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SIXTY YEARS OF THE ARCHIVES OF MECHANICS

Qualitative aspects of solutions in resonators

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We consider the system of micro-beam resonators in the thermoelastic theory of Lord and Shulmann. First, we prove the uniqueness and instability of solutions when the sign of a parameter is not prescribed. Existence of solutions and uniform bounds for the real part of the spectrum have been found. We finish the paper by proving the impossibility of the time localization of solutions.

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1. Introduction

It is well known that the usual theory of heat conduction based on Fourier's law predicts infinite speed of heat propagation. Heat transmission at low temperature has been observed to propagate by means of waves. These aspects have caused intense activity in the field of heat propagation. Extensive reviews on the so-called second-sound theories (hyperbolic heat conduction) are given in Chandrasekharaiah [1] and in the books of Müller and Ruggeri [10] and Jou *et al.* [5].

Instead of Fourier's law and leading to the classical hyperbolic-parabolic system of thermoelasticity together with the physical paradox of infinite propagation speed through the heat conduction part, we consider the model proposed by Lord and Shulmann [9]. Hetnarski and Ignaczak consider it using the nonclassical approach of thermoelasticity in their review [4]. Some mathe-

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matical results concerning alternative thermoelastic theories can be found in [14, 15, 16, 17, 18, 19, 20]. In [23] the thermoelastic damping in micro-beam resonators is considered in the case when the Lord and Shulmann thermoelastic theory is applied. The model that we consider here involves a system of two coupled partial differential equations. It is a coupling of the plate equation with heat conduction, the latter appearing in a hyperbolic model.

The system of Lord and Shulman has been studied before and, for example, the exponential stability has been obtained for bounded reference configurations as well as the nonlinear stability near the equilibrium, see [21, 22], for a coupling of classical elasticity with the hyperbolic heat conduction model (Cattaneo's law).

For the coupling of the plate equation with the classical heat-flow equation, i.e. heat conduction, is modeled by Fourier's law, see e.g. [11, 6, 12, 7, 8, 2].

Here we think of an isotropic and homogeneous thermoelastic material in $\mathbb{R}^3 \ni (x,y,z)$, which occupies a plate of thickness "h"; that is, $-h/2 \leq z \leq h/2$. If we consider the through-thickness displacement

$$u(x,y,t) := h^{-1} \int_{-h/2}^{h/2} u_3(x,y,z,t)dz, \tag{1.1}$$

we know that it satisfies the equation

$$D\triangle^2 u + d\alpha(1+\nu)\triangle\theta + \rho h u_{,tt} = 0, \tag{1.2}$$

where $D = (1-\nu^2)^{-1}EI$ is the bending stiffness, E is Young's modulus, ν is the Poisson ratio, $I = h^3/12$, α is the coefficient of thermal expansion and θ is the first moment of temperature which is defined as

$$\theta(x,y,t) = I^{-1} \int_{-h/2}^{h/2} z\eta(x,y,z,t)dz, \tag{1.3}$$

and which satisfies the equation

$$c(\theta_{,t} + \tau\theta_{,tt}) - K\triangle\theta + \frac{h}{I}K\theta - \frac{\alpha T E}{1-2\nu}\triangle(u_{,t} + \tau u_{,tt}) = 0, \tag{1.4}$$

where $\eta(x,y,z,t)$ is the temperature, τ is the relaxation parameter for the Cattaneo constitutive equation for the heat flux vector, K is the thermal conductivity and c is the heat capacity.

It seems that this system of equations has not been derived before, cf. [23], for the one-dimensional case. However, in view of the references [13, 23], this system corresponds to the field equations which govern the thermoelastic coupling for a plate in the context of the Lord-Shulman theory. We here will deal with the thermoelastic damping which is considered to be the significant loss mechanism in micro-scale resonators. In this sense we point out that a good review of the relevance of the thermomechanical damping in resonators can be found in [3].

In this paper we study four kinds of problem. One is to prove the uniqueness and the instability of solutions when we assume very relaxed conditions on the coefficients that determine the problem. Second problem is to determine a suitable frame where the thermoealstic problem in micro-beam resonators is well-posed. Third is to investigate the exponential stability of the solutions and the fourth one is to prove the impossibility of localization of solutions.

This paper is organized as follows: in Section 2 we set down the field equations and the boundary and initial conditions of the problem we consider in this paper. The uniqueness and instability result is proved in Section 3. In Section 4 we prove the existence result. In Section 5 we prove for the case of bounded reference configurations that the spectrum of the governing differential operator lies strictly in the left complex half-plane. The last Section 6 is devoted to the proof of impossibility of the localization of solutions.

2. Preliminaries

We consider the system which governs the micro-beam resonators in dimensionless form for the Lord-Shulman theory of thermoelasticity. The system of equations is (see [23] for the one-dimensional case)

$$a\Delta^2 u + \Delta\theta + \ddot{u} = F, \tag{2.1}$$

$$\Delta\theta - m\theta + d\Delta\dot{\hat{u}} = c\dot{\hat{\theta}} + G, \tag{2.2}$$

where

$$\hat{f} = f + \tau\dot{f}. \tag{2.3}$$

In this system we assume that m, τ, c and d are positive. In the next section we do not require the positivity of the parameter a, but it will be imposed in later sections. F and G are external supply terms like the external force or heat supply.

From now on, we consider a bounded reference configuration $B \subset \mathbb{R}^n$, the boundary of which fits the requirements of the divergence theorem. The space dimension $n \geq 2$ may be arbitrary but $n = 2$ corresponds to the model outlined in the Introduction.

In this paper we study solutions $(u,\theta) = (u(\mathbf{x},t),\theta(\mathbf{x},t))$, $x \in B$, $t \geq 0$. We study the qualitative behavior of classical solutions subject to the initial conditions

$$u(\mathbf{x},0) = u^0(\mathbf{x}), \quad \dot{u}(\mathbf{x},0) = v^0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \dot{\theta}(\mathbf{x},0) = \vartheta^0(\mathbf{x}), \tag{2.4}$$

and the boundary conditions

$$u(\mathbf{x},t) = \Delta u(\mathbf{x},t) = \theta(\mathbf{x},t) = 0, \;\; \mathbf{x} \in \partial B \;\times\; [0,\infty), \tag{2.5}$$

or

$$u(\mathbf{x},t) = \nabla u(\mathbf{x},t).\mathbf{n}(\mathbf{x}) = \theta(\mathbf{x},t) = 0, \;\; \mathbf{x} \in \partial B \;\times\; [0,\infty). \tag{2.6}$$

Standard notation is used, e.g. for the Laplace operator Δ, the gradient ∇, or the Sobolev spaces H^m and H_0^m.

3. Uniqueness and instability

In this section we present the problem of uniqueness and growth of the solutions of the system (2.1), (2.2) subject to the initial conditions (2.4) and the boundary conditions (2.5) or (2.6). It is worth noting that in this section, we assume that d and c are positive, but we do not impose any condition on a.

To obtain the uniqueness result, it is sufficient to prove that the only solution of the problem determined by the homogeneous version of the system (2.1), (2.2)

$$a\Delta^2 u + \Delta\theta + \ddot{u} \,=\, 0, \tag{3.1}$$

$$\Delta\theta - m\theta + d\Delta\dot{\hat{u}} \,=\, c\dot{\hat{\theta}}, \tag{3.2}$$

with homogeneous boundary conditions (2.5) or (2.6) and initial homogeneous conditions, is the null solution. The key is to define a suitable functional to which the logarithmic convexity is applicable. In this situation the energy equation gives

$$\begin{aligned} E(t) \,&\equiv \int_B \Big(d|\dot{\hat{u}}|^2 + da|\Delta\hat{u}|^2 + c\hat{\theta}^2 + \tau(|\nabla\theta|^2 + m\theta^2) + 2\int_0^t (|\nabla\theta|^2 + m\theta^2)ds\Big)dV \\ &\equiv E(0) \quad (= 0). \end{aligned} \tag{3.3}$$

We now define the new functional

$$G(t) = \int_B \big(d|\hat{u}|^2 + \tau(|\nabla\eta|^2 + m\eta^2) + \int_0^t (|\nabla\eta|^2 + m\eta^2)ds\big)dV, \tag{3.4}$$

where

$$\eta(t,\mathbf{x}) := \int_0^t \theta(s,\mathbf{x})ds. \tag{3.5}$$

Differentiating Eq. (3.4) we see that

$$G'(t) = 2\int_B \big(d\hat{u}\dot{\hat{u}} + \tau(\nabla\eta\nabla\theta + m\eta\theta) + \frac{1}{2}(|\nabla\eta|^2 + m\eta^2)\big)dV, \tag{3.6}$$

$$\begin{aligned} G''(t) &= 2\int_B \Big(|d\dot{\hat{u}}|^2 + \tau(|\nabla\theta|^2 + m\theta^2)\Big)dV \\ &+ 2\int_B \big(d\ddot{\hat{u}}\hat{u} + \tau(\nabla\eta.\nabla\dot{\theta} + m\eta\dot{\theta}) + (\nabla\eta.\nabla\theta + m\eta\theta)\big)dV. \end{aligned} \tag{3.7}$$

We also see that

$$\int_B (d\ddot{\hat{u}}\hat{u} + ad|\Delta\hat{u}|^2)dV = -\int_B d\Delta\hat{\theta}\hat{u}dV, \tag{3.8}$$

and

$$\int_B \big(c(\hat{\theta})^2 + \tau(\nabla\eta\nabla\dot{\theta} + m\eta\dot{\theta}) + (\nabla\eta\nabla\theta + m\eta\theta)\big)dV = \int_B d\Delta\hat{\theta}\hat{u}dV. \tag{3.9}$$

Now using (3.8) and (3.9) in (3.7), we derive

$$G''(t) = 2\int_B \big(d(|\dot{\hat{u}}|^2 + \tau(|\nabla\theta|^2 + m\theta^2)\big)dV - 2\int_B \big(a|\Delta\hat{u}|^2 + c(\hat{\theta})^2\big)dV. \tag{3.10}$$

In view of the energy equation (3.3) we have

$$G''(t) = 4\int_B \big(d|\dot{\hat{u}}|^2 + \tau(|\nabla\theta|^2 + m\theta^2)\big)dV + 4\int_B\int_0^t (|\nabla\theta|^2 + m\theta^2)dsdV. \tag{3.11}$$

Hence

$$G''G - (G')^2 \geq 0, \tag{3.12}$$

where we have used the Cauchy–Schwarz inequality.

Inequality (3.12) implies that $t \mapsto \ln G(t)$ is a convex function of t and then

$$G(t) \leq \big[G(0)\big]^{1-t/T}\big[G(T)\big]^{t/T}. \tag{3.13}$$

It then follows that $G(t) \equiv 0$ on the interval $[0,T]$ and from (3.4) $\hat{u} \equiv 0$ in $B \times [0,T]$. In view of the initial conditions, we also obtain that $u \equiv 0$. So θ satisfies Equation (3.2) without the $d\dot{\hat{u}}$-term. It implies that $\theta \equiv 0$ in $B \times [0,T]$ and the uniqueness is shown.

Now, we give growth estimates for some solutions of the problem determined by the system (3.1), (3.2), boundary conditions (2.5) or (2.6) and initial conditions (2.4). The key is again to find a suitable functional to which logarithmic convexity is applicable. To this end, a modification of (3.4) is necessary. We take again η as in (3.5). However, due to non-zero initial conditions we have:

$$c\dot{\hat{\eta}} - d\Delta\hat{u} - [c\theta^0 + c\tau\vartheta^0 - d\Delta u^0 - d\tau\Delta v^0] = \Delta\eta - m\eta. \tag{3.14}$$

The data terms are incorporated into the equation by defining $Q(\mathbf{x})$ to be a solution to the equation:

$$\Delta Q - mQ = [c\theta^0 + cb\vartheta^0 - d\Delta u^0 - db\Delta v^0], \tag{3.15}$$

subject to the homogeneous boundary conditions

$$Q(\mathbf{x}) = 0, \mathbf{x} \in \partial B. \tag{3.16}$$

The existence of Q is guaranteed by the existing results for elliptic equations. Now, we define

$$\beta := \eta + Q, \tag{3.17}$$

and (3.14) becomes

$$c\dot{\hat{\beta}} - d\Delta\hat{u} = \Delta\beta - m\beta. \tag{3.18}$$

Basing on (3.4) we now define the functional

$$G_{\omega,t_0}(t) = G_{0,0}(t) + \omega(t+t_0)^2, \tag{3.19}$$

where ω and t_0 are positive constants to be selected, and

$$G_{0,0}(t) = \int_B \Big(d|\hat{u}|^2 + \tau(|\nabla\beta|^2 + m\beta^2) + \int_0^t (|\nabla\beta|^2 + m\beta^2)ds\Big)dV. \tag{3.20}$$

In this situation, we also obtain (3.8), but (3.9) becomes

$$\int_B \big(c(\hat{\theta})^2 + \tau(\nabla\beta\nabla\dot{\theta} + m\beta\dot{\theta}) + (\nabla\beta\nabla\theta + m\beta\theta)\big)dV = \int_B d\Delta\hat{\theta}\hat{u}dV. \tag{3.21}$$

One also derives the formula for the energy:

$$\begin{aligned} E(t) &\equiv \int_B \Big(d|\dot{\hat{u}}|^2 + ad|\Delta\hat{u}|^2 + c\hat{\theta}^2 + \tau(|\nabla\theta|^2 + m\theta^2) + 2\int_0^t (|\nabla\theta|^2 + m\theta^2)d\tau\Big)dV \\ &\equiv E(0). \end{aligned} \tag{3.22}$$

By differentiating $G(t)$ and using (3.8),(3.20) and the energy equation (3.21), it is not difficult to see that

$$\begin{aligned} G''_{\omega,t_0}(t) = 4\int_B \Big(d|\dot{\hat{u}}|^2 + \tau(|\nabla\theta|^2 + m\theta^2)\Big)dV \\ + 4\int_B\int_0^t (|\nabla\theta|^2 + m\theta^2)d\tau dV - 2(2E(0) + \omega). \end{aligned} \tag{3.23}$$

Cauchy–Schwarz's inequality implies that

$$G''_{\omega,t_0}G_{\omega,t_0} - \left(G'_{\omega,t_0} - \frac{\nu}{2}\right)^2 \geq 0, \tag{3.24}$$

if

$$\omega = -2E(0), \tag{3.25}$$

and

$$\nu = 2\int_B \left(|\nabla Q|^2 + mQ^2\right) dV. \tag{3.26}$$

If we take t_0 such that $G'_{\omega,t_0}(0) > \nu$, it may be proved that

$$G_{\omega,t_0}(t) \geq \frac{G_{\omega,t_0}(0)G'_{\omega,t_0}(0)}{G'_{\omega,t_0}(0) - \nu} \exp\left(\frac{G'_{\omega,t_0}(0) - \nu}{G(0)}\right)t - \frac{\nu G_{\omega,t_0}(0)}{G'_{\omega,t_0}(0) - \nu}. \tag{3.27}$$

Thus, the function $G_{0,0}(t)$ satisfies the inequality

$$\begin{aligned} G_{0,0}(t) \geq \frac{G_{\omega,t_0}(0)G'_{\omega,t_0}(0)}{G'_{\omega,t_0}(0) - \nu} \exp\left(\frac{G'_{\omega,t_0}(0) - \nu}{G(0)}\right)t \\ - \frac{\nu G_{\omega,t_0}(0)}{G'_{\omega,t_0}(0) - \nu} - \omega(t + t_0)^2. \end{aligned} \tag{3.28}$$

Theorem 1. *Let* (u,θ) *be a solution of the initial-boundary-value problem determined by Eqs.* (3.1), (3.2), (2.4) *and* (2.5) *or* (2.6)*, such that the initial conditions satisfy the condition* $E(0)<0$*. Then, as time increases, the function* $G_{0,0}$ *grows exponentially.*

4. Well-posedness

In this section we prove existence of solutions of the problem determined by the system (2.1), (2.2), the initial conditions (2.4) and the boundary conditions (2.6)

The well-posedness result for the system can be achieved by an appropriate choice of variables and spaces which reflect the spatial structure of the system.

For the transformation to a first-order system that finally will be characterized by a semigroup, we apply the differential operator " ˆ " from (2.3) to the differential Equation (2.1) and obtain (now $a>0$)

$$a\Delta^2\hat{u}+\Delta\hat{\theta}+\hat{\ddot{u}}=\hat{F}. \tag{4.1}$$

We remark that finding of a solution $(\hat{u},\theta)$ allows to determine the desired solutions (u,θ) of the original system.

Defining

$$\mathbf{V}:=(\hat{u},\hat{u}_t,\theta,\theta_t)'$$

we obtain

$$\mathbf{V}_t=A\mathbf{V}+\mathbf{F},\qquad V(0)=V^0, \tag{4.2}$$

with the (yet formal) differential operator A given by the symbol

$$A_f:=\begin{pmatrix} 0 & 1 & 0 & 0\\ a\Delta^2 & 0 & -\Delta & -\tau\Delta\\ 0 & 0 & 0 & 1\\ 0 & \dfrac{d}{c\tau}\Delta & \dfrac{1}{c\tau}(\Delta-m) & -\dfrac{1}{\tau}\end{pmatrix},$$

the right-hand side $\mathbf{F}$ given by

$$\mathbf{F}:=(0,\hat{F},0,G)'$$

and the initial value

$$\mathbf{V}^0(\mathbf{x}):=(\hat{u},\hat{u}_t,\theta,\theta_t)'(\mathbf{x},0),$$

with its components being given in terms of the originally prescribed initial data by using the differential equations.

As the underlying Hilbert space we choose

$$\mathcal{H} := (H_0^2(B))^n \times (L^2(B))^n \times H_0^1(B) \times L^2(B)$$

with inner product

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} := \big(d\langle V^2, W^2\rangle + ad\langle \Delta V^1, \Delta W^1\rangle\big)\\ + \tau(\langle \nabla V^3, \nabla W^3\rangle + \tau m\langle V^3, W^3\rangle + c\langle V^3 + \tau V^4, W^3 + \tau W^4\rangle,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual $L^2(B)$-inner product. The operator A is now given as

$$A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}, \qquad AV := A_f V,$$

with

$$D(A) := \{V \in \mathcal{H} \mid V^2 \in H_0^2(B)^n,\ V^4 \in H_0^1(B),\ A_f V \in \mathcal{H}\}.$$

The operator is obviously densely defined and dissipative, i.e.

$$\forall\, V \in D(A): \quad \operatorname{Re}\langle AV, V\rangle_{\mathcal{H}} \leq 0.$$

The latter follows since we have chosen the setting with the inner product just in a way that we have

$$\langle AV, V\rangle_{\mathcal{H}} = -\langle \nabla V^3, \nabla V^3\rangle - m\langle V^3, V^3\rangle. \tag{4.3}$$

As a consequence we also see that the operator A is invertible.

LEMMA 2. 0 belongs to the resolvent set $\varrho(A)$, and A^{-1} is compact.

P r o o f. The solvability of $AV = F$ is equivalent to solving

$$V^2 = F^1, \tag{4.4}$$

$$-a\Delta^2 V^1 - \Delta V^3 - \tau\nabla V^4 = F^2, \tag{4.5}$$

$$V^4 = F^3, \tag{4.6}$$

$$\frac{d}{\tau c}\Delta V^2 + \frac{1}{\tau c}(\Delta - m)V^3 - \frac{1}{\tau}V^4 = F^4. \tag{4.7}$$

Eliminating V^2 and V^4, we have to solve

$$-a\Delta^2 V^1 - \Delta V^3 = F^2 + \tau\Delta F^3, \tag{4.8}$$

$$\frac{1}{c\tau}(\Delta - m)V^3 = -\frac{d}{\tau c}\Delta F^1 + \frac{1}{\tau}F^3 + F^4. \tag{4.9}$$

(i) First assume that $F^3 \in H^2(B) \cap H_0^1(B)$. Then (4.9) determines $V^3 \in H^2(B) \cap H_0^1(B)$, and then (4.8) determines $V^1 \in H^4(B) \cap H_0^2(B)$. Together with (4.4) and (4.6) we have found $V \in D(A)$ solving $AV = F$. Moreover, the elliptic estimates for (4.8) and (4.9) allow us to conclude

$$|V|_{\mathcal{H}} \leq C|F|_{\mathcal{H}}, \tag{4.10}$$

with a positive constant C which does not depend on V (resp. F).

(ii) Now let $F \in \mathcal{H}$ be arbitrary. We take a sequence $(F_n^3)_n \subset H^2(B) \cap H_0^1(B)$ with $F_n^3 \to F^3$ in $H_0^1(B)$. Then we can apply part (i) to $F_n := (F^1, F^2, F_n^3, F^4)'$ and conclude, using (4.10), that V_n with $AV_n = F_n$ converges to $V \in \mathcal{H}$ with $V^2 \in H_0^2(B)$ and $V^4 \in H_0^1(B)$. Moreover, for any $\Phi \in (C_0^\infty(B))^4$ we get, denoting by A_f^* the formal adjoint of A_f in $\mathcal{H}$,

$$\langle V, A_f^*\Phi\rangle_{\mathcal{H}} \leftarrow \langle V_n, A_f^*\Phi\rangle_{\mathcal{H}} = \langle AV_n, \Phi\rangle_{\mathcal{H}} \to \langle F, \Phi\rangle_{\mathcal{H}}.$$

Hence we have proved $V \in D(A)$ and $AV = F$. Moreover, we get the estimate (4.10) for any $F \in \mathcal{H}$.

This proves $0 \in \varrho(A)$, and the proof shows that (4.10) can be extended to

$$|V|_{\mathcal{H}} + \|V^1\|_{H^4} + \|V^2\|_{H^2} + \|V^3\|_{H^2} + \|V^4\|_{H^1} \leq C|F|_{\mathcal{H}}. \tag{4.11}$$

Using Rellich's selection theorem we get the compactness of A^{-1}. □

As a standard conclusion now, from the dissipativity and Lemma 2, we obtain that A generates a C_0-semigroup, and hence the initial (boundary) value problem (4.2) is uniquely solvable:

THEOREM 3. *For any $F \in C^0([0,\infty), D(A))$ or $F \in C^1([0,\infty), \mathcal{H})$ and any $V^0 \in D(A)$ there is a unique solution V to (4.2) with $V \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(A))$.*

We remark that the boundary condition (2.5) can be treated similarly. Also we note that the well-posedness consideration in this section naturally extended to unbounded domains.

5. Spectral bounds

We look at the homogeneous differential equation

$$V_t = AV,$$

arising for the boundary conditions (2.5), with A being defined in analogy to the operator A in the previous section (cf. the remark following Theorem 3).

Due to the boundary conditions, we can make the following expansion for $V = (V^1, V^2, V^3, V^4)'$:

$$V(t,x) = \sum_{j=1}^{\infty} (\alpha_j(t), \gamma_j(t), \delta_j(t), \varepsilon_j(t))^t \, w_j(x),$$

where $(w_j)_j$ denote the eigenfunctions of the Laplace operator, under the Dirichlet boundary conditions corresponding to the eigenvalue λ_j,

$$-\Delta v_j = \lambda_j w_j, \qquad w = 0 \qquad \text{on} \quad \partial B,$$

with

$$0 < \lambda_1 \leq \cdots \leq \lambda_j \to \infty \qquad (\text{as} \quad j \to \infty).$$

Then the coefficients satisfy the condition

$$\alpha_j' = \gamma_j, \qquad \gamma_j' = -a\lambda_j^2 \alpha_j + \lambda_j \delta_j + \tau \lambda_j \varepsilon_j,$$

$$\delta_j' = \varepsilon_j, \qquad \varepsilon_j' = -\frac{d}{c\tau}\lambda_j \gamma_j - \frac{1}{c\tau}(\lambda_j + m)\delta_j - \frac{1}{\tau}\varepsilon_j.$$

Eliminating γ_j and ε_j we obtain

$$\alpha_j'' = -a\lambda_j^2 \alpha_j + \lambda_j \delta_j + \tau \lambda_j \delta_j',$$

$$\delta_j'' = -\frac{d}{c\tau}\lambda_j \alpha_j' - \frac{1}{c\tau}(\lambda_j + m)\delta_j - \frac{1}{\tau}\delta_j'.$$

Differentiating and eliminating α_j, we obtain a fourth-order differential equation for δ_j:

$$(5.1) \qquad c\tau \delta_j'''' + c\delta_j''' + (\lambda_j + m + ac\tau\lambda_j^2 + d\tau\lambda_j^2)\delta_j'' + (ac\lambda_j^2 + d\lambda_j^2)\delta_j' + a\lambda_j^2(\lambda_j + m)\delta_j = 0.$$

We remark that α_j, γ_j, and ε_j satisfy the same differential equation. The characteristic polynomial P_j of this equation is given by

$$(5.2) \qquad P_j(\beta) = \beta^4 + \frac{1}{\tau}\beta^3 + \frac{1}{c\tau}(\lambda_j + m + \tau(ac + d)\lambda_j^2)\beta^2 + \frac{1}{c\tau}(ac + d)\lambda_j^2 \beta + \frac{a}{c\tau}(\lambda_j^3 + m\lambda_j^2).$$

The zeros of P_j are denoted by $\beta_1(j), \ldots, \beta_4(j)$ or, short, $\beta_1, \ldots, \beta_4$. Let $\mathcal{S}$ denote the spectral set of all zeros,

$$\mathcal{S} := \{\beta_k(j) \mid j = 1, 2, 3 \ldots; \ \ k = 1, 2, 3, 4\}.$$

We shall prove that it lies strictly in the left complex half-plane.

THEOREM 4.

$$\exists\, \omega > 0: \quad \sup\{\operatorname{Re}\beta \mid \beta \in \mathcal{S}\} \leq -\omega.$$

P r o o f. Let $\beta \in \mathcal{S}$. Since A is dissipative we have

$$\operatorname{Re}\beta \leq 0. \tag{5.3}$$

Next let us show that there are no purely imaginary eigenvalues. For this purpose let $\beta = i\mu$ with $\mu \in \mathbb{R} \setminus \{0\}$. Then μ satisfies the equation

$$\mu^4 - \frac{i}{\tau}\mu^3 - \frac{1}{c\tau}(\lambda_j + m + \tau(ac+d)\lambda_j^2)\mu^2 + \frac{i}{c\tau}(ac+d)\lambda_j^2\mu + \frac{a}{c\tau}(\lambda_j^3 + m\lambda_j^2) = 0. \tag{5.4}$$

First we look at the imaginary part in Equation (5.4) and conclude that

$$\mu^2 = \frac{ac+d}{c}\lambda_j^2. \tag{5.5}$$

Taking real parts in Equation (5.4) and using (5.5) we get

$$\lambda_j = -m \leq 0,$$

which is a contradiction and hence proves that there are no purely imaginary eigenvalues. It remains to show that

$$\exists\, \omega_1 > 0\ \exists\, j_0\ \forall\, j \geq j_0\ \forall\, k = 1,2,3,4: \quad \operatorname{Re}\beta_k(j) \leq -\omega_1. \tag{5.6}$$

In order to prove (5.6) we note that the characteristic equation $P_j(\beta) = 0$ can be rewritten as

$$\beta^4 - (\beta_1+\beta_2+\beta_3+\beta_4)\beta^3 + (\beta_1\beta_2+\beta_1\beta_3+\beta_1\beta_4+\beta_2\beta_3+\beta_2\beta_4+\beta_3\beta_4)\beta^2 + (\beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_4 + \beta_1\beta_3\beta_4 + \beta_2\beta_3\beta_4)\,\beta + \beta_1\beta_2\beta_3\beta_4 = 0, \tag{5.7}$$

and we may assume, without any loss of generality, that

$$\beta_2 = \overline{\beta}_1, \qquad \beta_4 = \overline{\beta}_3.$$

Comparing (5.7) with (5.2) we obtain

$$\operatorname{Re}\beta_1 + \operatorname{Re}\beta_3 = -\frac{1}{2\tau}, \tag{5.8}$$

$$4\operatorname{Re}\beta_1\operatorname{Re}\beta_3 + |\beta_1|^2 + |\beta_3|^2 = -\frac{1}{c\tau}(\lambda_j + m + \tau(ac+d)\lambda_j^2), \tag{5.9}$$

$$|\beta_1|^2\operatorname{Re}\beta_3 + |\beta_3|^2\operatorname{Re}\beta_1 = -\frac{ac+d}{2c\tau}\lambda_j^2, \tag{5.10}$$

$$|\beta_1|^2|\beta_3|^2 = \frac{a}{c\tau}(\lambda_j^3 + m\lambda_j^2). \tag{5.11}$$

We conclude from (5.8), in view of (5.3), that

$$\operatorname{Re}\beta_{1,2} = \mathbf{O}(1), \qquad \operatorname{Re}\beta_{3,4} = \mathbf{O}(1) \qquad (\text{as} \quad j \to \infty). \tag{5.12}$$

This combined with (5.9) yields

$$\lim_{j\to\infty} \frac{|\beta_1|^2 + |\beta_3|^2}{\lambda_j^2} = \frac{ac+d}{c}. \tag{5.13}$$

Equation (5.11) implies

$$\lim_{j\to\infty} \frac{|\beta_1|^2|\beta_3|^2}{\lambda_j^3} = \frac{a}{c\tau}. \tag{5.14}$$

From (5.13) and (5.14) we obtain

$$|\beta_1|^2 = \frac{ac+d}{c}\lambda_j^2 + \mathbf{o}(\lambda_j^2), \qquad |\beta_3|^2 = \frac{a}{\tau(ac+d)}\lambda_j + \mathbf{o}(\lambda_j). \tag{5.15}$$

Combining Eqs. (5.15), (5.10) and (5.12) we get

$$\frac{|\beta_1|^2 \operatorname{Re}\beta_3}{\lambda_j^2} + \frac{|\beta_3|^2 \operatorname{Re}\beta_1}{\lambda_j^2} = -\frac{ac+d}{2c\tau},$$

implying

$$\operatorname{Re}\beta_4 = \operatorname{Re}\beta_3 \longrightarrow -\frac{1}{2\tau}, \tag{5.16}$$

which, together with (5.8), yields

$$\operatorname{Re}\beta_2 = \operatorname{Re}\beta_1 \longrightarrow -\frac{1}{2\tau}. \tag{5.17}$$

If we choose (any, but fixed) ω_1 satisfying the condition

$$0 < \omega_1 < \frac{1}{2\tau},$$

Eqs. (5.16) and (5.17) prove (5.6) (with j_0 depending on ω_1).
Now ω can be chosen as

$$\omega := \min\{\omega_1, -\omega_2\}$$

where

$$\omega_2 := \max\{\operatorname{Re}\beta_k(j) \mid j = 1, \ldots, j_0; \;\; k = 1,2,3,4\},$$

and $\omega_2 < 0$ because of (5.3) and the non-existence of purely imaginary eigenvalues. $\square$

As a corollary we get an estimate on the spectrum $\sigma(A)$ of A, showing that it lies strictly in the left complex half-plane.

COROLLARY 5.

$$\sup\{\operatorname{Re}\beta \mid \beta\in\sigma(A)\} \le -\omega < 0.$$

P r o o f. Since A^{-1} is compact, by Lemma 2 we have

$$\sigma(A)=\sigma_p(A) \qquad \text{(point spectrum)}.$$

For a possible eigenvalue β with eigenfunction V we can expand V into the series

$$V(x)=\sum_{j=1}^{\infty}(\alpha_j,\gamma_j,\delta_j,\varepsilon_j)'\,w_j(x),$$

with complex numbers $\alpha_j,\gamma_j,\delta_j,\varepsilon_j$. It follows

$$P_j(\beta)\alpha_j=P_j(\beta)\gamma_j=P_j(\beta)\delta_j=P_j(\beta)\varepsilon_j=0,$$

that is, if β is an eigenvalue then it necessarily belongs to the spectral set $\mathcal{S}$. $\square$

This result confirmes the expectation that the semigroup is exponentially stable, but the formal proof of this property is still missing. Standard approaches (multiplier methods, uniform boundedness of resolvents) failed up to now, and the problem remains as a challenge for future investigations.

6. Impossibility of localization

In the previous section we have proved that the decay of solutions is expected to be controlled by a negative exponential. A natural question is to ask if the decay is fast enough to guarantee that the solution vanishes in a finite time. In this section, we prove the impossibility of localization of solutions with respect to the time variable. This would give information concerning the lower bound for the decay of the solutions. That is, the aim of this section is to establish the following result:

THEOREM 6. *Let (u,θ) be a solution of the problem determined by Eqs.* (3.1), (3.2), (2.4), (2.5), *which vanishes for all $t\ge t_0$ for some $t_0>0$. Then (u,θ) is the null solution.*

P r o o f. The impossibility of localization of solutions is equivalent to the uniqueness for the backward in time problem. Therefore we consider

$$a\Delta^2 u+\Delta\theta+\ddot{u}=0, \tag{6.1}$$

$$-\Delta\theta+m\theta+d\Delta\dot{\ddot{u}}=c\dot{\ddot{\theta}}, \tag{6.2}$$

where we have used the notation

$$\tilde{f} = f - \tau \dot{f}. \tag{6.3}$$

It is sufficient to prove that the only solution for null initial data for the system (6.1), (6.2) is the null solution.

We define the new energy term

$$E^*(t) := \frac{1}{2} \int_B \big(d|\dot{\tilde{u}}|^2 + ad|\Delta \tilde{u}|^2 + c(\tilde{\theta})^2 + \tau(|\nabla\theta|^2 + m\theta^2)\big)dv.$$

We easily obtain, using the boundary conditions,

$$\frac{dE^*}{dt} = \int_B (|\nabla\theta|^2 + m\theta^2)dv. \tag{6.4}$$

This implies the existence of a positive constant C such that for all $t \geq 0$

$$\frac{dE^*}{dt} \leq CE^*(t). \tag{6.5}$$

Thus, we obtain the estimate

$$E^*(t) \leq E^*(0)\exp(Ct), \tag{6.6}$$

and for null initial data we deduce that $E(t) = 0$ for all $t \geq 0$. It follows that $\theta = 0$ and $\tilde{u} = 0$. In view of the initial conditions, the solution of the ordinary differential equation $\tilde{u} = 0$ is $u = 0$, and then the uniqueness of solutions is proved. □

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