lateral edges (2.14) (for open cylindrical shells), and the resultant force and resultant moment (2.18) on \mathcal{C}_0 .

Let us denote the problem formulated above by $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$. We mention that in the case when $\mathbf{f} = \mathbf{l} = \mathbf{N}^{(\gamma)} = \mathbf{M}^{(\gamma)} = \mathbf{0}$, this problem is the well-known relaxed Saint–Venant’s problem for Cosserat shells. On the other hand, if the prescribed loads \mathbf{f} , \mathbf{l} , $\mathbf{N}^{(\gamma)}$ and $\mathbf{M}^{(\gamma)}$ are independent of the axial coordinate z , then $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$ is called the Almansi–Michell problem, by analogy with the corresponding situation from the three-dimensional theory of elasticity. Also, if \mathbf{f} , \mathbf{l} , $\mathbf{N}^{(\gamma)}$ and $\mathbf{M}^{(\gamma)}$ are polynomials in the axial coordinate z , then $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$ is known as the Almansi problem. In Sec. 3 we shall give a solution to the problems of Almansi and Michell.

In the remainder of this paper, we consider anisotropic and inhomogeneous Cosserat shells, with constitutive coefficients which are independent of the axial coordinate z . Thus, we assume that the coefficients $C_{(n)}^{\alpha\dots}$ of the strain energy density \mathcal{W} in (2.5) are functions of the circumferential coordinate s only.

We denote by $\mathcal{D}(\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$ the set of all displacement fields $v = (\mathbf{u}, \boldsymbol{\delta}) \in C^1(\bar{\mathcal{S}}) \cap C^2(\mathcal{S})$ which satisfy the equations of equilibrium (2.12), together with the conditions on the lateral edges (2.14) for open shells, and the continuity conditions (2.15) for closed shells. Then, the problem

$P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$ can be formulated in the following form: find a displacement field $v \in \mathcal{D}(\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$ which satisfies the end edge conditions (2.18).

2.3. Some results concerning Saint–Venant’s problem

In this section, we summarize some results concerning the relaxed Saint–Venant’s problem for anisotropic cylindrical shells which have been obtained previously in [12]. The results and the notations presented in this section will be useful in the subsequent considerations.

The relaxed Saint–Venant’s problem is a particular situation of the problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$, in which we have

$$\mathbf{f} = \mathbf{l} = \mathbf{0}, \quad \mathbf{N}^{(\gamma)} = \mathbf{M}^{(\gamma)} = \mathbf{0}, \quad \gamma = 1, 2.$$

We shall denote by $P(\mathcal{R}^0, \mathcal{M}^0)$ the relaxed Saint–Venant’s problem, and by $K(\mathcal{R}^0, \mathcal{M}^0)$ the set of all solutions to this problem. Also, we shall use the notation $\mathcal{D}_0 = \mathcal{D}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. We observe that $P(\mathcal{R}^0, \mathcal{M}^0)$ can be decomposed into two problems:

(P_1): the extension-bending-torsion problem, characterized by $\mathcal{R}_\alpha^0 = 0$,

(P_2): the flexure problem, characterized by $\mathcal{R}_3^0 = \mathcal{M}_i^0 = 0$.

We denote by $K_I(\mathcal{R}_3^0, \mathcal{M}_1^0, \mathcal{M}_2^0, \mathcal{M}_3^0)$ and $K_{II}(\mathcal{R}_1^0, \mathcal{R}_2^0)$ the sets of all solutions $v = (\mathbf{u}, \boldsymbol{\delta})$ to the problems (P_1) and (P_2), respectively.

In [12], we have determined an exact solution of the extension-bending-torsion problem (P_1) expressed as a linear combination of four displacement fields $v^{(1)}, v^{(2)}, v^{(3)}$ and $v^{(4)}$. These displacement fields $v^{(k)}$ are known and their expressions are given in the Appendix. For any constants a_k , $k = 1, 2, 3, 4$, we denote by \hat{a} the four-dimensional vector $\hat{a} = (a_1, a_2, a_3, a_4)$ and let $v\{\hat{a}\}$ be the displacement field

$$v\{\hat{a}\} = a_1v^{(1)} + a_2v^{(2)} + a_3v^{(3)} + a_4v^{(4)}.$$

According to [12], the displacement field $v\{\hat{a}\}$ has the following properties: (i) $\partial v\{\hat{a}\}/\partial x_3$ is a rigid displacement field; (ii) $v\{\hat{a}\}$ satisfies the equations of equilibrium for zero body loads and the boundary conditions on the lateral edges for zero external loads, i.e. $v\{\hat{a}\} \in \mathcal{D}_0$; and (iii) the resultant force and resultant moment corresponding to the field $v\{\hat{a}\}$ are given by

$$(2.19) \quad \begin{aligned} \mathcal{R}(v\{\hat{a}\}) &= -\left(\sum_{k=1}^4 D_{3k}a_k\right)\mathbf{e}_3, \\ \mathcal{M}(v\{\hat{a}\}) &= \epsilon_{\beta\alpha}\left(\sum_{k=1}^4 D_{\beta k}a_k\right)\mathbf{e}_\alpha - \left(\sum_{k=1}^4 D_{4k}a_k\right)\mathbf{e}_3, \end{aligned}$$

where the coefficients D_{kr} are defined by

$$(2.20) \quad D_{kr} = \frac{1}{\bar{z}} \langle v^{(k)}, v^{(r)} \rangle, \quad k, r \in \{1, 2, 3, 4\}.$$

We have $\det(D_{kr})_{4 \times 4} \neq 0$. The solution of the relaxed Saint–Venant’s problem is presented by the following result.

THEOREM 1. (i) *The extension-bending-torsion problem (P_1) for cylindrical Cosserat shells admits a solution $v^0 \in K_I(\mathcal{R}_3^0, \mathcal{M}_1^0, \mathcal{M}_2^0, \mathcal{M}_3^0)$ such that $\frac{\partial v^0}{\partial x_3}$ is a rigid displacement field. This solution is given by*

$$(2.21) \quad v^0 = v\{\hat{a}\},$$

where the constants a_k are determined by the system of equations

$$(2.22) \quad \left(\sum_{r=1}^4 D_{kr} a_r \right)_{k=1, \dots, 4} = (\mathcal{M}_2^0, -\mathcal{M}_1^0, -\mathcal{R}_3^0, -\mathcal{M}_3^0).$$

(ii) *The flexure problem (P_2) admits a solution $v^F \in K_{II}(\mathcal{R}_1^0, \mathcal{R}_2^0)$ of the form*

$$(2.23) \quad v^F = \int_0^{x_3} v\{\hat{b}\} dx_3 + v\{\hat{c}\} + w(s),$$

where $\hat{b} = (b_1, b_2, b_3, b_4)$, $\hat{c} = (c_1, c_2, c_3, c_4)$ are constants and $w(s)$ is a displacement field which depends only on s . For this solution, the constants b_k are given by

$$(2.24) \quad \left(\sum_{r=1}^4 D_{kr} b_r \right)_{k=1, \dots, 4} = (-\mathcal{R}_1^0, -\mathcal{R}_2^0, 0, 0).$$

Theorem 1 has been proved in Sec. 5 (see Theorems 4 and 5) of [12]. We mention that the constants \hat{c} and the field $w(s)$ appearing in (2.23) have also been determined in [12] and their expressions are recorded in the Appendix.

We mention that v^0 and v^F represent exact solutions to the linear equations of problems (P_1) and (P_2) . In this context, we notice that the solutions v^0 and v^F possess properties which are analogous to those of the classical Saint–Venant’s solutions in linear elasticity (see e.g., [9]).

3. Solution to the problems of Almansi and Michell

In this section, we shall give a solution to the Almansi and Michell problems $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$, as defined in Sec. 2.2. Thus, we study the deformation of cylindrical shells subject to some assigned body loads \mathbf{f} , \mathbf{l} , and some

external loads on the lateral edges $\mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)}$ (for open shells). The resultant force \mathcal{R}^0 and resultant moment \mathcal{M}^0 act on the end edge \mathcal{C}_0 .

Let us confine our attention first to the Almansi–Michell problem and assume that the fields $\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)}$ are independent of the axial coordinate x_3 . For this case, the Theorem 1 of [12] admits the following consequence.

COROLLARY 2. *Let v be a solution of the Almansi–Michell problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$. If $\frac{\partial v}{\partial x_3} \in C^1(\bar{\mathcal{S}}) \cap C^2(\mathcal{S})$, then $\frac{\partial v}{\partial x_3}$ is a solution of the relaxed Saint–Venant’s problem $P(\mathbf{G}, \mathbf{Q})$ where*

$$(3.1) \quad \begin{aligned} \mathbf{G} &= \int_{\mathcal{C}_0} \mathbf{f} dl + (1 - \varepsilon)(\mathbf{N}^{(1)} + \mathbf{N}^{(2)}), \\ \mathbf{Q} &= \epsilon_{\alpha\beta} \mathcal{R}_\beta^0 \mathbf{e}_\alpha + \int_{\mathcal{C}_0} (\mathbf{R} \times \mathbf{f} + \mathbf{D} \times \mathbf{l}) dl \\ &\quad + (1 - \varepsilon) \sum_{\gamma=1}^2 (\mathbf{R}^{(\gamma)}(0) \times \mathbf{N}^{(\gamma)} + \mathbf{D}^{(\gamma)} \times \mathbf{M}^{(\gamma)}), \end{aligned}$$

where we denote by

$$\mathbf{R}^{(\gamma)}(z) = [\mathbf{R}(s, z)]_{s=s_\gamma}, \quad \mathbf{D}^{(\gamma)} = [\mathbf{D}(s)]_{s=s_\gamma} \quad (\gamma = 1, 2),$$

and ε is a parameter which takes the values $\varepsilon = 0$ for open cylindrical shells and $\varepsilon = 1$ for closed shells.

Suggested by Corollary 2 and Theorem 1, we search for the solution of the Almansi–Michell problem in the form

$$(3.2) \quad v = \int_0^{x_3} \int_0^{x_3} v\{\hat{b}\} dx_3 dx_3 + \int_0^{x_3} v\{\hat{c}\} dx_3 + v\{\hat{a}\} + x_3 w(s) + \tilde{w}(s),$$

where $\hat{a}, \hat{b}, \hat{c}$ are sets of constants to be determined, while $w(s)$ and $\tilde{w}(s)$ are unknown displacement fields of class $C^2[0, \bar{s}]$ which depend on s only. Indeed, for the field (3.2) we have (modulo a rigid displacement)

$$(3.3) \quad \frac{\partial v}{\partial x_3} = \int_0^{x_3} v\{\hat{b}\} dx_3 + v\{\hat{c}\} + w(s),$$

and the Theorem 1 asserts that (3.3) can be a solution of Saint–Venant’s relaxed problem for a suitable choice of \hat{b}, \hat{c} and $w(s)$. Let us prove the following result.

THEOREM 3. *Denote by X the set of all displacement fields of the form (3.2). Then, there exists a field $v = (\mathbf{u}, \boldsymbol{\delta}) \in X$ such that v is a solution of the Almansi–Michell problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$.*

P r o o f. Assume that v is a displacement field of the form (3.2). We want to determine the constants \hat{a} , \hat{b} , \hat{c} and the fields $w(s)$, $\tilde{w}(s)$ such that v will be a solution. By virtue of Corollary 2 and (3.3), if v is a solution of the problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$, then we have

$$(3.4) \quad \int_0^{x_3} v\{\hat{b}\} dx_3 + v\{\hat{c}\} + w(s) \in K(\mathbf{G}, \mathbf{Q}).$$

From (3.4) and Theorem 1 (see also the Appendix), we deduce that $w(s)$ can be determined as the solution of the cross-section plane problem, while the constants \hat{b} and \hat{c} are given by

$$(3.5) \quad \begin{aligned} \left(\sum_{r=1}^4 D_{kr} b_r \right)_{k=1, \dots, 4} &= (-G_\alpha, 0, 0), \\ \left(\sum_{r=1}^4 D_{kr} c_r \right)_{k=1, \dots, 4} &= (\epsilon_{\alpha\beta}(Q_\beta - \mathcal{M}_\beta(\hat{w})), \mathcal{R}_3(\hat{w}) - G_3, \mathcal{M}_3(\hat{w}) - Q_3), \end{aligned}$$

where \hat{w} is expressed by $\hat{w} = w(s) + \int_0^z v\{\hat{b}\} dz$. In what follows, we assume that \hat{b} , \hat{c} and $w(s)$ are known. Let us determine $\tilde{w}(s)$ such that the field (3.2) satisfies $v \in \mathcal{D}(\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$. The equations of equilibrium (2.12) reduce to

$$(3.6) \quad \mathcal{A}(\tilde{w}) = -\mathcal{F} - \mathcal{A}(V\{\hat{b}, \hat{c}, w\}),$$

where we have denoted:

$$V\{\hat{b}, \hat{c}, w\} = \int_0^z \int_0^z v\{\hat{b}\} dz dz + \int_0^z v\{\hat{c}\} dz + z w(s).$$

Also, from (2.14) and (2.15) we obtain the boundary conditions:

(i) for open shells

$$(3.7) \quad \begin{aligned} \mathbf{N}(\tilde{w}) &= \mathbf{N}^{(\gamma)} - \mathbf{N}(V\{\hat{b}, \hat{c}, w\}), \\ \mathbf{M}(\tilde{w}) &= \mathbf{M}^{(\gamma)} - \mathbf{M}(V\{\hat{b}, \hat{c}, w\}) \quad \text{on } L_\gamma \ (\gamma = 1, 2); \end{aligned}$$

(ii) for closed shells

$$(3.8) \quad \tilde{w}(s_1) = \tilde{w}(s_2), \quad \tilde{w}'(s_1) = \tilde{w}'(s_2).$$

Since the operators \mathcal{A} and $\partial/\partial z$ commute, we obtain that

$$\frac{\partial}{\partial z} \mathcal{A}(V\{\hat{b}, \hat{c}, w\}) = \mathcal{A}\left(\int_0^z v\{\hat{b}\} dz + v\{\hat{c}\} + w(s)\right) = 0,$$

by virtue of (3.4). Thus, the right-hand side of (3.6) is a function independent of z . Similarly, we see that the right-hand sides of relations (3.7) do not depend on z . Consequently, the Eqs. (3.6)–(3.8) constitute a cross-section plane problem of the type (5.1)–(5.3) for the unknown field $\tilde{w}(s)$ (see the Appendix). We know that the problem (3.6)–(3.8) admits a solution $\tilde{w}(s)$ if and only if the conditions corresponding to (5.4) are satisfied. But, in our case the conditions (5.4) reduce to

$$\mathcal{R}\left(\int_0^z v\{\hat{b}\} dz + v\{\hat{c}\} + w(s)\right) = \mathbf{G}, \quad \mathcal{M}_3\left(\int_0^z v\{\hat{b}\} dz + v\{\hat{c}\} + w(s)\right) = Q_3,$$

which hold true by virtue of (3.4). Hence, we can determine the field $\tilde{w}(s)$ such that $v \in \mathcal{D}(\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$.

Finally, we find the constants \hat{a} by imposing the end edge conditions $\mathcal{R}_i(v) = \mathcal{R}_i^0$, $\mathcal{M}_i(v) = \mathcal{M}_i^0$. We notice that the relations $\mathcal{R}_\alpha(v) = \mathcal{R}_\alpha^0$ are verified in view of (3.4). Then, the remaining conditions $\mathcal{R}_3(v) = \mathcal{R}_3^0$, $\mathcal{M}_i(v) = \mathcal{M}_i^0$ can be put in the form of the algebraic system

$$(3.9) \quad \left(\sum_{r=1}^4 D_{kr} a_r\right)_{k=1,\dots,4} = (\epsilon_{\alpha\beta}(\mathcal{M}_\beta^0 - \mathcal{M}_\beta(\bar{w})), \mathcal{R}_3(\bar{w}) - \mathcal{R}_3^0, \mathcal{M}_3(\bar{w}) - \mathcal{M}_3^0),$$

where $\bar{w} = V\{\hat{b}, \hat{c}, w\} + \tilde{w}$. From (3.9) we determine \hat{a} and the proof is complete. \square

Let us turn our attention now to the Almansi problem. As we have mentioned above, in the case of the Almansi problem the external loads $\mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}$ and $\mathbf{M}^{(\gamma)}$ are polynomials in the axial coordinate, i.e. we have

$$(3.10) \quad \begin{aligned} \mathbf{f} &= \sum_{m=0}^n \mathbf{f}_{(m)} x_3^m, & \mathbf{l} &= \sum_{m=0}^n \mathbf{l}_{(m)} x_3^m, \\ \mathbf{N}^{(\gamma)} &= \sum_{m=0}^n \mathbf{N}_{(m)}^\gamma x_3^m, & \mathbf{M}^{(\gamma)} &= \sum_{m=0}^n \mathbf{M}_{(m)}^\gamma x_3^m, \end{aligned}$$

where $\mathbf{f}_{(m)}$ and $\mathbf{l}_{(m)}$ are given functions which depend only on s , while $\mathbf{N}_{(m)}^\gamma$ and $\mathbf{M}_{(m)}^\gamma$ are known constants ($\gamma = 1, 2$).

In what follows, we shall present a method to construct the solution of the Almansi problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$.

Let us denote by (A_0) the Almansi–Michell problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}_{(0)}, \mathbf{l}_{(0)}, \mathbf{N}_{(0)}^\gamma, \mathbf{M}_{(0)}^\gamma)$ and by (A_m) the particular Almansi problem $P(\mathbf{0}, \mathbf{0}, \mathbf{f}_{(m)}x_3^m, \mathbf{l}_{(m)}x_3^m, \mathbf{N}_{(m)}^\gamma x_3^m, \mathbf{M}_{(m)}^\gamma x_3^m)$, $m = 1, \dots, n$. Let $S_m(\mathbf{f}_{(m)}x_3^m, \mathbf{l}_{(m)}x_3^m, \mathbf{N}_{(m)}^\gamma x_3^m, \mathbf{M}_{(m)}^\gamma x_3^m)$ denote the set of all solutions to the problem $P(\mathbf{0}, \mathbf{0}, \mathbf{f}_{(m)}x_3^m, \mathbf{l}_{(m)}x_3^m, \mathbf{N}_{(m)}^\gamma x_3^m, \mathbf{M}_{(m)}^\gamma x_3^m)$, $m = 0, 1, \dots, n$. By the linearity of the theory, it is sufficient to know the solution of each problem (A_m) in order to solve our initial Almansi problem. We notice that we have already determined a solution of the problem (A_0) in Theorem 3.

To obtain a solution of the problems (A_m) for all $m \geq 1$, we employ the method of induction. As our induction hypothesis, assume that we know a solution of the problem (A_m) . This implies that we can find a displacement field v^* such that

$$(3.11) \quad v^* \in S_m(\mathbf{f}_{(m+1)}x_3^m, \mathbf{l}_{(m+1)}x_3^m, \mathbf{N}_{(m+1)}^\gamma x_3^m, \mathbf{M}_{(m+1)}^\gamma x_3^m).$$

Let us determine a solution v of the problem (A_{m+1}) . To this end, we shall use the following result, which is a consequence of Theorem 1 from [12].

LEMMA 4. *Let v be a solution of the problem (A_{m+1}) such that $\frac{\partial v}{\partial x_3} \in C^1(\bar{\mathcal{S}}) \cap C^2(\mathcal{S})$. Then, we have*

$$(3.12) \quad \frac{1}{m+1} \frac{\partial v}{\partial x_3} \in S_m(\mathbf{f}_{(m+1)}x_3^m, \mathbf{l}_{(m+1)}x_3^m, \mathbf{N}_{(m+1)}^\gamma x_3^m, \mathbf{M}_{(m+1)}^\gamma x_3^m).$$

The next theorem establishes the existence of a solution for the problem (A_{m+1}) and its proof shows how to construct such a solution.

THEOREM 5. *Let v^* be the displacement field given by (3.11). Then, there exists a solution v to the problem (A_{m+1}) of the form*

$$(3.13) \quad v = (m+1) \left(\int_0^{x_3} v^* dx_3 + v\{\hat{a}\} + w^*(s) \right),$$

where \hat{a} are some constants and $w^*(s)$ is a displacement field which depends only on s .

P r o o f. Let us determine first the function $w^*(s)$ such that the field (3.13) satisfies $v \in \mathcal{D}(\mathbf{f}_{(m+1)}x_3^{m+1}, \mathbf{l}_{(m+1)}x_3^{m+1}, \mathbf{N}_{(m+1)}^\gamma x_3^{m+1}, \mathbf{M}_{(m+1)}^\gamma x_3^{m+1})$. The equations of equilibrium (2.12) can be written in this case as

$$(3.14) \quad \mathcal{A}(w^*(s)) = -\frac{1}{m+1} (\mathbf{f}_{(m+1)}z^{m+1}, \mathbf{l}_{(m+1)}z^{m+1}) - \mathcal{A} \left(\int_0^z v^* dz \right).$$

We notice that

$$\begin{aligned} \frac{\partial}{\partial z} \left[\mathcal{A} \left(\int_0^z v^* dz \right) + \frac{1}{m+1} (\mathbf{f}_{(m+1)} z^{m+1}, \mathbf{l}_{(m+1)} z^{m+1}) \right] \\ = \mathcal{A}(v^*) + (\mathbf{f}_{(m+1)} z^m, \mathbf{l}_{(m+1)} z^m) = 0, \end{aligned}$$

by virtue of (3.11). Hence, the right-hand side of relation (3.14) does not depend on z , and the Eq. (3.14) reduces to

$$(3.15) \quad \mathcal{A}(w^*(s)) = - \left[\mathcal{A} \left(\int_0^z v^* dz \right) \right] (s, 0), \quad s \in [0, \bar{s}].$$

Similarly, the conditions on the lateral edges for open shells reduce to

$$\begin{aligned} \mathbf{N}(w^*(s_\gamma)) &= - \left[\mathbf{N} \left(\int_0^z v^* dz \right) \right] (s_\gamma, 0), \\ \mathbf{M}(w^*(s_\gamma)) &= - \left[\mathbf{M} \left(\int_0^z v^* dz \right) \right] (s_\gamma, 0) \quad \text{on } L_\gamma. \end{aligned}$$

For closed cylindrical shells, the continuity conditions (2.15)_{1,2} become

$$w^*(s_1) = w^*(s_2), \quad w^{*'}(s_1) = w^{*'}(s_2).$$

It is clear that the above conditions together with the Eqs. (3.15) represent a cross-section plane problem of the type (5.1)–(5.3) for the determination of $w^*(s)$. The necessary and sufficient conditions (5.4) for the existence of a solution $w^*(s)$ are equivalent to the relations $\mathcal{R}(v^*) = \mathbf{0}$, $\mathcal{M}_3(v^*) = 0$, which are valid by virtue of (3.11). Thus, from the Eqs. (3.15) we can find the field $w^*(s)$ with the desired properties.

To complete the proof, we have to determine the constants \hat{a} such that the field (3.13) satisfies $\mathcal{R}(v^*) = \mathbf{0}$, $\mathcal{M}(v^*) = \mathbf{0}$. These end edge conditions lead us to the following relations:

$$\begin{aligned} \left(\sum_{r=1}^4 D_{kr} a_r \right)_{k=1, \dots, 4} \\ = \left(\epsilon_{\beta\alpha} \mathcal{M}_\beta \left(\int_0^z v^* dz + w^* \right), \mathcal{R}_3 \left(\int_0^z v^* dz + w^* \right), \mathcal{M}_3 \left(\int_0^z v^* dz + w^* \right) \right), \end{aligned}$$

which can be used to find the constants a_k , $k = 1, \dots, 4$. \square

Let us mention that the form (3.13) for the solution v of the problem (A_{m+1}) has been suggested by the result (3.12) presented in Lemma 4.

By the method of induction, we know how to solve any problem (A_m) , for $m = 0, 1, \dots, n$. In conclusion, we can determine the solution to the Almansi problem $P(\mathcal{R}^0, \mathcal{M}^0, \mathbf{f}, \mathbf{l}, \mathbf{N}^{(\gamma)}, \mathbf{M}^{(\gamma)})$.

The results established in this paper concerning the problems of Almansi and Michell, can be useful in the treatment of practical problems for loaded cylindrical shells. Since the solution procedure is valid for the case of general anisotropy, we can particularize it for different material symmetries. Provided we know the constitutive coefficients which characterize a specific anisotropic shell, we can apply the above method to obtain the displacement field corresponding to the given external loads. We mention that the constitutive coefficients for isotropic and orthotropic Cosserat shells have been determined completely in [1, 2], in terms of the classical elasticity constants. In the next section, we present the case of orthotropic cylindrical Cosserat shells and determine the equilibrium of a circular tube under the action of a hydrostatic pressure which depends linearly on the axial coordinate.

The Cosserat theory has the advantage that it can be easily extended to include some other effects, such as thermal and porosity effects [12, 13]. Also, we remark a strong analogy between the properties of solutions for Cosserat shells and the corresponding classical results of the three-dimensional elasticity.

4. Deformation of an orthotropic circular cylindrical tube

In this section we shall restrict our attention to orthotropic and homogeneous shells. We use the model of Cosserat surfaces and assume that the axes of orthotropy coincide with the directions of the vectors $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\} \equiv \{\boldsymbol{\tau}, \mathbf{e}_3, \mathbf{n}\}$.

We consider cylindrical shells made of an orthotropic material with constitutive coefficients denoted by $c_{11}, c_{22}, c_{33}, c_{12}, c_{13}, c_{23}, c_{44}, c_{55}$ and c_{66} , such that

$$\begin{aligned} t_{11} &= c_{11}\varepsilon_{11} + c_{12}\varepsilon_{22} + c_{13}\varepsilon_{33}, \\ t_{22} &= c_{12}\varepsilon_{11} + c_{22}\varepsilon_{22} + c_{23}\varepsilon_{33}, \\ t_{33} &= c_{13}\varepsilon_{11} + c_{23}\varepsilon_{22} + c_{33}\varepsilon_{33}, \\ t_{23} &= 2c_{44}\varepsilon_{23}, \quad t_{31} = 2c_{55}\varepsilon_{31}, \quad t_{12} = 2c_{66}\varepsilon_{12}, \end{aligned}$$

where t_{ij} and ε_{kr} are the components of the stress tensor and strain tensor in the three-dimensional linear theory, referred to the orthonormal basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$.

Let h be the constant thickness of the cylindrical shell. It is convenient to introduce the notations

$$I = \frac{h^3}{12}, \quad \gamma = \frac{I}{hr^2} = \frac{1}{12} \left(\frac{h}{r} \right)^2,$$

$$d_{11} = c_{11} - \frac{c_{13}c_{13}}{c_{33}}, \quad d_{22} = c_{22} - \frac{c_{23}c_{23}}{c_{33}}, \quad d_{12} = c_{12} - \frac{c_{13}c_{23}}{c_{33}}.$$

Then, the strain energy function (2.5)₁ for orthotropic and homogeneous cylindrical Cosserat shells has the form (see [2], Sec. 4.27)

$$\begin{aligned} 2\mathcal{W} = & h[c_{11}(e_{ss})^2 + c_{22}(e_{zz} + \gamma r \rho_{zz})^2 + c_{33}(\gamma_n)^2 + 2c_{12}e_{ss}(e_{zz} + \gamma r \rho_{zz}) \\ & + 2c_{13}e_{ss}\gamma_n + 2c_{23}(e_{zz} + \gamma r \rho_{zz})\gamma_n + c_{44}(\gamma_z + \gamma r \rho_{nz})^2 + c_{55}(\gamma_s)^2 \\ & + c_{66}(2e_{sz} + \gamma r \rho_{sz})^2] + \gamma h[d_{11}(r\rho_{ss} - 2e_{ss} + \gamma_n)^2 + d_{22}(r\rho_{zz})^2 \\ & + 2d_{12}(r\rho_{ss} - 2e_{ss} + \gamma_n)(r\rho_{zz}) + c_{66}(r\rho_{sz} + r\rho_{zs} - 2e_{sz})^2], \end{aligned}$$

and the relations (2.5)₂₋₄ can be used to derive the constitutive equations in our case. We mention that the expressions of $d_{\alpha\beta}$ and the other coefficients which appear in the constitutive equations have been determined in [2], Chapter 4, by comparison between the solutions to some corresponding problems in the two different approaches (three-dimensional and Cosserat surface). In this respect, see also the paper [14].

Let us formulate the specific problem which will be treated subsequently. We investigate the deformation of a circular cylindrical orthotropic shell under the action of a hydrostatic pressure, acting on its surface. Assume that the magnitude of this pressure depends only on the axial coordinate. The cylindrical tube is also subjected to extension, bending and torsion, due to some resultant forces and moments acting on its end edges. Thus, the system of external loads acting on the closed cylindrical shell is given by

$$(4.1) \quad \mathbf{f} = \mathbf{0}, \quad \mathbf{l} = -h(P_1x_3 + P_0)\mathbf{n}, \quad \mathcal{R}^0 = \mathcal{R}_3^0\mathbf{e}_3, \quad \mathcal{M}^0 = \mathcal{M}_i^0\mathbf{e}_i,$$

where the resultants \mathcal{R}_3^0 and \mathcal{M}_i^0 are prescribed, while P_0 and P_1 are given constants. The component l_n specified in (4.1) accounts for the hydrostatic pressure ($P_1x_3 + P_0$) acting on the major surfaces of the shell (see [1], Sec. 24iii), which is linear in the axial coordinate.

We shall determine the equilibrium of the circular tube subject to the external loads (4.1), using the results presented in Sec. 3. For a circular cross-section, the functions $x_\alpha(s)$ which appear in the parametric representation (2.9) are given by

$$(4.2) \quad x_1(s) = r_0 \cos \frac{s}{r_0}, \quad x_2(s) = r_0 \sin \frac{s}{r_0}, \quad s \in [0, 2\pi r_0],$$

where r_0 denotes the radius of the cylindrical surface. We have $r(s) = r_0$ and $\bar{s} = 2\pi r_0$.

In the case of circular cylindrical orthotropic shells, the field $v\{\hat{a}\}$ has been determined in [12], Sec. 6, and it has the following expression:

$$(4.3) \quad \begin{aligned} u_\alpha &= -\frac{1}{2} a_\alpha x_3^2 - a_4(\epsilon_{\alpha\beta} x_\beta) x_3 + (D a_\gamma x_\gamma + G a_3) x_\alpha, \\ u_3 &= (a_\alpha x_\alpha + a_3) x_3, \quad \delta_3 = r_0^{-1} (a_\alpha x_\alpha) x_3, \\ \delta_\alpha &= r_0^{-1} [-a_4(\epsilon_{\alpha\beta} x_\beta) x_3 + (E a_\gamma x_\gamma + H a_3) x_\alpha + F(\epsilon_{\gamma\delta} a_\gamma x_\delta) \epsilon_{\alpha\beta} x_\beta], \end{aligned}$$

where the constant coefficients D , E , F , G and H are given by the equations

$$(4.4) \quad \begin{aligned} c_{11}D + c_{13}E &= -c_{12}(1 + \gamma), \\ c_{13}D + c_{33}E + c_{55}(D + F) &= -c_{23}(1 + \gamma), \\ (c_{55} + \gamma d_{11})(D + F) - \gamma d_{11}E &= \gamma d_{12}, \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} (c_{11} + \gamma d_{11})G + (c_{13} - \gamma d_{11})H &= -c_{12}, \\ (c_{13} - \gamma d_{11})G + (c_{33} + \gamma d_{11})H &= -c_{23}. \end{aligned}$$

The resultants corresponding to the field (4.3) are expressed by (2.19) where the coefficients D_{kr} have the following values:

$$(4.6) \quad \begin{aligned} D_{11} = D_{22} &= \frac{1}{2} r_0^2 \bar{s} h \{ (1 + \gamma) [c_{22}(1 + \gamma) + c_{12}D + c_{23}E] \\ &\quad + \gamma [d_{22} - d_{12}(D + F - E)] \}, \\ D_{33} &= \bar{s} h (c_{22} + c_{12}G + c_{23}H), \\ D_{44} &= h r_0^2 c_{66} (1 + 3\gamma + \gamma^2), \\ D_{kr} &= 0 \quad \text{for } k \neq r. \end{aligned}$$

Following the method described in Sec. 3, we divide our problem into two problems, with respect to the dependence on the axial coordinate. Thus, we determine separately the deformation of the cylindrical tube due to the following systems of external loads:

$$(4.7) \quad \mathbf{f} = \mathbf{0}, \quad \mathbf{l} = -h P_0 \mathbf{n}, \quad \mathcal{R}^0 = \mathcal{R}_3^0 \mathbf{e}_3, \quad \mathcal{M}^0 = \mathcal{M}_i^0 \mathbf{e}_i,$$

and

$$(4.8) \quad \mathbf{f} = \mathbf{0}, \quad \mathbf{l} = -h(P_1 x_3) \mathbf{n}, \quad \mathcal{R}^0 = \mathbf{0}, \quad \mathcal{M}^0 = \mathbf{0}.$$

This split is necessary because the loads (4.7) give rise to a problem of Almansi–Michell type, while (4.8) represents an Almansi problem (since the external load \mathbf{l} depends on x_3). In view of the linearity of the theory, the solution of our initial problem is then obtained as a sum of the solutions for the two auxiliary problems.

We observe that, for the external loads (4.7), we have a problem of Almansi–Michell type, and consequently we can apply the technique presented by Theorem 3 to derive the solution of this problem. On the other hand, in order to solve the problem for the external loads (4.8), we follow the recurrence process described by Lemma 4 and Theorem 5 for the treatment of Almansi’s problem. After some simple but lengthy calculations, we obtain the following solution for our initial problem corresponding to the loads (4.1):

$$\begin{aligned}
 u_\alpha &= -\frac{1}{2} a_\alpha x_3^2 - a_4 (\epsilon_{\alpha\beta} x_\beta) x_3 + (D a_\gamma x_\gamma + G a_3) x_\alpha + A (P_1 x_3 + P_0) x_\alpha, \\
 u_3 &= (a_\alpha x_\alpha + a_3) x_3 + C \left(\frac{1}{2} P_1 x_3 + P_0 \right) x_3, \\
 (4.9) \quad \delta_\alpha &= r_0^{-1} [-a_4 (\epsilon_{\alpha\beta} x_\beta) x_3 + (E a_\gamma x_\gamma + H a_3) x_\alpha \\
 &\quad + F (\epsilon_{\gamma\delta} a_\gamma x_\delta) \epsilon_{\alpha\beta} x_\beta + B (P_1 x_3 + P_0) x_\alpha], \\
 \delta_3 &= r_0^{-1} \left[(a_\alpha x_\alpha) x_3 - r_0^2 P_1 \left(A \left(1 + \gamma \frac{d_{12}}{c_{44}} \right) + B \gamma \left(1 - \frac{d_{12}}{c_{44}} \right) \right) \right].
 \end{aligned}$$

Here, the notations D , E , F , G , H are specified by (4.4) and (4.5), while the constant coefficients A , B and C are given by the relations

$$\begin{aligned}
 (4.10) \quad & (c_{11} + \gamma d_{11}) A + (c_{13} - \gamma d_{11}) B + c_{12} C = 0, \\
 & (c_{13} - \gamma d_{11}) A + (c_{33} + \gamma d_{11}) B + c_{23} C = -1, \\
 & c_{12} A + c_{23} B + c_{22} C = 0.
 \end{aligned}$$

The constants a_k are expressed in terms of the applied resultants by

$$(4.11) \quad a_\alpha = \frac{\epsilon_{\alpha\beta} \mathcal{M}_\beta^0}{D_{11}}, \quad a_3 = -\frac{\mathcal{R}_3^0}{D_{33}}, \quad a_4 = -\frac{\mathcal{M}_3^0}{D_{44}},$$

where D_{kr} are given in (4.6).

Finally, we consider the same problem in the limiting case of a very thin shell, i.e. $h \ll r$. If we apply the above results in the limit $\gamma \equiv \frac{1}{12} \left(\frac{h}{r_0} \right)^2 \rightarrow 0$, then we obtain that the constants A , B , C , D , E , F , G and H are given by

$$D = -F = G = -\frac{c_{12} c_{33} - c_{13} c_{23}}{c_{11} c_{33} - (c_{13})^2}, \quad E = H = \frac{c_{12} c_{13} - c_{11} c_{23}}{c_{11} c_{33} - (c_{13})^2}$$

and

$$\begin{aligned}c_{11}A + c_{13}B + c_{12}C &= 0, \\c_{13}A + c_{33}B + c_{23}C &= -1, \\c_{12}A + c_{23}B + c_{22}C &= 0.\end{aligned}$$

The form of the solution (4.9) is simplified and becomes

$$\begin{aligned}u_\alpha &= -\frac{1}{2}a_\alpha x_3^2 - a_4(\epsilon_{\alpha\beta}x_\beta)x_3 + D(a_\gamma x_\gamma + a_3)x_\alpha + A(P_1x_3 + P_0)x_\alpha, \\u_3 &= (a_\alpha x_\alpha + a_3)x_3 + C\left(\frac{1}{2}P_1x_3 + P_0\right)x_3, \quad \delta_3 = r_0^{-1}[(a_\alpha x_\alpha)x_3 - r_0^2AP_1], \\ \delta_\alpha &= r_0^{-1}[-a_4(\epsilon_{\alpha\beta}x_\beta)x_3 + E(a_\gamma x_\gamma + a_3)x_\alpha - D(\epsilon_{\gamma\delta}a_\gamma x_\delta)\epsilon_{\alpha\beta}x_\beta + B(P_1x_3 + P_0)x_\alpha].\end{aligned}$$

Here, the constants a_k are determined by the relations (4.11), in which the coefficients D_{11} , D_{33} and D_{44} are expressed by

$$\begin{aligned}D_{11} &= \frac{1}{2} \frac{r_0^2 h \bar{s}}{c_{11}c_{33} - (c_{13})^2} \det(c_{ij})_{3 \times 3}, \\D_{33} &= \frac{h \bar{s}}{c_{11}c_{33} - (c_{13})^2} \det(c_{ij})_{3 \times 3}, \\D_{44} &= hr_0^2 c_{66}.\end{aligned}$$

We mention that, for isotropic and homogeneous Cosserat shells, the problems of Saint–Venant, Almansi and Michell have been solved in [8, 15] and the solutions have been given in a closed form. The particular cases of *circular* cylindrical shells and Cosserat *plates* have also been treated thoroughly. Using the identification of the constitutive coefficients in the isotropic case, we have compared these results (see [13], Sec. 6.2) with the corresponding solutions from the classical theory (see e.g. [3–5]), and we have found a good agreement between the two approaches.

5. The Almansi–Michell problem for orthotropic plates

We emphasize that the results presented in this paper are exact in the context of linear theory and they involve no approximations. In order to show that the results obtained by this method are correct, we shall make a comparison with the exact three-dimensional solution of the Almansi–Michell problem for initially flat shells.

In this section we confine our attention to the special case when the reference configuration is a flat surface. This means that $B_{\alpha\beta} = 0$ and the cylindrical shell

reduces to a rectangular plate. We can choose the coordinate frame $Ox_1x_2x_3$ such that the reference surface is situated in the x_1Ox_3 plane (see Fig. 2) and is given by the parametric equations (2.9), with

$$(5.1) \quad x_1(s) = s - \frac{\bar{s}}{2}, \quad x_2(s) = 0, \quad s \in [0, \bar{s}].$$

Thus, in our case we have $\mathbf{A}_1 = \boldsymbol{\tau} = \mathbf{e}_1$, $\mathbf{A}_2 = \mathbf{e}_3$, $\mathbf{A}_3 = \mathbf{n} = -\mathbf{e}_2$, and the reference configuration occupies the region $\{(x_1, x_3); -\bar{s}/2 < x_1 < \bar{s}/2, 0 < x_3 < \bar{z}\} \subset x_1Ox_3$.

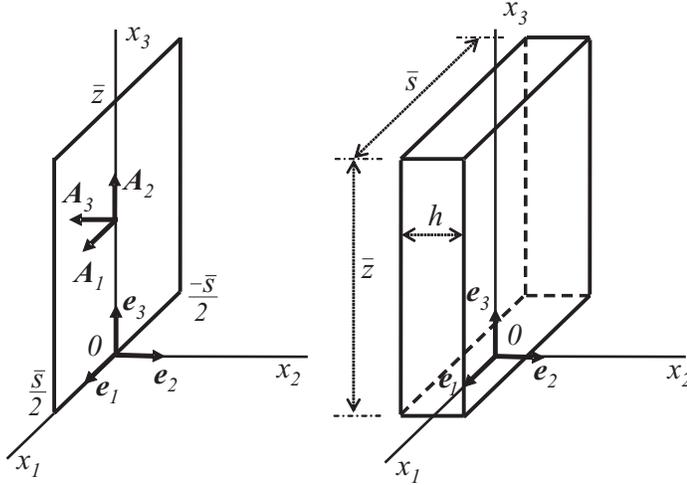


FIG. 2. Reference configurations of the Cosserat plate and the corresponding three-dimensional plate.

Since the curvature radius r is $+\infty$ for flat surfaces, the geometrical equations (2.11) written for Cosserat plates become

$$\begin{aligned} e_{ss} &= \frac{\partial u_s}{\partial s}, & e_{zz} &= \frac{\partial u_z}{\partial z}, & e_{sz} &= e_{zs} = \frac{1}{2} \left(\frac{\partial u_s}{\partial z} + \frac{\partial u_z}{\partial s} \right), \\ \gamma_s &= \delta_s + \frac{\partial u_n}{\partial s}, & \gamma_z &= \delta_z + \frac{\partial u_n}{\partial z}, & \gamma_n &= \delta_n, \\ \rho_{ss} &= \frac{\partial \delta_s}{\partial s}, & \rho_{zz} &= \frac{\partial \delta_z}{\partial z}, & \rho_{sz} &= \frac{\partial \delta_s}{\partial z}, \\ \rho_{zs} &= \frac{\partial \delta_z}{\partial s}, & \rho_{ns} &= \frac{\partial \delta_n}{\partial s}, & \rho_{nz} &= \frac{\partial \delta_n}{\partial z}. \end{aligned}$$

If the Cosserat plate is made of an homogeneous and orthotropic material in which the axes of orthotropy coincide with the directions of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, then

the strain energy function has the form (see [14], and also [2], Sec. 4.27)

$$\begin{aligned} 2\mathcal{W} = & h[c_{11}(e_{ss})^2 + c_{22}(e_{zz})^2 + c_{33}(\gamma_n)^2 + 2c_{12}e_{ss}e_{zz} + 2c_{13}e_{ss}\gamma_n \\ & + 2c_{23}e_{zz}\gamma_n + c_{44}(\gamma_z)^2 + c_{55}(\gamma_s)^2 + 4c_{66}(e_{sz})^2] + I[d_{11}(\rho_{ss})^2 \\ & + d_{22}(\rho_{zz})^2 + 2d_{12}\rho_{ss}\rho_{zz} + c_{66}(\rho_{sz} + \rho_{zs})^2]. \end{aligned}$$

From the constitutive equations (2.5)₂₋₄ we get in our case

$$\begin{aligned} N_{ss} &= h(c_{11}e_{ss} + c_{12}e_{zz} + c_{13}\gamma_n), & N_{zz} &= h(c_{12}e_{ss} + c_{22}e_{zz} + c_{23}\gamma_n), \\ N_{sz} &= N_{zs} = 2hc_{66}e_{sz}, & V_s &= hc_{55}\gamma_s, & V_z &= hc_{44}\gamma_z, \\ V_n &= h(c_{13}e_{ss} + c_{23}e_{zz} + c_{33}\gamma_n), & M_{ss} &= I(d_{11}\rho_{ss} + d_{12}\rho_{zz}), \\ M_{sz} &= M_{zs} = Ic_{66}(\rho_{sz} + \rho_{zs}), & M_{zz} &= I(d_{12}\rho_{ss} + d_{22}\rho_{zz}), & M_{sn} &= M_{zn} = 0. \end{aligned}$$

For this particular (flat) reference configuration, we shall present the solution of the Almansi–Michell problem in terms of the displacement field in a closed form. In other words, let us determine the equilibrium of the plate under the action of the body loads $\mathbf{f}(x_1)$, $\mathbf{l}(x_1)$, the tractions $\mathbf{N}^{(\gamma)}$ and director couples $\mathbf{M}^{(\gamma)}$ on the lateral edges $x_1 = \pm \frac{\bar{s}}{2}$, the resultant force $\mathcal{R}^0 = \mathcal{R}_i^0 \mathbf{e}_i$, the resultant moment $\mathcal{M}^0 = \mathcal{M}_i^0 \mathbf{e}_i$ acting on the end edge $x_3 = 0$. Here, $\mathbf{f} = f_i \mathbf{e}_i$ and $\mathbf{l} = l_i \mathbf{e}_i$ are arbitrary functions of x_1 , while $\mathbf{N}^{(\gamma)}$, $\mathbf{M}^{(\gamma)}$, \mathcal{R}^0 and \mathcal{M}^0 are given constant vectors. (The loads $\mathbf{N}^{(1)}$ and $\mathbf{M}^{(1)}$ act on the edge $x_1 = -\bar{s}/2$, and the loads $\mathbf{N}^{(2)}$ and $\mathbf{M}^{(2)}$ act on the edge $x_1 = +\bar{s}/2$.)

To find the solution of our Almansi–Michell problem, we follow the procedure described in Sec. 3 by Theorem 3. First, we notice that the displacement field $v\{\hat{a}\}$ corresponding to Cosserat plates is given by [12]

$$(5.2) \quad \begin{aligned} u_1 &= -\frac{1}{2} a_1 (Ax_1^2 + x_3^2) - a_3 Ax_1, & u_2 &= \frac{1}{2} a_2 (Ax_1^2 - x_3^2) + a_4 x_1 x_3, \\ u_3 &= (a_1 x_1 + a_3) x_3, \\ \delta_1 &= a_2 Ax_1 + a_4 x_3, & \delta_2 &= B(a_1 x_1 + a_3), & \delta_3 &= -a_2 x_3 + a_4 p(x_1), \end{aligned}$$

where we denote by $p(x_1)$ the function

$$(5.3) \quad p(x_1) = x_1 - \frac{2}{\vartheta} \cdot \frac{\sinh(\vartheta x_1)}{\cosh(\vartheta \bar{s}/2)}, \quad \text{with } \vartheta = \frac{2}{h} \sqrt{\frac{3c_{44}}{c_{66}}},$$

and the constants A , B , C are given by the expressions

$$A = \frac{c_{12}c_{33} - c_{13}c_{23}}{c_{11}c_{33} - (c_{13})^2}, \quad B = \frac{c_{11}c_{23} - c_{12}c_{13}}{c_{11}c_{33} - (c_{13})^2}, \quad C = \frac{\det(c_{ij})_{3 \times 3}}{c_{11}c_{33} - (c_{13})^2}.$$

Then, if we search the solution of our Almansi–Michell problem in the form (3.2) and we follow the steps presented in the proof of Theorem 3, we arrive at the solution $v = \{\mathbf{u}, \boldsymbol{\delta}\}$ given by

$$\begin{aligned}
 u_1 &= -\frac{1}{2} Ax_1^2 \left(\frac{1}{2} b_1 x_3^2 + c_1 x_3 + a_1 \right) - \frac{1}{2} x_3^2 \left(\frac{1}{12} b_1 x_3^2 + \frac{1}{3} c_1 x_3 + a_1 \right) \\
 &\quad - Ax_1 (c_3 x_3 + a_3) + \tilde{u}_1(x_1), \\
 u_2 &= \frac{1}{2} Ax_1^2 \left(\frac{1}{2} b_2 x_3^2 + c_2 x_3 + a_2 \right) - \frac{1}{2} x_3^2 \left(\frac{1}{12} b_2 x_3^2 + \frac{1}{3} c_2 x_3 + a_2 \right) \\
 &\quad + x_1 x_3 \left(\frac{1}{2} c_4 x_3 + a_4 \right) + \tilde{u}_2(x_1), \\
 (5.4) \quad u_3 &= \frac{1}{6} b_1 x_1 x_3 \left[x_3^2 + x_1^2 \left(A - \frac{C}{c_{66}} \right) + \frac{3C\bar{s}^2}{4c_{66}} \right] \\
 &\quad + \frac{1}{2} x_3^2 (c_1 x_1 + c_3) + x_3 (a_1 x_1 + a_3) + \tilde{u}_3(x_1), \\
 \delta_1 &= Ax_1 \left(\frac{1}{2} b_2 x_3^2 + c_2 x_3 + a_2 \right) + x_3 \left(\frac{1}{2} c_4 x_3 + a_4 \right) + \tilde{\delta}_1(x_1), \\
 \delta_2 &= Bx_1 \left(\frac{1}{2} b_1 x_3^2 + c_1 x_3 + a_1 \right) + B(c_3 x_3 + a_3) + \tilde{\delta}_2(x_1), \\
 \delta_3 &= -x_3 \left(\frac{1}{6} b_2 x_3^2 + \frac{1}{2} c_2 x_3 + a_2 \right) + b_2 x_3 q(x_1) + (c_4 x_3 + a_4) p(x_1) + \tilde{\delta}_3(x_1),
 \end{aligned}$$

where the function $q(x_1)$ is defined by

$$q(x_1) = \frac{1}{2} Ax_1^2 - \frac{A\bar{s}}{\vartheta} \cdot \frac{\cosh(\vartheta x_1)}{\sinh(\vartheta \bar{s}/2)} + \frac{1}{\vartheta^2} \left(2A - \frac{C}{c_{66}} \right).$$

The functions $\tilde{u}_i(x_1)$ and $\tilde{\delta}_i(x_1)$ which appear in (5.4) are known, and they can be computed directly by solving simple linear boundary-value problems for ordinary differential equations. More precisely, $\tilde{u}_1(x_1)$ and $\tilde{\delta}_2(x_1)$ can be found from the equations

$$\begin{aligned}
 (5.5) \quad &c_{11} \tilde{u}_1''(x_1) - c_{13} \tilde{\delta}_2'(x_1) \\
 &= -\frac{1}{h} f_1(x_1) - \frac{1}{2} b_1 \left[c_{12} Ax_1^2 - \frac{C}{c_{66}} (c_{12} + c_{66}) \left(x_1^2 - \frac{\bar{s}^2}{4} \right) \right], \\
 &c_{13} \tilde{u}_1(x_1) - c_{33} \tilde{\delta}_2(x_1) \\
 &= \frac{1}{h} l_2(x_1) + \frac{1}{2} b_1 c_{23} x_1 \left[\frac{1}{3} x_1^2 \left(A - \frac{C}{c_{66}} \right) + \frac{C}{4c_{66}} \bar{s}^2 \right], \\
 &(c_{11} \tilde{u}_1' - c_{13} \tilde{\delta}_2) \left(\pm \frac{\bar{s}}{2} \right) = \frac{1}{h} N_1^{(\gamma)} - \frac{1}{6} b_1 c_{12} \left(A + \frac{2C}{c_{66}} \right) \cdot \left(\pm \frac{\bar{s}}{2} \right)^3,
 \end{aligned}$$

and the functions $\tilde{u}_2(x_1)$ and $\tilde{\delta}_1(x_1)$ are determined by

$$\begin{aligned}
 & c_{55}(\tilde{u}_2''(x_1) - \tilde{\delta}_1'(x_1)) \\
 & \quad = -\frac{1}{h} f_2(x_1) + c_{44} \left[c_4(p(x_1) - x_1) + b_2 \left(q(x_1) - \frac{1}{2} Ax_1^2 \right) \right], \\
 (5.6) \quad & Id_{11} \tilde{\delta}_1''(x_1) + hc_{55}(\tilde{u}_2'(x_1) - \tilde{\delta}_1(x_1)) \\
 & \quad = -l_1(x_1) - I[(d_{12} + c_{66})(c_4 p'(x_1) + b_2 q'(x_1)) + c_{66}(c_4 + b_2 Ax_1)], \\
 & (\tilde{u}_2' - \tilde{\delta}_1) \left(\pm \frac{\bar{s}}{2} \right) = \frac{1}{hc_{55}} N_2^{(\gamma)}, \\
 & \tilde{\delta}_1' \left(\pm \frac{\bar{s}}{2} \right) = \frac{1}{Id_{11}} M_1^{(\gamma)} - A \left[c_4 p \left(\pm \frac{\bar{s}}{2} \right) + b_2 q \left(\pm \frac{\bar{s}}{2} \right) \right].
 \end{aligned}$$

The functions $\tilde{u}_3(x_1)$ and $\tilde{\delta}_3(x_1)$ satisfy the following uncoupled boundary-value problems which are easy to solve:

$$\begin{aligned}
 (5.7) \quad & \tilde{u}_3''(x_1) = -\frac{1}{hc_{66}} f_3(x_1) + \left(A - \frac{C}{c_{66}} \right) (c_1 x_1 + c_3), \\
 & \tilde{u}_3' \left(\pm \frac{\bar{s}}{2} \right) = \frac{1}{hc_{66}} N_3^{(\gamma)} + A \left[\frac{1}{2} c_1 \left(\pm \frac{\bar{s}}{2} \right)^2 + c_3 \left(\pm \frac{\bar{s}}{2} \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (5.8) \quad & \tilde{\delta}_3''(x_1) - \vartheta^2 \tilde{\delta}_3(x_1) = -\frac{1}{Ic_{66}} l_3(x_1) - c_2 \left(\frac{1}{2} \vartheta^2 Ax_1^2 + A - \frac{C}{c_{66}} \right), \\
 & \tilde{\delta}_3' \left(\pm \frac{\bar{s}}{2} \right) = \frac{1}{Ic_{66}} M_3^{(\gamma)} - c_2 A \cdot \left(\pm \frac{\bar{s}}{2} \right).
 \end{aligned}$$

The constants $b_1, b_2, c_1, \dots, c_4$ and a_1, \dots, a_4 which appear in the Eqs. (5.4)–(5.8) are expressed in terms of the loads applied, by the relations

$$\begin{aligned}
 (5.9) \quad & b_1 = \frac{-12}{h\bar{s}^3 C} \left(\int_{-\bar{s}/2}^{\bar{s}/2} f_1(x_1) dx_1 + N_1^{(2)} - N_1^{(1)} \right), \\
 & b_2 = \frac{-12}{h^3 \bar{s} C} \left(\int_{-\bar{s}/2}^{\bar{s}/2} f_2(x_1) dx_1 + N_2^{(2)} - N_2^{(1)} \right), \\
 & c_1 = -\frac{12}{h\bar{s}^3 C} \left[\mathcal{R}_1^0 + \int_{-\bar{s}/2}^{\bar{s}/2} x_1 f_3(x_1) dx_1 - \frac{\bar{s}}{2} (N_3^{(1)} + N_3^{(2)}) \right],
 \end{aligned}$$

$$\begin{aligned}
(5.9) \quad c_2 &= -\frac{12}{h^3 \bar{s} C} \left(\mathcal{R}_2^0 - \int_{-\bar{s}/2}^{\bar{s}/2} l_3(x_1) dx_1 + M_3^{(1)} - M_3^{(2)} \right), \\
c_3 &= \frac{-1}{h \bar{s} C} \left(\int_{-\bar{s}/2}^{\bar{s}/2} f_3(x_1) dx_1 + N_3^{(2)} - N_3^{(1)} \right), \\
c_4 &= -\frac{1}{D} \left[\int_{-\bar{s}/2}^{\bar{s}/2} (x_1 f_2(x_1) + l_1(x_1)) dx_1 \right. \\
&\quad \left. + \frac{\bar{s}}{2} (N_2^{(1)} + N_2^{(2)}) - (M_1^{(1)} - M_1^{(2)}) \right], \\
a_1 &= \frac{12}{h \bar{s}^3 C} \left(\mathcal{M}_2^0 - h \int_{-\bar{s}/2}^{\bar{s}/2} x_1 (c_{12} \tilde{u}'_1 - c_{23} \tilde{\delta}_2) dx_1 \right) \\
&\quad - \frac{\bar{s}^2 b_1 c_{22}}{40} \left(A + \frac{4C}{c_{66}} \right), \\
a_2 &= -\frac{12}{h^3 \bar{s} C} \left[\mathcal{M}_1^0 - Id_{11} \left(\tilde{\delta}_1 \left(\frac{\bar{s}}{2} \right) - \tilde{\delta}_1 \left(\frac{-\bar{s}}{2} \right) \right) \right. \\
&\quad \left. - \frac{I \bar{s} b_2 d_{22}}{12} \left(A \bar{s}^2 - \frac{h^2 C}{c_{44}} \right) \right], \\
a_3 &= \frac{-1}{h \bar{s} C} \left(\mathcal{R}_3^0 - h \int_{-\bar{s}/2}^{\bar{s}/2} (c_{12} \tilde{u}'_1 - c_{23} \tilde{\delta}_2) dx_1 \right), \\
a_4 &= -\frac{1}{D} \left[\mathcal{M}_3^0 - Ic_{66} \left(\vartheta^2 \int_{-\bar{s}/2}^{\bar{s}/2} x_1 \tilde{\delta}_3 dx_1 - \tilde{\delta}_3 \left(\frac{\bar{s}}{2} \right) + \tilde{\delta}_3 \left(\frac{-\bar{s}}{2} \right) \right) \right],
\end{aligned}$$

where D is the torsional rigidity in the Cosserat theory which is given by

$$(5.10) \quad D = \frac{h^3 c_{66}}{3} \left(\frac{\bar{s}}{2} + p \left(\frac{\bar{s}}{2} \right) \right) = \frac{h^3 \bar{s}}{3} c_{66} \left[1 - \frac{2}{\vartheta \bar{s}} \tanh \left(\frac{\vartheta \bar{s}}{2} \right) \right].$$

Thus, the displacement field (5.4) represents the general solution of the Almansi–Michell problem corresponding to orthotropic Cosserat plates. For comparison purposes, we shall consider now the deformation of three-dimensional plates, i.e. orthotropic rectangular parallelepipeds.

The problems of Almansi and Michell for three-dimensional cylinders have been considered in several books, e.g. [10, 11, 16]. Let us consider a cylindrical body which occupies the region $\{(x_1, x_2, x_3); -\bar{s}/2 < x_1 < \bar{s}/2, -h/2 < x_2 < h/2,$

$0 < x_3 < \bar{z}$ referred to the Cartesian coordinate frame $Ox_1x_2x_3$ (see Fig. 2). The body is subjected to resultant forces and moments acting on the bases $x_3 = 0, \bar{z}$, to the assigned body forces \mathbf{f}^* and to a prescribed stress vector field \mathbf{p} acting on the lateral boundaries $x_1 = \pm\bar{s}/2$ and $x_2 = \pm h/2$. We assume that the loads have the form

$$(5.11) \quad \mathbf{f}^* = f_3^*(x_1, x_2) \mathbf{e}_3, \quad \mathbf{p} = p_3(x_1, x_2) \mathbf{e}_3, \quad \mathcal{R}^0 = \mathcal{R}_i^0 \mathbf{e}_i, \quad \mathcal{M}^0 = \mathcal{M}_i^0 \mathbf{e}_i,$$

where \mathcal{R}^0 and \mathcal{M}^0 denote the resultant force and resultant moment about O , acting on the basis $x_3 = 0$. We observe that the fields \mathbf{f}^* and \mathbf{p} do not depend on x_3 and thus we have an Almansi–Michell problem. The solution of this problem for orthotropic and homogeneous three-dimensional cylinders is presented in [16], Sec. 4.8, in the form of the displacement field $\mathbf{u}^* = u_i^* \mathbf{e}_i$ given by the relations

$$(5.12) \quad \begin{aligned} u_1^* &= -\frac{1}{2} c_1^* x_3 \left(\frac{1}{3} x_3^2 + Ax_1^2 - Bx_2^2 \right) - \frac{1}{2} a_1^* (x_3^2 + Ax_1^2 - Bx_2^2) \\ &\quad - Ax_1 x_2 (c_2^* x_3 + a_2^*) - Ax_1 (c_3^* x_3 + a_3^*) - a_4^* x_2 x_3, \\ u_2^* &= -\frac{1}{2} c_2^* x_3 \left(\frac{1}{3} x_3^2 - Ax_1^2 + Bx_2^2 \right) - \frac{1}{2} a_2^* (x_3^2 - Ax_1^2 + Bx_2^2) \\ &\quad - Bx_1 x_2 (c_1^* x_3 + a_1^*) - Bx_2 (c_3^* x_3 + a_3^*) + a_4^* x_1 x_3, \\ u_3^* &= \frac{1}{2} x_3^2 (c_1^* x_1 + c_2^* x_2 + c_3^*) + x_3 (a_1^* x_1 + a_2^* x_2 + a_3^*) \\ &\quad + a_4^* \varphi(x_1, x_2) + \chi(x_1, x_2), \end{aligned}$$

This solution is exact in the context of a linear theory, and we shall use it for comparison with our solution for plates in the Cosserat approach. The constants c_i^* and a_k^* which appear in (5.12) are expressed in terms of the loads by

$$(5.13) \quad \begin{aligned} a_1^* &= \frac{12}{h\bar{s}^3 C} \mathcal{M}_2^0, & a_2^* &= -\frac{12}{h^3 \bar{s} C} \mathcal{M}_1^0, & a_3^* &= -\frac{1}{h\bar{s} C} \mathcal{R}_3^0, \\ c_1^* &= \frac{-12}{h\bar{s}^3 C} \left(\mathcal{R}_1^0 + \int_{\Sigma} x_1 f_3^* da + \int_{\partial\Sigma} x_1 p_3 dl \right), \\ c_2^* &= \frac{-12}{h^3 \bar{s} C} \left(\mathcal{R}_2^0 + \int_{\Sigma} x_2 f_3^* da + \int_{\partial\Sigma} x_2 p_3 dl \right), \\ c_3^* &= \frac{-1}{h\bar{s} C} \left(\int_{\Sigma} f_3^* da + \int_{\partial\Sigma} p_3 dl \right), \\ a_4^* &= \frac{-1}{D_0} \left(\mathcal{M}_3^0 + \int_{\Sigma} \left(c_{44} x_1 \frac{\partial \chi}{\partial x_2} - c_{66} x_2 \frac{\partial \chi}{\partial x_1} \right) da \right), \end{aligned}$$

where Σ denotes the rectangular cross-section $\Sigma = \{(x_1, x_2); -\bar{s}/2 < x_1 < \bar{s}/2, -h/2 < x_2 < h/2\}$ and $\partial\Sigma$ is its boundary. The function $\varphi(x_1, x_2)$ is the so-called torsion function which is given as the solution of the boundary-value problem

$$(5.14) \quad \begin{aligned} c_{66} \frac{\partial^2 \varphi}{\partial x_1^2} + c_{44} \frac{\partial^2 \varphi}{\partial x_2^2} &= 0 \quad \text{in } \Sigma, \\ \frac{\partial \varphi}{\partial x_1} &= x_2 \quad \text{for } x_1 = \pm \frac{\bar{s}}{2}, \quad \text{and} \quad \frac{\partial \varphi}{\partial x_2} = -x_1 \quad \text{for } x_2 = \pm \frac{h}{2}, \end{aligned}$$

while D_0 is the torsional rigidity of three-dimensional cylinders

$$(5.15) \quad D_0 = \int_{\Sigma} \left[c_{44} x_1 \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) - c_{66} x_2 \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) \right] da.$$

The function $\chi(x_1, x_2)$ appearing in (5.12) is determined by the boundary-value problem

$$(5.16) \quad \begin{aligned} c_{66} \frac{\partial^2 \chi}{\partial x_1^2} + c_{44} \frac{\partial^2 \chi}{\partial x_2^2} &= -f_3^* + (c_{66}A + c_{44}B - C)(c_1x_1 + c_2x_2 + c_3) \quad \text{in } \Sigma, \\ \frac{\partial \chi}{\partial x_1} &= Ax_1 \left(\frac{1}{2} c_1x_1 + c_2x_2 + c_3 \right) - \frac{1}{2} c_1 Bx_2^2 \pm \frac{p_3}{c_{66}} \quad \text{for } x_1 = \pm \frac{\bar{s}}{2}, \\ \frac{\partial \chi}{\partial x_2} &= Bx_2 \left(c_1x_1 + \frac{1}{2} c_2x_2 + c_3 \right) - \frac{1}{2} c_2 Ax_1^2 \pm \frac{p_3}{c_{44}} \quad \text{for } x_2 = \pm \frac{h}{2}. \end{aligned}$$

Let us compare now the solution (5.12)–(5.16) in the three-dimensional theory with the corresponding solution (5.4)–(5.10) for Cosserat plates, and to show a good agreement between the two approaches. In making the identification of the displacement fields in the two approaches, we employ the relations (see [2], p. 122)

$$(5.17) \quad \mathbf{u} = \frac{1}{h} \int_{-h/2}^{h/2} \mathbf{u}^* dx_2, \quad -\boldsymbol{\delta} = \frac{1}{h} \int_{-h/2}^{h/2} \frac{\partial \mathbf{u}^*}{\partial x_2} dx_2,$$

and for the two load systems we have the expressions

$$(5.18) \quad \begin{aligned} \mathbf{f} &= \int_{-h/2}^{h/2} \mathbf{f}^*(x_1, x_2) dx_2 + \left[\mathbf{p} \left(x_1, \frac{h}{2} \right) + \mathbf{p} \left(x_1, \frac{-h}{2} \right) \right], \\ \mathbf{N}^{(\gamma)} &= \int_{-h/2}^{h/2} \mathbf{p} \left(\pm \frac{\bar{s}}{2}, x_2 \right) dx_2, \end{aligned}$$

$$(5.18)_{\text{[cont.]}} \quad -\mathbf{1} = \int_{-h/2}^{h/2} x_2 \mathbf{f}^* dx_2 + \frac{h}{2} \left[\mathbf{p} \left(x_1, \frac{h}{2} \right) - \mathbf{p} \left(x_1, \frac{-h}{2} \right) \right],$$

$$-\mathbf{M}^{(\gamma)} = \int_{-h/2}^{h/2} x_2 \mathbf{p} \left(\pm \frac{\bar{s}}{2}, x_2 \right) dx_2.$$

The minus sign appearing in the left-hand sides of relations (5.17)₂ and (5.18)_{3,4} is due to our choice of coordinate axes and the inverse orientation $\mathbf{A}_3 = -\mathbf{e}_2$.

Using the special form of the load system (5.11) appearing in the relation (5.18), we obtain in this case that $f_\alpha = 0$, $l_\alpha = 0$, $N_\alpha^{(\gamma)} = 0$, $M_\alpha^{(\gamma)} = 0$, and from (5.9) and (5.5), (5.6) we deduce that $b_1 = b_2 = c_4 = 0$, and $\tilde{u}_\alpha = 0$, $\tilde{\delta}_\alpha = 0$. Consequently, by comparison between (5.9) and (5.13) we get

$$(5.19) \quad a_i = a_i^*, \quad c_i = c_i^*, \quad i = 1, 2, 3.$$

In view of (5.19), we can verify that the relations

$$u_\alpha = \frac{1}{h} \int_{-h/2}^{h/2} u_\alpha^* dx_2, \quad -\delta_\alpha = \frac{1}{h} \int_{-h/2}^{h/2} \frac{\partial u_\alpha^*}{\partial x_2} dx_2,$$

are exactly satisfied by the solutions (5.4) and (5.12) in the two different approaches (modulo a rigid displacement). According to (5.17), we still have to check the equalities

$$(5.20) \quad u_3 = \frac{1}{h} \int_{-h/2}^{h/2} u_3^* dx_2, \quad -\delta_3 = \frac{1}{h} \int_{-h/2}^{h/2} \frac{\partial u_3^*}{\partial x_2} dx_2, \quad \text{and} \quad a_4 = a_4^*,$$

for the complete agreement of the two solutions to hold. In what follows, we shall observe that the relations (5.20) are satisfied approximately, but nevertheless they are satisfied exactly in the thin plate limit, i.e. when $h \ll \bar{s}$. Indeed, in view of (5.4)_{3,6}, (5.9)₁₀, (5.12) and (5.13)₇, the relations (5.20) are verified approximately provided we have

$$(5.21) \quad \tilde{u}_3(x_1) \simeq \frac{I}{2h} c_1 B x_1 + \frac{1}{h} \int_{-h/2}^{h/2} \chi(x_1, x_2) dx_2,$$

$$\tilde{\delta}_3(x_1) \simeq \frac{I}{2h} c_2 B - \frac{1}{h} \left[\chi \left(x_1, \frac{h}{2} \right) - \chi \left(x_1, \frac{-h}{2} \right) \right],$$

$$\vartheta^2 \int_{-\bar{s}/2}^{\bar{s}/2} x_1 \tilde{\delta}_3(x_1) dx_1 - \tilde{\delta}_3 \left(\frac{\bar{s}}{2} \right) + \tilde{\delta}_3 \left(\frac{-\bar{s}}{2} \right) \simeq \int_{\Sigma} \left(c_{66} x_2 \frac{\partial \chi}{\partial x_1} - c_{44} x_1 \frac{\partial \chi}{\partial x_2} \right) da$$

and

$$(5.22) \quad p(x_1) \simeq -\frac{1}{h} \left[\varphi \left(x_1, \frac{h}{2} \right) - \varphi \left(x_1, \frac{-h}{2} \right) \right], \quad D \simeq D_0.$$

Let us mention that the good agreement (5.22) between the torsion functions and torsional rigidities in the two approaches has been proved in [2], Sec. 3.15 and 4.15, where it is shown that for a thickness range such that $\frac{h}{\bar{s}} \sqrt{\frac{c_{66}}{c_{44}}} < 2$, the value of D is very close to D_0 . The limit values of the quantities in (5.22) coincide as $\frac{h}{\bar{s}} \rightarrow 0$, and they are given by

$$(5.23) \quad p(x_1) = -\frac{1}{h} \left[\varphi \left(x_1, \frac{h}{2} \right) - \varphi \left(x_1, \frac{-h}{2} \right) \right] = x_1, \quad D = D_0 = \frac{h^3 \bar{s}}{3} c_{66}.$$

Finally, for verifying the relations (5.21), we have to solve the boundary-value problems (5.7), (5.8) and (5.16). In this purpose, let us confine ourself to a simpler case when the load functions (5.11) are given by the expression

$$(5.24) \quad f_3^*(x_1, x_2) = g_1 x_1 + g_2 x_2 + g_3, \quad p_3(x_1, x_2) = p_3^0,$$

where g_i and p_3^0 are constants. Then, the functions $\tilde{u}_3(x_1)$, $\tilde{\delta}_3(x_1)$ can be calculated from the ordinary differential equations (5.7), (5.8), and we obtain the following expressions in the thin plate limit:

$$(5.25) \quad \begin{aligned} \tilde{u}_3(x_1) &= \frac{1}{2} A x_1^2 \left(\frac{1}{3} c_1 x_1 + c_3 \right) + \frac{2\mathcal{R}_1^0}{h\bar{s}^3 c_{66}} x_1 \left(x_1^2 - \frac{3\bar{s}^2}{4} \right) + \frac{p_3^0}{\bar{s}c_{66}} x_1^2, \\ \tilde{\delta}_3(x_1) &= \frac{1}{2} c_2 A x_1^2, \quad \text{for } h \ll \bar{s}. \end{aligned}$$

Taking into account (5.24), the boundary-value problem (5.16) can be solved by means of the Fourier series expansions. We obtain the solution

$$(5.26) \quad \begin{aligned} \chi(x_1, x_2) &= \frac{1}{2} A x_1^2 \left(\frac{1}{3} c_1 x_1 - c_2 x_2 + c_3 \right) \\ &\quad + \frac{1}{2} B x_2^2 \left(-c_1 x_1 + \frac{1}{3} c_2 x_2 + c_3 \right) + p_3^0 \left(\frac{x_1^2}{\bar{s}c_{66}} + \frac{x_2^2}{hc_{44}} \right) \\ &\quad + \sum_{n=0}^{+\infty} \left[H_{2n+1}(x_1) \sin \frac{(2n+1)\pi x_2}{h} + G_{2n+1}(x_2) \sin \frac{(2n+1)\pi x_1}{\bar{s}} \right], \end{aligned}$$

denoting by $\eta = \sqrt{c_{66}/c_{44}}$ and

$$\begin{aligned} H_{2n+1}(x_1) &= \frac{(-1)^n \cdot 4h^2}{(2n+1)^3 \pi^3} \left[\frac{h(g_2 + c_2 C - 2c_2 c_{66} A)}{(2n+1)\pi c_{44}} + \bar{s} c_2 A \eta \frac{\cosh((2n+1)\pi \eta x_1/h)}{\sinh((2n+1)\pi \eta \bar{s}/2h)} \right], \end{aligned}$$

$$\begin{aligned} G_{2n+1}(x_2) &= \frac{(-1)^n \cdot 4\bar{s}^2}{(2n+1)^3 \pi^3} \left[\frac{\bar{s}(g_1 + c_1 C - 2c_1 c_{44} B)}{(2n+1)\pi c_{66}} + \frac{h c_1 B}{\eta} \frac{\cosh((2n+1)\pi x_2/\eta \bar{s})}{\sinh((2n+1)\pi h/2\eta \bar{s})} \right]. \end{aligned}$$

By virtue of (5.25) and (5.26) we observe that the relations (5.21) are satisfied exactly in the limit as $h/\bar{s} \rightarrow 0$.

In conclusion, the solution of our problem in the theory of Cosserat plates is in a very good agreement with the exact three-dimensional solution for thin plates and moreover, the solutions in the two approaches coincide in the thin plate limit, i.e. when $h/\bar{s} \rightarrow 0$.

To recapitulate, the solution of the Almansi–Michell problem for very thin orthotropic plates ($h \ll \bar{s}$) with the load system (5.11), (5.24) is given by (from (5.4), (5.23), (5.25))

$$\begin{aligned} u_1 &= -\frac{1}{2} c_1 x_3 \left(A x_1^2 + \frac{1}{3} x_3^2 \right) - \frac{1}{2} a_1 (A x_1^2 + x_3^2) - A x_1 (c_3 x_3 + a_3), \\ u_2 &= \frac{1}{2} c_2 x_3 \left(A x_1^2 - \frac{1}{3} x_3^2 \right) + \frac{1}{2} a_2 (A x_1^2 - x_3^2) + a_4 x_1 x_3, \\ u_3 &= \frac{1}{2} A x_1^2 \left(\frac{1}{3} c_1 x_1 + c_3 \right) + \frac{1}{2} x_3^2 (c_1 x_1 + c_3) + x_3 (a_1 x_1 + a_3) \\ &\quad + \frac{2\mathcal{R}_1^0}{h\bar{s}^3 c_{66}} x_1 \left(x_1^2 - \frac{3\bar{s}^2}{4} \right) + \frac{p_3^0}{\bar{s} c_{66}} x_1^2, \end{aligned}$$

$$\delta_1 = A x_1 (c_2 x_3 + a_2) + a_4 x_3,$$

$$\delta_2 = B x_1 (c_1 x_3 + a_1) + B (c_3 x_3 + a_3),$$

$$\delta_3 = \frac{1}{2} c_2 A x_1^2 - x_3 \left(\frac{1}{2} c_2 x_3 + a_2 \right) + a_4 x_1,$$

where the constants c_i and a_k are expressed in terms of the loads by

$$\begin{aligned} a_1 &= \frac{12}{h\bar{s}^3 C} \mathcal{M}_2^0, \quad a_2 = -\frac{12}{h^3 \bar{s} C} \mathcal{M}_1^0, \quad a_3 = -\frac{1}{h\bar{s} C} \mathcal{R}_3^0, \quad a_4 = -\frac{3}{h^3 \bar{s} c_{66}} \mathcal{M}_3^0, \\ c_1 &= -\frac{1}{C} \left(g_1 + \frac{12}{h\bar{s}^3} \mathcal{R}_1^0 \right), \quad c_2 = -\frac{1}{C} \left(g_2 + \frac{12}{h^3 \bar{s}} \mathcal{R}_2^0 \right), \quad c_3 = -\frac{1}{C} \left(g_3 + 2p_3^0 \frac{h+\bar{s}}{h\bar{s}} \right). \end{aligned}$$

From the above comparison we observe that the theory of Cosserat surfaces produce very good results for thin bodies, and the derivation of solutions is much simpler than that in the three-dimensional theory, since we reduce the problem to ordinary differential equations instead of two-dimensional boundary-value problems. Moreover, in the case of curved shells the advantage of the Cosserat approach is even more visible, because it can easily handle problems in which the axes of orthotropy vary along the circumference of the cylindrical thin body, as can be seen from the example in Sec. 4.

Appendix A

The cross-section plane problem

We recall that the solution of the relaxed Saint–Venant’s problem for anisotropic three-dimensional cylinders reduce to the solution of some generalized plane strain problems associated with the cross-section of the cylinder (see e.g. [9]). As a counterpart of these generalized plane strain problems we consider, in the case of cylindrical shells, the following problem (called the *cross-section plane problem*): find the displacement field $v(s) = (\mathbf{u}(s), \boldsymbol{\delta}(s))$ which depends only on s , and which satisfies the equations

$$(A.1) \quad \mathcal{A}(v(s)) = -\mathcal{F}(s),$$

and the boundary conditions:

(i) for open cylindrical shells

$$(A.2) \quad \mathbf{N}(v(s_\gamma)) = \mathbf{N}^{(\gamma)}, \quad \mathbf{M}(v(s_\gamma)) = \mathbf{M}^{(\gamma)} \quad \text{on } L_\gamma \quad (\gamma = 1, 2),$$

(ii) for closed cylindrical shells

$$(A.3) \quad v(s_1) = v(s_2), \quad v'(s_1) = v'(s_2),$$

where $\mathcal{F}(s) = (f_s, f_z, f_n, l_s, l_z, l_n)$ represents the prescribed body loads, while the external loads on the lateral edges $\mathbf{N}^{(\gamma)}$ and $\mathbf{M}^{(\gamma)}$ are given constants.

According to Theorem 2 from [12], the necessary and sufficient conditions for the existence of the solution $v(s)$ to the cross-section plane problem (A.1)–(A.3) are the following:

$$(A.4) \quad \int_{\mathcal{C}_0} \mathbf{f} \, dl + (1 - \varepsilon) (\mathbf{N}^{(1)} + \mathbf{N}^{(2)}) = \mathbf{0},$$

$$\left[\int_{\mathcal{C}_0} (\mathbf{R} \times \mathbf{f} + \mathbf{D} \times \mathbf{l}) \, dl + (1 - \varepsilon) \sum_{\gamma=1}^2 (\mathbf{R}^{(\gamma)}(0) \times \mathbf{N}^{(\gamma)} + \mathbf{D}^{(\gamma)} \times \mathbf{M}^{(\gamma)}) \right] \cdot \mathbf{e}_3 = 0,$$

where ε takes the values $\varepsilon = 0$ for open cylindrical shells and $\varepsilon = 1$ for closed shells.

The solution of the problem (A.1)–(A.3) can be determined as in Sec. 4 of [12], provided the conditions (A.4) are satisfied.

The displacement fields $v^{(k)}$

Let us define the fields $v^{(k)}$, $k = 1, 2, 3, 4$, which are employed to construct the solution of the relaxed Saint–Venant’s problem for cylindrical Cosserat shells.

First, we introduce four displacement fields denoted by $v_c^{(k)}$, $k = 1, \dots, 4$, and given by

$$(A.5) \quad \begin{aligned} v_c^{(\alpha)} &= \left(-\frac{1}{2} z^2 \mathbf{e}_\alpha + z x_\alpha \mathbf{e}_3, z \epsilon_{\alpha\beta} x'_\beta \mathbf{e}_3 \right), & v_c^{(3)} &= (z \mathbf{e}_3, \mathbf{0}), \\ v_c^{(4)} &= (-z \epsilon_{\alpha\beta} x_\beta \mathbf{e}_\alpha, z x'_\alpha \mathbf{e}_\alpha). \end{aligned}$$

We observe that the strain measures corresponding to the fields (A.5) are independent of the axial coordinate x_3 and that $\partial v_c^{(k)}/\partial x_3$ is a rigid displacement field.

For each $k = 1, \dots, 4$, let us consider the cross-section plane problem (A.1)–(A.3) for the given data

$$(A.6) \quad \mathcal{F}(s) = \mathcal{A}(v_c^{(k)}), \quad \mathbf{N}^{(\gamma)} = -\mathbf{N}(v_c^{(k)}(s_\gamma)), \quad \mathbf{M}^{(\gamma)} = -\mathbf{M}(v_c^{(k)}(s_\gamma)), \quad \gamma = 1, 2.$$

We can verify that the fields (A.6) satisfy the conditions (A.4), so that the problem (A.1)–(A.3) with the system of external loads (A.6) admits a solution, denoted by $w^{(k)}(s)$.

Then, we define the displacement fields

$$(A.7) \quad v^{(k)} = v_c^{(k)} + w^{(k)}(s), \quad k = 1, 2, 3, 4.$$

Let us remark that the fields $v^{(k)}$ have the following properties (see [12], Sec. 5):

$$v^{(k)} \in \mathcal{D}_0 \quad \text{and} \quad \mathcal{R}_\alpha(v^{(k)}) = 0, \quad k = 1, \dots, 4, \quad \alpha = 1, 2.$$

The field $w(s)$ and the constants \hat{c}

For the sake of completeness, we present here the expressions for the field $w(s)$ and the constants \hat{c} which appear in the solution (2.23) of the flexure problem (P_2). The displacement field $w(s)$ is determined by solving the cross-section plane problem (A.1)–(A.3) for the external loads given by (see Theorem 5 of [12])

$$\mathcal{F}(s) = \mathcal{A} \left(\int_0^{x_3} v \{\hat{b}\} dx_3 \right), \quad \mathbf{N}^{(\gamma)} = -\mathbf{N} \left(\int_0^{x_3} v \{\hat{b}\} dx_3 \right) (s_\gamma),$$

$$\mathbf{M}^{(\gamma)} = -\mathbf{M} \left(\int_0^{x_3} v \{\hat{b}\} dx_3 \right) (s_\gamma), \quad \gamma = 1, 2.$$

Then, the constants $\hat{c} = (c_1, c_2, c_3, c_4)$ are given by the system of equations

$$\left(\sum_{r=1}^4 D_{kr} c_r \right)_{k=1, \dots, 4} = (-\mathcal{M}_2(\hat{w}), \mathcal{M}_1(\hat{w}), \mathcal{R}_3(\hat{w}), \mathcal{M}_3(\hat{w})),$$

where we denote by $\hat{w}(s) = w(s) + \int_0^{x_3} v \{\hat{b}\} dx_3$. In this manner, we have specified precisely the solution v^F of the flexure problem, introduced by Theorem 1.

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