Arch. Mech., **60**, 5, pp. 385–401, Warszawa 2008
SIXTY YEARS OF THE ARCHIVES OF MECHANICS

Some exact solutions for the rotational flow of a generalized second-grade fluid between two circular cylinders

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The velocity field and the associated tangential stress corresponding to the flow of a generalized second-grade fluid between two infinite coaxial circular cylinders, are determined by means of the Laplace and Hankel transforms. At time $t = 0$, the fluid is at rest and at $t = 0^+$ the cylinders suddenly begin to rotate about their common axis with a constant angular acceleration. The solutions that have been obtained satisfy the governing differential equations and all the imposed initial and boundary conditions. The similar solutions for a second-grade fluid and Newtonian fluid are recovered from our general solutions. The influence of the fractional coefficient on the velocity of the fluid is also analyzed by graphical illustrations.

Key words: generalized second-grade fluid, velocity field, shear stress, fractional calculus, Hankel and Laplace transforms.

1. Introduction

In many fields, such as food industry, drilling operations, polymer chemical industry and bio-engineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particles, oils, red cells and other long-chain molecules. Generally, the viscosity function varies non-linearly with the shear rate and the elasticity is felt through elongational effects and time-dependent effects. In these cases, the fluids have been treated as viscoelastic fluids. Because of the difficulty to suggest a single model, which exhibits all properties of viscoelastic fluids, they cannot be described as simply as Newtonian fluids. For this reason, many models or constitutive equations have been proposed and most of them are empirical or semi-empirical.

The second-grade fluids are the common, non-Newtonian viscoelastic fluids in industrial fields, such as polymer solutions. The ordinary linear constitutive model for a second-grade fluid has the following form:

$$\tau(t) = \mu\varepsilon(t) + E\frac{d\varepsilon(t)}{dt}, \tag{1.1}$$

where τ is the stress, ε is the strain, μ is the viscosity coefficient and E is the viscoelasticity constant. This mathematical model provides a reasonable qualitative description, however, it is not satisfactory from a quantitative viewpoint (see Caputo and Mainardi [1]).

Several authors [2–4] suggested that integral-order models for viscoelastic materials seem to be inadequate from both the qualitative and quantitative point of view. At the same time, they proposed fractional-order laws of deformation for modelling the viscoelastic behavior of real materials.

Caputo and Mainardi [1] formulated the following fractional-order model:

$$\sigma(t) + a\frac{d^\alpha\sigma}{dt^\alpha} = m\varepsilon(t) + b\frac{d^\alpha\varepsilon}{dt^\alpha}; \qquad 0 < \alpha \le 1, \tag{1.2}$$

where α, a, m and b are constants which depend on the nature of material. This model includes the classical law when $\alpha = 1$, $a = 0$, $m = \mu$ and $b = E$. Bagley and Torvik [5, 6] and Rogers [7] have shown that law (1.2) is very useful for modeling of most viscoelastic materials. In addition to experimental findings, they proved that the four-parameter model (1.2) seems to be satisfactory for most real materials.

Bagley and Torvik [5, 6], Koeller [8], Xu and Tan [9, 10] proposed the fractional derivative approach to viscoelasticity in order to describe the properties of numerous viscoelastic materials. They suggest the general form of the model as

$$\sigma(t) = E_0\varepsilon(t) + E_1 D_t^\beta[\varepsilon(t)], \tag{1.3}$$

where $D_t^\beta[\varepsilon(t)]$ is defined by [11]

$$D_t^\beta[\varepsilon(t)] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{\varepsilon(\tau)}{(t-\tau)^\beta}d\tau; \qquad 0 < \beta < 1, \tag{1.4}$$

where $\Gamma(\cdot)$ is the Gamma function.

According to the molecular theory for dilute polymer solutions due to Rouse [12], the stress is

$$\sigma(t) = \mu_s D_t^1[\varepsilon(t)] + \left[\frac{3}{2}(\mu_0 - \mu_s)nkT\right]^{1/2} D_t^{1/2}[\varepsilon(t)], \tag{1.5}$$

where n is the number of molecules per unit volume of the polymer solution, k is the Boltzmann constant, T is the absolute temperature, μ_s is the steady-flow viscosity of the solvent in the solution and μ_0 is the steady-flow viscosity of the solution. Thus, the Rouse theory provides us the presence of fractional derivative along with the first derivative of classical viscoelasticity in the relation between stress and strain for some polymers. Ferry *et al.* [13] modified the Rouse theory in concentrated polymer solutions and polymer solids with no cross-linking and obtained that

$$\sigma(t) = \left(\frac{3\mu\rho RT}{2M}\right)^{1/2} D_t^{1/2}[\varepsilon(t)], \tag{1.6}$$

where M is the molecular weight, ρ is the density, μ is the viscosity and R is the universal gas constant.

Thus, the fractional calculus approach to viscoelasticity for the study of viscoelastic material properties is justified, at least for polymer solutions and for polymer solids without cross-linking.

For a second-grade fluid, the Cauchy stress tensor $\mathbf{T}$ is given by the constitutive equation [14, 15]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.7}$$

where p is the pressure, $\mathbf{I}$ is the unit tensor, μ is the dynamic viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined by

$$\mathbf{A}_1 = \mathrm{grad}\mathbf{v} + (\mathrm{grad}\mathbf{v})^T, \tag{1.8}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\mathrm{grad}\mathbf{v}) + (\mathrm{grad}\mathbf{v})^T\mathbf{A}_1, \tag{1.9}$$

where d/dt denotes the material time derivative, $\mathbf{v}$ is the velocity field and grad is the gradient operator.

If the second-grade fluid given by Eq. (1.7) is compatible with thermodynamics, then the material moduli must meet the following restrictions [16]:

$$\mu \geq 0, \qquad \alpha_1 \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 = 0. \tag{1.10}$$

For a generalized second-grade fluid, Eq. (1.7) still holds but $\mathbf{A}_2$ is defined as follows [5, 18–20]

$$\mathbf{A}_2 = D_t^{\beta}\mathbf{A}_1 + \mathbf{A}_1(\mathrm{grad}\mathbf{v}) + (\mathrm{grad}\mathbf{v})^T\mathbf{A}_1, \tag{1.11}$$

and D_t^{β} is the Riemann–Liouville fractional calculus operator defined by (1.4). For $\beta = 1$, we have $D_t^1 f(t) = df(t)/dt$ and hence Eq. (1.11) is reduced to Eq. (1.9).

In this paper, we study the motion of a generalized second-grade fluid between two infinite concentric circular cylinders, both cylinders are rotating around their common axis ($r = 0$), with constant angular accelerations. We have obtained the velocity field and the resulting shear stress by means of the Laplace and Hankel transforms.

Making $\beta = 1$, respectively, $\beta = 1$ and $\alpha_1 = 0$ in our general solutions, we obtain the velocity field and the resulting shear stress corresponding to the flow of second-grade fluids, respectively, Newtonian fluids, performing the same motions. By using the graphical illustrations, we have studied the effect of fractional derivative on the velocity field.

2. Rotational flow between concentric cylinders

Let us consider an incompressible second-grade fluid at rest in an annular region between two straight circular cylinders of radii R_1 and R_2 ($> R_1$), as shown in Fig. 1. At time $t = 0^+$, both cylinders suddenly begin to rotate about their common axis, with constant angular accelerations. Owing to the shear, the fluid is gradually moved and its velocity in cylindrical coordinates (r, θ, z) is given by [15, 20, 21]

$$\mathbf{v} = \mathbf{v}(r,t) = \omega(r,t)\mathbf{e}_\theta, \tag{2.1}$$

where $\mathbf{e}_\theta$ is the transverse unit vector. For these flows, the constraint of incompressibility is automatically satisfied.

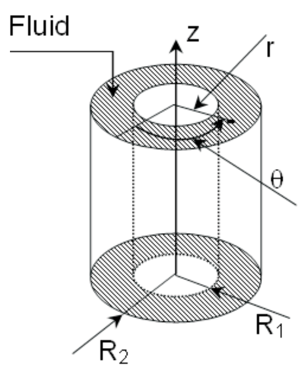

Fig. 1. Flow geometry.

Based on the above suppositions, the constitutive equation of generalized second-grade fluid, corresponding to this motion is

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{2.2}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress which is different from zero. In absence of body forces and a pressure gradient in the axial direction, the balance of linear momentum leads to the following equation:

$$\rho\frac{\partial\omega(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t), \tag{2.3}$$

where ρ is the constant density of the fluid.

Eliminating τ from Eqs. (2.2) and (2.3), we get the governing equation

$$\frac{\partial\omega(r,t)}{\partial t} = (\nu + \alpha D_t^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t); \quad r \in (R_1, R_2),\ t > 0, \tag{2.4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, and $\alpha = \alpha_1/\rho$ is the fractional viscoelastic constant. Consequently, the velocity field corresponding to this motion does not depend upon the material module α_2.

We consider the initial and boundary conditions

$$\omega(r,0) = 0, \tag{2.5}$$

$$\omega(R_1,t) = R_1\Omega_1 t, \qquad \omega(R_2,t) = R_2\Omega_2 t \quad \text{for } t > 0. \tag{2.6}$$

To solve this problem, we shall use as in [21, 22] the Laplace and Hankel transforms.

2.1. Calculation of the velocity field

Applying the Laplace transform to Eqs. (2.4) and (2.6) and using the Eq. (2.5), we obtain the following ordinary differential equation [17]:

$$(\nu + \alpha q^{\beta})\left[\frac{\partial^2\overline{\omega}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\omega}(r,q)}{\partial r} - \frac{\overline{\omega}(r,q)}{r^2}\right] - q\overline{\omega}(r,q) = 0, \tag{2.7}$$

where the image function $\overline{\omega}(r,q) = \int_0^\infty \omega(r,t)e^{-qt}dt$ of $\omega(r,t)$ has to satisfy the conditions

$$\overline{\omega}(R_1,q) = \frac{R_1\Omega_1}{q^2}, \qquad \overline{\omega}(R_2,q) = \frac{R_2\Omega_2}{q^2}, \tag{2.8}$$

where q is the transform parameter.

We denote by $\overline{\omega}_H(r_n,q) = \int_{R_1}^{R_2} r\overline{\omega}(r,q)B_1(rr_n)dr$ the Hankel transform of the function $\overline{\omega}(r,q)$, where

$$B_1(rr_n) = J_1(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_1(rr_n),$$

and r_n are the positive roots of the transcendental equation $B_1(R_1 r) = 0$, and $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of order one of the first and second kind. Applying the Hankel transform to Eq. (2.7), taking into account the conditions (2.8) and using the following relations:

$$\frac{d}{dr}[B_1(rr_n)] = r_n[J_0(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_0(rr_n)] - \frac{1}{r}B_1(rr_n) \tag{2.9}$$

and

$$J_0(z)Y_1(z) - J_1(z)Y_0(z) = -\frac{2}{\pi z}; \tag{2.10}$$

now we find that

$$(\nu+\alpha q^{\beta})\left\{\frac{2[R_2\Omega_2J_1(R_1r_n) - R_1\Omega_1J_1(R_2r_n)]}{\pi q^2 J_1(R_1r_n)} - r_n^2\overline{\omega}_H(r_n,q)\right\} - q\overline{\omega}_H(r_n,q) = 0,$$

or equivalently

$$\overline{\omega}_H(r_n,q) = \frac{2[R_2\Omega_2J_1(R_1r_n) - R_1\Omega_1J_1(R_2r_n)]}{\pi J_1(R_1r_n)}\frac{\nu+\alpha q^{\beta}}{q^2[q+\alpha r_n^2 q^{\beta}+\nu r_n^2]}. \tag{2.11}$$

Equation (2.11) can be written in the following equivalent form:

$$\overline{\omega}_H(r_n,q) = \overline{\omega}_{1H}(r_n,q) + \overline{\omega}_{2H}(r_n,q), \tag{2.12}$$

where

$$\overline{\omega}_{1H}(r_n,q) = \frac{2}{\pi r_n^2}\frac{R_2\Omega_2J_1(R_1r_n) - R_1\Omega_1J_1(R_2r_n)}{J_1(R_1r_n)}\frac{1}{q^2}, \tag{2.13}$$

and

$$\overline{\omega}_{2H}(r_n,q) = -\frac{2}{\pi r_n^2}\frac{R_2\Omega_2J_1(R_1r_n) - R_1\Omega_1J_1(R_2r_n)}{J_1(R_1r_n)} \times \frac{1}{q[q+\alpha r_n^2 q^{\beta}+\nu r_n^2]}. \tag{2.14}$$

The inverse Hankel transforms of the functions $\overline{\omega}_{1H}$ and $\overline{\omega}_{2H}$ are

$$\overline{\omega}_1(r,q) = \frac{\Omega_1R_1^2(R_2^2-r^2) + \Omega_2R_2^2(r^2-R_1^2)}{(R_2^2-R_1^2)r}\frac{1}{q^2}, \tag{2.15}$$

and

$$\overline{\omega}_2(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}\overline{\omega}_{2H}(r_n,q). \tag{2.16}$$

Now, we find that the function $\overline{\omega}(r,q)$ has the form

$$\overline{\omega}(r,q)=\frac{\Omega_1R_1^2(R_2^2-r^2)+\Omega_2R_2^2(r^2-R_1^2)}{(R_2^2-R_1^2)r}\frac{1}{q^2} -\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\frac{1}{q[q+\alpha r_n^2q^\beta+\nu r_n^2]}. \tag{2.17}$$

We introduce the notation

$$F(q)=\frac{1}{q[q+\alpha r_n^2q^\beta+\nu r_n^2]}, \tag{2.18}$$

and rewrite Eq. (2.18) in the equivalent form

$$F(q)=\frac{q^{-1-\beta}}{(q^{1-\beta}+\alpha r_n^2)+\nu r_n^2q^{-\beta}}=\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{q^{-1-\beta-k\beta}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}. \tag{2.19}$$

In order to determine the inverse Laplace transform of the function $\overline{\omega}(r,q)$, we will use the following formulae [23]:

$$L^{-1}\left\{\frac{1}{q^a}\right\}=\frac{t^{a-1}}{\Gamma(a)};\qquad a>0,$$

$$L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=G_{a,b,c}(d,t) =\sum_{j=0}^{\infty}\frac{\Gamma(c+j)d^j}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]},\qquad Re(ac-b)>0.$$

So we find that the velocity field $\omega(r,t)$ has the following form:

$$\omega(r,t)=\frac{\Omega_1R_1^2(R_2^2-r^2)+\Omega_2R_2^2(r^2-R_1^2)}{(R_2^2-R_1^2)r}t -\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)} \times\sum_{k=0}^{\infty}(-\nu r_n^2)^kG_{1-\beta,-1-\beta-k\beta,k+1}(-\alpha r_n^2,t), \tag{2.20}$$

or, equivalently

$$\begin{aligned}\omega(r,t) &= \frac{\Omega_1 R_1^2(R_2^2-r^2)+\Omega_2 R_2^2(r^2-R_1^2)}{(R_2^2-R_1^2)r}t\\ &\quad -\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\\ &\quad \times\sum_{j,k=0}^{\infty}\frac{(-\nu r_n^2)^k(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{t^{(1-\beta)j+k+1}}{\Gamma[(1-\beta)j+k+2]}.\end{aligned}\tag{2.21}$$

2.2. Calculation of the shear stress

The shear stress $\tau(r,t)$ is obtained from Eqs. (2.2) and (2.17). Applying the Laplace transform to Eq. (2.2) we find

$$\overline{\tau}(r,q)=(\mu+\alpha_1q^\beta)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{\omega}(r,q).\tag{2.22}$$

Now, differentiating Eq. (2.17) with respect to r and replacing the values of $\partial\overline{\omega}(r,q)/\partial r$ and that of $\overline{\omega}(r,q)$ itself in Eq. (2.22), we get

$$\begin{aligned}\overline{\tau}(r,q) &= \frac{2R_1^2R_2^2(\Omega_2-\Omega_1)}{(R_2^2-R_1^2)r^2}\left(\mu\frac{1}{q^2}+\alpha_1\frac{1}{q^{2-\beta}}\right)\\ &\quad +\pi\sum_{n=1}^{\infty}\left[\frac{2}{r}B_1(rr_n)-r_nB(rr_n)\right]\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\\ &\quad \times\sum_{j,k=0}^{\infty}\frac{(-\nu r_n^2)^k(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\left[\mu\frac{1}{q^{(1-\beta)j+k+2}}+\alpha_1\frac{1}{q^{(1-\beta)j+k+2-\beta}}\right],\end{aligned}\tag{2.23}$$

where

$$B(rr_n)=J_0(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_0(rr_n).$$

Applying inverse Laplace transform to the image function $\overline{\tau}(r,q)$, we find the shear stress

$$\begin{aligned}\tau(r,t) &= \frac{2R_1^2R_2^2(\Omega_2-\Omega_1)}{(R_2^2-R_1^2)r^2}\left(\mu t+\frac{\alpha_1t^{1-\beta}}{\Gamma(2-\beta)}\right)\\ &\quad +\pi\sum_{n=1}^{\infty}\left[\frac{2}{r}B_1(rr_n)-r_nB(rr_n)\right]\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\\ &\quad \times\sum_{j,k=0}^{\infty}\frac{(-\nu r_n^2)^k(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\left[\mu\frac{t^{(1-\beta)j+k+1}}{\Gamma[(1-\beta)j+k+2]}+\alpha_1\frac{t^{(1-\beta)j+k+1-\beta}}{\Gamma[(1-\beta)j+k+2-\beta]}\right].\end{aligned}\tag{2.24}$$

3. Limiting case $(\beta = 1)$

Assuming $\beta = 1$ in Eq. (2.20) we obtain the velocity field

$$\begin{aligned}(3.1)\qquad \omega(r,t) &= \frac{\Omega_1 R_1^2(R_2^2 - r^2) + \Omega_2 R_2^2(r^2 - R_1^2)}{(R_2^2 - R_1^2)r}t \\ &- \pi \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n)[R_2\Omega_2 J_1(R_1 r_n) - R_1\Omega_1 J_1(R_2 r_n)]B_1(R_1 r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \\ &\times \sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-2-k,k+1}(-\alpha r_n^2, t),\end{aligned}$$

corresponding to an ordinary second-grade fluid, performing the same motion. Similarly, from (2.24) we obtain the associated shear stress

$$\begin{aligned}(3.2)\qquad \tau(r,t) &= \frac{2R_1^2 R_2^2(\Omega_2 - \Omega_1)}{(R_2^2 - R_1^2)r^2}(\mu t + \alpha_1) \\ &+ \pi \sum_{n=1}^{\infty}\left[\frac{2}{r}B_1(r r_n) - r_n B(r r_n)\right]\frac{J_1(R_1 r_n)[R_2\Omega_2 J_1(R_1 r_n) - R_1\Omega_1 J_1(R_2 r_n)]}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \\ &\times \sum_{j,k=0}^{\infty}\frac{(-\nu r_n^2)^k(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\left[\mu\frac{t^{k+1}}{\Gamma(k+2)} + \alpha_1\frac{t^k}{\Gamma(k+1)}\right].\end{aligned}$$

The above relations can be simplified if we use the following relations:

$$\begin{aligned}\sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-2-k,k+1}(-\alpha r_n^2, t) &= \sum_{k=0}^{\infty}(-\nu r_n^2)^k \sum_{j=0}^{\infty}\frac{(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{t^{k+1}}{\Gamma(k+2)} \\ = \sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k t^{k+1}}{\Gamma(k+2)}\frac{1}{(1+\alpha r_n^2)^{k+1}} &= -\frac{1}{\nu r_n^2}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)^{k+1} \\ &= \frac{1}{\nu r_n^2}\left[1 - \exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right].\end{aligned}$$

As a result, we find that the velocity field (3.1) has the form

$$\begin{aligned}(3.3)\qquad \omega(r,t) &= \frac{\Omega_1 R_1^2(R_2^2 - r^2) + \Omega_2 R_2^2(r^2 - R_1^2)}{(R_2^2 - R_1^2)r}t \\ &-\frac{\pi}{\nu}\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)[R_2\Omega_2 J_1(R_1 r_n) - R_1\Omega_1 J_1(R_2 r_n)]}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}\frac{B_1(r r_n)}{r_n^2}\left[1 - \exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right],\end{aligned}$$

and the shear stress (3.2) has the form

$$\tau(r,t)=\frac{2R_1^2R_2^2(\Omega_2-\Omega_1)}{(R_2^2-R_1^2)r^2}(\mu t+\alpha_1)+\pi\sum_{n=1}^{\infty}\left[\frac{2}{r}B_1(rr_n)-r_nB(rr_n)\right]\times\frac{J_1(R_1r_n)[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times\left\{\frac{\mu}{\nu r_n^2}\left[1-\exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)\right]+\frac{\alpha_1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)\right\}. \tag{3.4}$$

Equations (3.3) and (3.4) are identical with those obtained by Fetecau *et al.* [20, Eqs. (3.12) and (3.16) for $\lambda\to 0$]. Making $\alpha\to 0$ in Eqs. (3.3) and (3.4), the similar solutions corresponding to the Newtonian fluid, performing the same motion, are recovered. Making $\Omega_1=0$ and $\Omega_2=\Omega$ or $\Omega_1=\Omega$ and $\Omega_2=0$ in Eqs. (2.20) and (2.24), we obtain the velocity field and the adequate shear stress corresponding to the flow between two cylinders, one of them being at rest.

4. Conclusion and numerical results

In this paper we have established exact solutions for the velocity field and shear stress, corresponding to the flow of a generalized second-grade fluid between two concentric circular cylinders. The motion is produced by the two cylinders which at time $t=0^+$ begin to rotate around their common axis with angular velocities $\Omega_1 t$ and $\Omega_2 t$. The solutions, obtained by means of Laplace and Hankel transforms, are presented, in integral and series forms, in terms of the generalized G-function, and satisfy all the imposed initial and boundary conditions. For $\beta=1$ or $\beta=1$ and $\alpha=0$, similar solutions for the ordinary second-grade fluids, respectively, Newtonian fluids are recovered. The velocity field and adequate shear stress corresponding to the flow between two cylinders, one of them being at rest, are obtained as particular cases of our general solutions. Assuming $\Omega_1=0$ and $\Omega_2=\Omega$ in Eqs. (2.20), for instance, we obtain the velocity field

$$\omega(r,t)=\frac{\Omega R_2^2(r^2-R_1^2)}{(R_2^2-R_1^2)r}t-\pi R_2\Omega\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\times\sum_{j,k=0}^{\infty}\frac{(-\nu r_n^2)^k(-\alpha r_n^2)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{t^{(1-\beta)j+k+1}}{\Gamma[(1-\beta)j+k+2]}, \tag{4.1}$$

corresponding to the flow between cylinders, the inner cylinder being at rest.

Finally, the numerical results are given to illustrate the influence of the fractional parameter β on the velocity $\omega(r,t)$. In all figures we considered

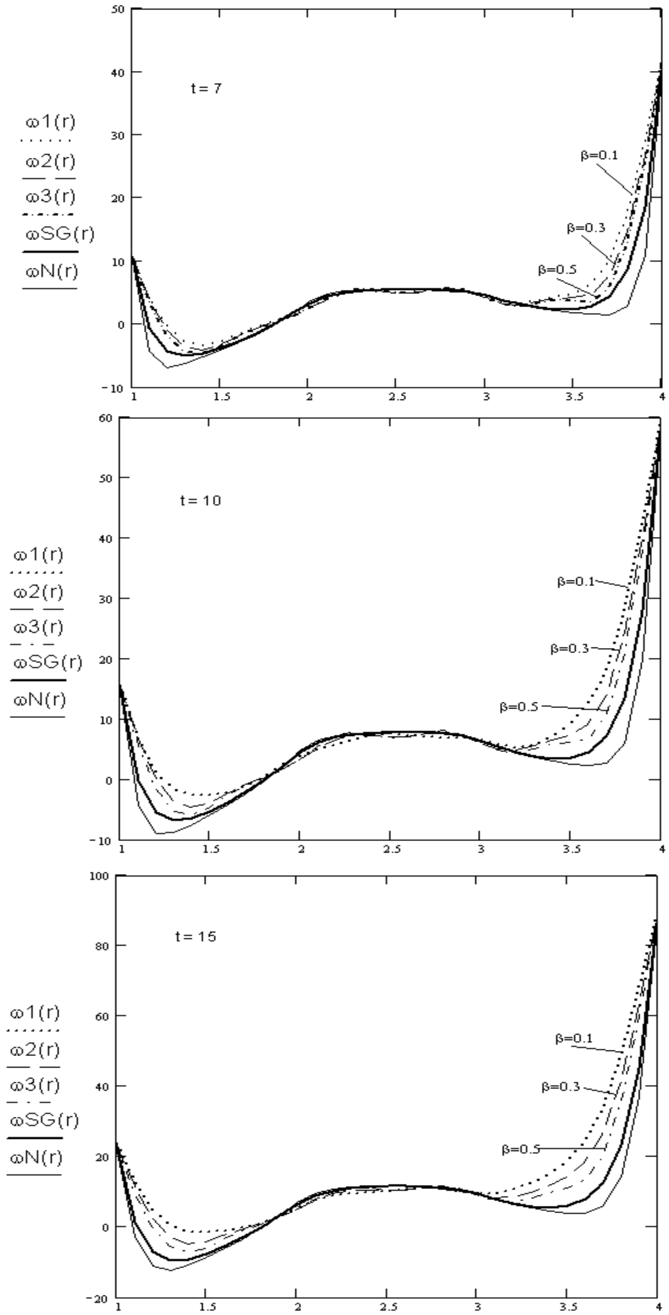

FIG. 2. Velocity profiles $\omega(r)$ for different values of the fractional coefficient β.

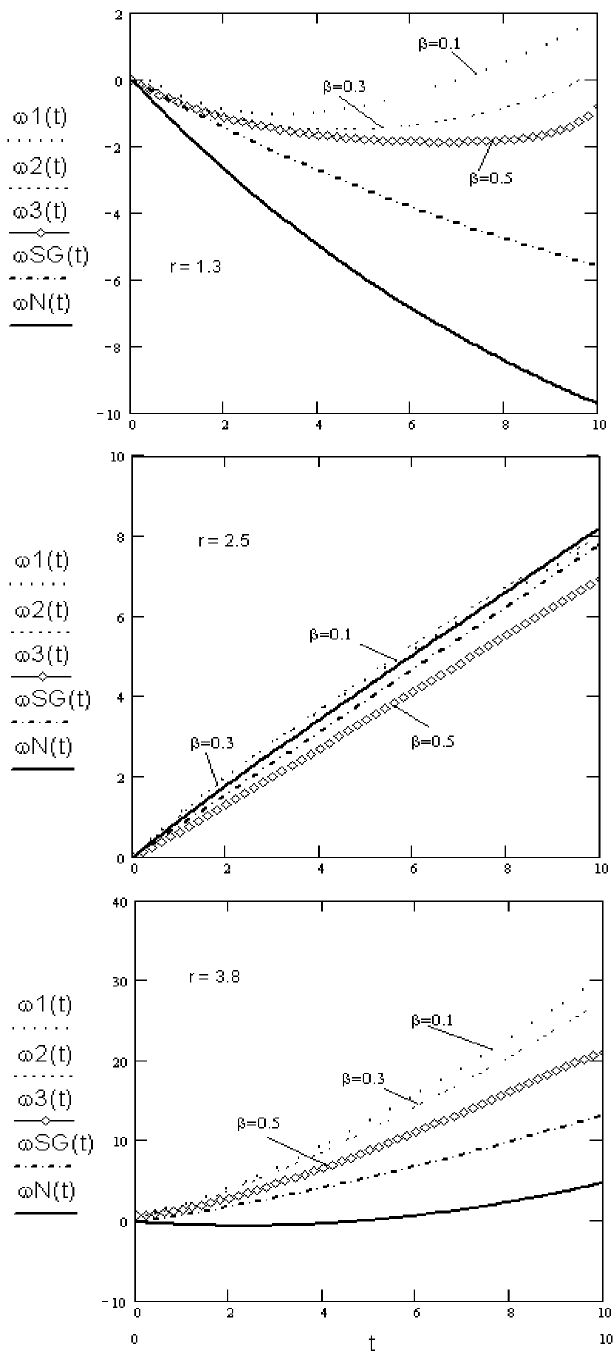

Fig. 3. Time variation of the velocity.

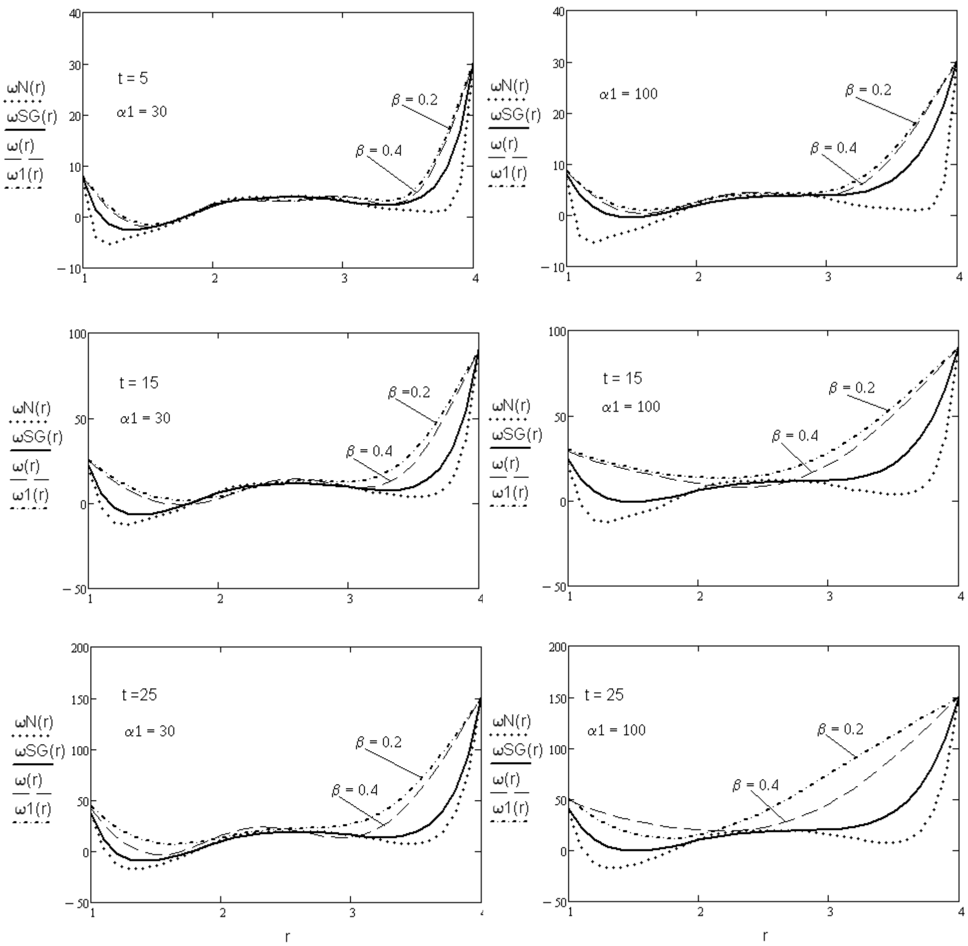

FIG. 4. Velocity profiles for different values of parameters $\alpha 1$ and β.

$R_1 = 1$, $R_2 = 4$, $\Omega_1 = 3$, $\Omega_2 = 1.5$. The profiles of the velocity corresponding to the motion of the Newtonian fluid (curve ωN(r)), second-grade fluid (curve $\omega\mathrm{SG}(r)$)) and generalized second-grade fluid (curves $\omega(r)$, $\omega1(r)$, $\omega2(r)$ and $\omega3(r)$), are plotted for different values of time t and fractional coefficient β.

In Fig. 2, with $\rho = 1260$, $\alpha_1 = 11.34$ and $\mu = 1.48$, the profiles of the velocity are plotted for fractional coefficient $\beta \in \{0.1, 0.3, 0.5\}$. It is clear from these figures that the velocity of the fluid increases when the fractional coefficient decreases. Moreover, in these cases, the influence of β is stronger near the boundary of the domain. The generalized second-grade fluid flows faster than

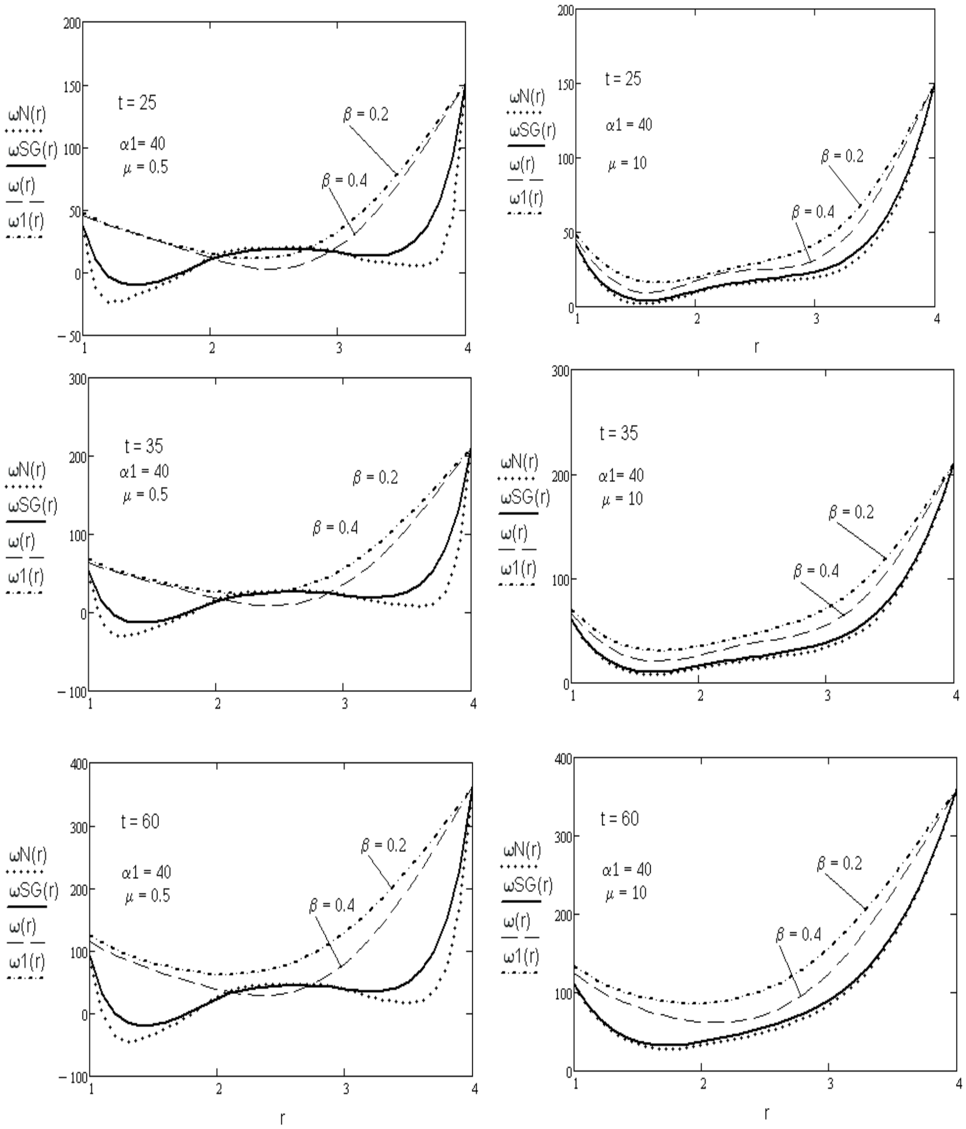

Fig. 5. Velocity profiles for different values of coefficients μ and β.

the second-grade and Newtonian fluids. Figure 3 depicts the histories of the velocity field $\omega(r,t)$ at the positions $r = \{1.3, 2.5, 3.8\}$ for $t \in [0,10]$ and the same values of the fractional coefficient which are used in Fig. 2.

Figure 4 is drawn for $\rho = 1260$, $\mu = 1.48$, $\alpha_1 \in \{30, 100\}$ and $\beta \in \{0.2, 0.4\}$. The influence of the fractional coefficient on the velocity is not modified, the

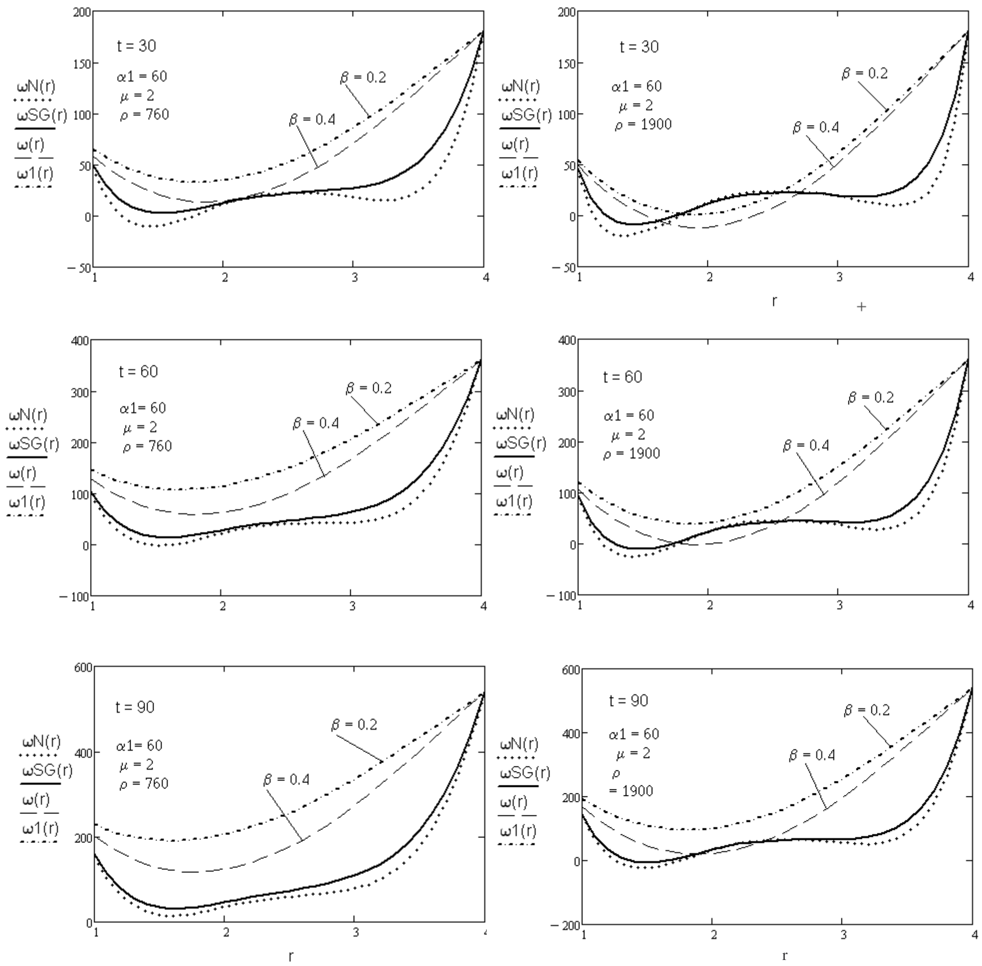

Fig. 6. Velocity profiles for different values of coefficients ρ and β.

characteristics of the flow being similar to those of the previous cases. However, we note that the modification in the value of material constant α_1 leads to a significant modification in the velocity value. The second-grade fluid with fractional derivative flows much faster than the second-grade one and this difference is significant at higher values of the time t.

In Fig. 5, we consider $\rho = 1260$, $\alpha_1 = 60$, $\mu \in \{0.5, 10\}$ and $\beta \in \{0.2, 0.4\}$. It is easy to see that the three fluids maintain the previous flow properties. It is important to note that the variation of the material constant μ has a significant influence on the flow speed. For higher values of μ, speeds differ significantly from the point of view of their values. Again, these differences are larger for higher values of the time t.

Figure 6 is drawn for $\alpha_1 = 60$, $\mu = 2$, $\rho \in \{760, 1900\}$ and $\beta \in \{0.2, 0.4\}$. The flow maintains the above general character it should be mentioned that, in case of lower values of the density ρ, velocity increases when the fractional coefficient decreases and the differences are higher for higher values of t. The units of the parameters in Figs. 2–6 are SI units and the roots r_n have been approximated by $n\pi/(R2 - R1)$ [24].

Acknowledgements

We would like to thank to referees for their useful comments and suggestions regarding an earlier version of this paper. The author Georgiana Bolat acknowledges the support from the Ministry of Education and Research, CNCSIS, through PN II-Ideas, Grant No. 26/28-09-2007, Cod CNCSIS ID_593.

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Received April 28, 2008; revised version September 1, 2008.