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Spatial estimates concerning the harmonic vibrations in rectangular plates with voids

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This paper studies the spatial behaviour of the amplitude of a harmonic vibration in a rectangular plate of Mindlin type, made of a homogeneous and isotropic elastic material with voids. Provided the frequency of vibration is lower than the critical value, some appropriate measures are introduced relating the amplitude of resulting harmonic vibration, and the corresponding first-order differential inequalities are established under mild conditions on the elastic coefficients. The case of a semi-infinite plate is also studied and some Phragmén–Lindelöf alternatives are established.

1. Introduction

In [1] Nunziato and Cowin have presented a general theory of elastic materials with voids. In this theory, the bulk density is written as a product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom and it was employed previously by Goodman and Cowin [2] to develop a continuum theory of granular materials. The linear theory of elastic materials with voids has been developed by Cowin and Nunziato [3]. The first investigations in the theory of thermoelastic materials with voids are due to Nunziato and Cowin [1] and Ieşan [4]. The intended applications of the theory concern geological materials and manufactured porous materials. A presentation of this theory can be found in [5] and [6].

The classical Kirchhoff's theory of bending of elastic plates neglects the effects of transverse shear forces. Mindlin [7] and Reissner [8] (see also [9]) developed the theories of elastic plates which take into consideration the effect of transverse shear forces. The time-harmonic oscillations in elastic and thermoelastic plates of Mindlin type were studied by Schiavone and Constanda [10] and Schiavone and Tait [11]. In [12] Ciarletta studies the spatial behaviour of the transient and steady-state solutions in thin plates with shear deformation. Some methods to study the spatial behaviour of the transient solutions in bending of Mindlin-type thermoelastic plates was presented by D'APICE and CHIRIŢĂ [13] and by D'APICE [14].

Using the Nunziato-Cowin theory, SCARPETTA [15] and BÎRSAN [16] established the equations of bending of elastic and thermoelastic plates with voids. The existence and uniqueness of solutions in equilibrium and dynamic bending theory of elastic plates with voids were established in [17]. Some results concerning the steady-state solutions for thermoelastic porous plates were presented in [18].

In the present paper we study the spatial behaviour of the solutions describing harmonic vibrations in the bending theory of Mindlin-type rectangular finite (semi-infinite) plates with voids. First, we consider a finite rectangular plate made of a homogeneous and isotropic porous elastic material, constrained on the lateral sides and one of the ends, and the other end is subjected to boundary data which are time-harmonic with angular frequency ω. For the study of spatial behaviour of the amplitude of vibration that takes place in the plate, we introduce two appropriate measures and then we establish, for each of them, a first-order differential inequality, provided the frequency of the harmonic vibration is lower than the barrier frequency. The integration of these differential inequalities furnishing some estimates describes the exponential decay of the amplitude. The results are extended to a semi-infinite rectangular plate. The measures and methods used are inspired by the studies [12, 19, 20] in classical elasticity and termoelasticity, and by the paper [21–24] in linear porous elasticity. Our study includes the class of materials with negative Poisson's ratio (see LAKES [25, 26]). This type of materials have interesting proprieties such as high energy absorption and fracture resistance and possible applications of these materials were presented in the papers [27–29].

2. Formulation of problem

Throughout this paper we consider the region $\overline{S}\times[-h_0/2, h_0/2]$ of the physical space $\mathbb{R}^3$, where S is a domain in $\mathbb{R}^2$ whose boundary ∂S is a closed Lyapunov curve and $0 < h_0 = \text{constant} \ll \operatorname{diam} S$. We call this region a plate with the thickness h_0.

We assume that B is the interior of the right cylinder defined above, and that it is filled with an isotropic and homogeneous, linearly elastic material with voids.

We choose a rectangular Cartesian system $Ox_1x_2x_3$ so that the plane Ox_1x_2 is its middle plane. The Latin subscripts are understood to range over the integers $1, 2, 3$, whereas Greek subscripts are confined to the range $1, 2$; summation over repeated subscripts is implied and comma followed by a subscript is used to denote partial derivative with respect to the corresponding Cartesian coordinate. Moreover, we use a superposed dot to denote partial differentiation with respect to time. We also consider a fixed time $T > 0$.

Let ρ be the bulk mass density and χ the equilibrated inertia [2] in the reference state.

In the bending theory of Mindlin-type plates with voids, the displacements $\mathbf{u}$ and the change in volume fraction φ can be written in the form [15, 16]

$$\begin{aligned} u_\alpha &= x_3 v_\alpha(x_1, x_2, t), \\ u_3 &= v_3(x_1, x_2, t), \\ \varphi &= x_3 \psi(x_1, x_2, t) \qquad \text{in } B \times (0, T). \end{aligned} \tag{2.1}$$

The equations of motion for elastic plates with voids in the bending theory [15, 16], in absence of the external body loads, are the following:

$$\begin{aligned} N_{\beta\alpha,\beta} - N_{\alpha 3} &= \rho h^2 \ddot{v}_\alpha, \\ N_{\beta 3,\beta} &= \rho \ddot{v}_3, \\ H_{\alpha,\alpha} + G - H_3 &= \rho\chi h^2 \ddot{\psi} \qquad \text{in } S \times (0, T), \end{aligned} \tag{2.2}$$

where $h^2 = h_0^2/12$. For the meaning of the functions $N_{\beta i}$, H_i and G which appear in the above relations, we refer to [9, 10] and [16].

For elastic porous plates made from an isotropic and homogeneous material, we have the following constitutive equations:

$$\begin{aligned} N_{\alpha\beta} &= h^2[\lambda v_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(v_{\alpha,\beta} + v_{\beta,\alpha}) + \beta\psi\delta_{\alpha\beta}], \\ N_{\alpha 3} &= \mu(v_\alpha + v_{3,\alpha}), \\ H_\beta &= \alpha h^2 \psi_{,\beta}, \qquad H_3 = \alpha\psi, \\ G &= -h^2(\beta v_{\gamma,\gamma} + \xi\psi) \qquad \text{in } \overline{S} \times [0, T], \end{aligned} \tag{2.3}$$

where $\lambda, \mu, \alpha, \beta$ and ξ are constitutive constants and δ_{ij} is the Kronecker's delta.

The internal energy density $\mathcal{E}$ per unit area of the middle plane, associated with the kinematic fields v_i and ψ, is defined by

$$\begin{aligned} 2\mathcal{E} = h^2 \left[\lambda v_{\alpha,\alpha} v_{\beta,\beta} + \mu(v_{\alpha,\beta} + v_{\beta,\alpha}) v_{\alpha,\beta} + 2\beta v_{\alpha,\alpha} + \xi\psi^2\right] \\ + \alpha h^2 \psi_{,\beta}\psi_{,\beta} + \alpha\psi^2 + \mu(v_\alpha + v_{3,\alpha})(v_\alpha + v_{3,\alpha}), \end{aligned} \tag{2.4}$$

and it is positive definite if

$$\mu > 0, \quad \alpha > 0, \quad \lambda + \mu > 0, \quad \xi > \frac{\beta^2}{\lambda + \mu}. \tag{2.5}$$

In this paper we will not impose all these inequalities. More precisely, the first two inequalities will be assumed, while the last two will be changed.

In what follows we assume that

$$S = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 \in [0, L],\ x_2 \in [0, l]\}, \qquad l > 0,\ L > 0. \tag{2.6}$$

To the Eqs. (2.2) we add the initial conditions

$$v_r = a_r^0, \qquad \dot{v}_r = b_r^0, \qquad \psi = c^0, \qquad \dot{\psi} = d^0, \qquad \text{in } \overline{S} \times \{0\} \tag{2.7}$$

and the boundary conditions

$$\begin{aligned} v_r(x_1, 0, t) &= 0, \qquad v_r(x_1, l, t) = 0, \\ \psi(x_1, 0, t) &= 0, \qquad \psi(x_1, l, t) = 0, \qquad x_1 \in [0, L], \\ v_r(L, x_2, t) &= 0, \qquad v_r(0, x_2, t) = \widetilde{w}(x_2) \exp(-i\omega t), \\ \psi(L, x_2, t) &= 0, \qquad \psi(0, x_2, t) = \widetilde{\phi}(x_2) \exp(-i\omega t), \qquad x_2 \in [0, l], \end{aligned} \tag{2.8}$$

where $\widetilde{w}_r$ and $\widetilde{\phi}$ are prescribed continuous functions, ω is a positive definite constant and i is the imaginary unit, that is $i = \sqrt{-1}$.

It is easy to see that

$$\begin{aligned} v_r(x_1, x_2, t) &= V_r(x_1, x_2, t) + w_r(x_1, x_2) \exp(-i\omega t), \\ \psi(x_1, x_2, t) &= \Psi(x_1, x_2, t) + \phi(x_1, x_2) \exp(-i\omega t), \end{aligned} \tag{2.9}$$

where (V_r, Ψ) absorbs the initial conditions and satisfies the null boundary conditions and the Eqs. (2.2) and (2.3), while (w_r, ϕ) satisfies the boundary value problem consisting of the field equations:

$$\begin{aligned} S_{\beta\alpha,\beta} - S_{\alpha 3} &= -\rho h^2 \omega^2 w_\alpha, \\ S_{\beta 3,\beta} &= -\rho \omega^2 w_3, \\ T_{\alpha,\alpha} + \Gamma - T_3 &= -\rho h^2 \chi \omega^2 \phi, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} S_{\alpha\beta} &= h^2[\lambda w_{\gamma,\gamma} \delta_{\alpha\beta} + \mu(w_{\alpha,\beta} + w_{\beta,\alpha}) + \beta\phi\delta_{\alpha\beta}], \\ S_{\alpha 3} &= \mu(w_\alpha + w_{3,\alpha}), \\ T_\beta &= \alpha h^2 \phi_{,\beta}, \qquad T_3 = \alpha\phi \\ \Gamma &= -h^2(\beta w_{\gamma,\gamma} + \xi\phi), \end{aligned} \tag{2.11}$$

and of the boundary conditions

$$
\begin{aligned}
&w_r(x_1,0) = 0, \qquad w_r(x_1,l) = 0, \\
&\phi(x_1,0) = 0, \qquad \phi(x_1,l) = 0, \qquad x_1 \in [0,L], \\
&w_r(L,x_2) = 0, \qquad w_r(0,x_2) = \widetilde{w}(x_2), \\
&\phi(L,x_2) = 0, \qquad \phi(0,x_2) = \widetilde{\phi}(x_2), \qquad x_2 \in [0,l].
\end{aligned} \tag{2.12}
$$

From (2.10) and (2.11) we deduce the following partial differential equations for the functions w_r and ϕ:

$$
\begin{aligned}
h^2[\mu\Delta w_\alpha + (\lambda+\mu)w_{\gamma,\gamma\alpha} + \beta\phi_{,\alpha}] - \mu(w_\alpha + w_{3,\alpha}) &= -\rho h^2\omega^2 w_\alpha, \\
\mu(\Delta w_3 + w_{\gamma,\gamma}) &= -\rho\omega^2 w_3, \\
\alpha h^2\phi_{,\gamma\gamma} - \beta h^2 w_{\gamma,\gamma} - (h^2\xi + \alpha)\phi &= -\rho\chi h^2\omega^2\phi.
\end{aligned} \tag{2.13}
$$

Next, we study the decay estimates for the amplitude (w_r, ϕ) of the steady-state vibration, satisfying Eqs. (2.13) under the boundary conditions (2.12), for the classes of material for which

$$
\mu > 0, \quad \alpha > 0, \quad \lambda + 2\mu > 0, \quad \xi > 2\beta^2 \min\left(\frac{1}{2\lambda+3\mu}, \frac{1}{\lambda+2\mu}\right). \tag{2.14}
$$

3. First estimate

In this section we will establish two results which describe the spatial behaviour of the amplitude of the considered vibration under the following assumption:

$$
\mu > 0, \quad \alpha > 0, \quad 2\lambda + 3\mu > 0, \quad \xi > \frac{2\beta^2}{2\lambda+3\mu}. \tag{3.1}
$$

In order to analyse the spatial behaviour under the above hypotheses concerning the constitutive constants of the material, we write the basic Eqs. $(2.13)_1$ in the form

$$
M_{\beta\alpha,\beta} - S_{\alpha 3} = -\rho h^2\omega^2 w_\alpha, \tag{3.2}
$$

where

$$
M_{\beta\alpha} = h^2[\mu w_{\alpha,\beta} + (\lambda+\mu)w_{\gamma,\gamma}\delta_{\alpha\beta} + \beta\phi\delta_{\alpha\beta}]. \tag{3.3}
$$

We associate with the amplitude of the steady-state vibration the cross-sectional functional

$$
K(x_1) = -\int_0^l (\bar{M}_{1\alpha}w_\alpha + M_{1\alpha}\bar{w}_\alpha + \bar{S}_{13}w_3 + S_{13}\bar{w}_3 + \bar{T}_1\phi + T_1\bar{\phi})dx_2 \tag{3.4}
$$

for every $x_1 \in [0,L]$, where the superposed bar denotes a complex conjugate.

Let us define the following quantities:

$$
\begin{aligned}
k_m &= \min\left\{\mu, \frac{1}{2}\left(\xi + 2\lambda + 3\mu - \sqrt{[\xi - (2\lambda + 3\mu)]^2 + 8\beta^2}\right)\right\},\\
k_M &= \max\left\{\mu, \frac{1}{2}\left(\xi + 2\lambda + 3\mu + \sqrt{[\xi - (2\lambda + 3\mu)]^2 + 8\beta^2}\right)\right\},
\end{aligned} \tag{3.5}
$$

and the barrier frequency

$$
\omega_1 = \min\left\{\frac{\pi}{l}\sqrt{\frac{k_m}{2\rho}}, \frac{\pi^2 h}{2l}\sqrt{\frac{k_m \mu}{\rho(k_m h^2 \pi^2 + \mu l^2)}}, \frac{1}{lh}\sqrt{\frac{l^2 h^2 k_m + \alpha(l^2 + \pi^2 h^2)}{\rho\chi}}\right\}. \tag{3.6}
$$

THEOREM 1. *Suppose that the hypothesis* (3.1) *holds true and* $\omega < \omega_1$. *The functional* $K(x_1)$ *represents an acceptable measure of the solution that satisfies the following exponential decay estimate*

$$
0 \le K(x_1) \le K(0)\exp(-\sigma x_1) \tag{3.7}
$$

for every $x_1 \in [0, L]$, *where the positive constant* σ *depends on the constitutive constants* $\lambda, \mu, \xi, \alpha, \beta$, *thickness* h_0 *and width* l *of the plate.*

P r o o f. By direct differentiation in (3.4), we get

$$
\begin{aligned}
K'(x_1) = -\int_0^l (&\bar{M}_{1\alpha,1} w_\alpha + M_{1\alpha,1}\bar{w}_\alpha + \bar{S}_{13,1} w_3 + S_{13,1}\bar{w}_3\\
&+ \bar{T}_{1,1}\phi + T_{1,1}\bar{\phi} + \bar{M}_{1\alpha} w_{\alpha,1} + M_{1\alpha}\bar{w}_{\alpha,1}\\
&+ \bar{S}_{13} w_{3,1} + S_{13}\bar{w}_{3,1} + \bar{T}_1 \phi_{,1} + T_1 \bar{\phi}_{,1})\,dx_2.
\end{aligned} \tag{3.8}
$$

From (3.2) and $(2.10)_2$ we deduce that

$$
\begin{aligned}
M_{1\alpha,1} &= -M_{2\alpha,2} + \mu(w_\alpha + w_{3,\alpha}) - \rho h^2 \omega^2 w_\alpha,\\
S_{13,1} &= -S_{23,2} - \rho\omega^2 w_3,\\
T_{1,1} &= -T_{2,2} + \alpha\phi + \beta h^2 w_{\gamma,\gamma} + \xi h^2 \phi - \rho h^2 \chi \omega^2 \phi,
\end{aligned} \tag{3.9}
$$

and hence, using integration by parts, the lateral boundary conditions $(2.12)_{1-4}$ and the relations (2.11) and (3.3), the relation (3.8) can be written in the form

$$
\begin{aligned}
K'(x_1) = -2\int_0^l \{&h^2[\mu w_{\alpha,\beta}\bar{w}_{\alpha,\beta} + (\lambda + \mu) w_{\gamma,\gamma}\bar{w}_{\rho,\rho} + \beta(\phi\bar{w}_{\rho,\rho} + \bar{\phi} w_{\rho,\rho})\\
&+ \xi\phi\bar{\phi}\,] + \mu(w_\alpha + w_{3,\alpha})(\bar{w}_\alpha + \bar{w}_{3,\alpha}) + \alpha h^2 \phi_{,\alpha}\bar{\phi}_{,\alpha} + \alpha\phi\bar{\phi}\\
&- \rho\omega^2 h^2 w_\alpha \bar{w}_\alpha - \rho\omega^2 w_3 \bar{w}_3 - \rho\omega^2 h^2 \chi\phi\bar{\phi}\}\,dx_2.
\end{aligned} \tag{3.10}
$$

Let us introduce the following bilinear form:

$$\begin{aligned}\mathcal{F}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\{(\lambda+2\mu)(a_1\bar{b}_1+\bar{a}_1b_1+a_2\bar{b}_2+\bar{a}_2b_2)+(\lambda+\mu)(a_1\bar{b}_2\\ +\bar{a}_1b_2+a_2\bar{b}_1+\bar{a}_2b_1)+\beta[\bar{a}_3(b_1+b_2)+a_3(\bar{b}_1+\bar{b}_2)\\ +\bar{b}_3(a_1+a_2)+b_3(\bar{a}_1+\bar{a}_2)]+\xi(a_3\bar{b}_3+\bar{a}_3b_3)\},\end{aligned} \tag{3.11}$$

for every $\mathbf{a}=\{a_1,a_2,a_3\}$ and $\mathbf{b}=\{b_1,b_2,b_3\}$, and we define the quadratic form

$$\begin{aligned}\mathcal{F}_1 &= (\lambda+2\mu)(w_{1,1}\bar{w}_{1,1}+w_{2,2}\bar{w}_{2,2})+(\lambda+\mu)(w_{1,1}\bar{w}_{2,2}+\bar{w}_{1,1}w_{2,2})\\ &\quad+\xi\phi\bar{\phi}+\beta[\bar{\phi}(w_{1,1}+w_{2,2})+\phi(\bar{w}_{1,1}+\bar{w}_{2,2})],\\ \mathcal{F}_2 &= \mu(w_{1,2}\bar{w}_{1,2}+w_{2,1}\bar{w}_{2,1}),\end{aligned} \tag{3.12}$$

in the variables $w_{1,1}, w_{2,2}, \phi$ and, respectively, $w_{1,2}, w_{2,1}$.

In view of the assumptions (3.1) we can say that the quadratic forms $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite quadratic forms.

The eigenvalues of the matrix associated to the quadratic form $\mathcal{F}_1$ are

$$\begin{aligned}k_1 &= \mu,\\ k_2 &= \frac{1}{2}\Big(\xi+2\lambda+3\mu+\sqrt{[\xi-(2\lambda+3\mu)]^2+8\beta^2}\Big),\\ k_3 &= \frac{1}{2}\Big(\xi+2\lambda+3\mu-\sqrt{[\xi-(2\lambda+3\mu)]^2+8\beta^2}\Big).\end{aligned} \tag{3.13}$$

Clearly, we have the inequalities

$$k_m(w_{\alpha\beta}\bar{w}_{\alpha\beta}+\phi\bar{\phi}) \le \mathcal{F}_1+\mathcal{F}_2 \le k_M(w_{\alpha\beta}\bar{w}_{\alpha\beta}+\phi\bar{\phi}), \tag{3.14}$$

where k_m and k_M, defined by the relations (3.5), are the lowest and the largest characteristic values, respectively, of the matrix associated to the quadratic form $\mathcal{F}_1$.

Thus, we have the inequality

$$\begin{aligned}-K'(x_1) \ge 2\int_0^l \{k_m h^2(w_{\alpha,1}\bar{w}_{\alpha,1}+w_{1,2}\bar{w}_{1,2}+w_{2,2}\bar{w}_{2,2}+\phi\bar{\phi})\\ +\mu(w_\alpha+w_{3,\alpha})(\bar{w}_\alpha+\bar{w}_{3,\alpha})+\alpha h^2\phi_{,\alpha}\bar{\phi}_{,\alpha}+\alpha\phi\bar{\phi}\\ -\rho\omega^2h^2w_\alpha\bar{w}_\alpha-\rho\omega^2w_3\bar{w}_3-\rho\omega^2h^2\chi\phi\bar{\phi}\}dx_2.\end{aligned} \tag{3.15}$$

Now, on the basis of the boundary conditions $(2.12)_{1-4}$, the Wirtinger's inequality holds

$$\int_0^l w_{r,2}\bar{w}_{r,2}dx_2 \ge \frac{\pi^2}{l^2}\int_0^l w_r\bar{w}_r dx_2, \qquad r=1,2,3 \text{ (not summed for } r). \tag{3.16}$$

CIARLETTA [12] established, in view of inequality (3.16) for $r = 2$, an inequality of the following type

$$E \geq \frac{a^2}{2} \int_0^l \left(w_{2,2}\bar{w}_{2,2} + \frac{\pi^2}{2(a^2\pi^2 + l^2)} w_{3,2}\bar{w}_{3,2} \right) dx_2, \tag{3.17}$$

where $a \in \mathbb{R}$ and

$$E = \int_0^l [a^2 w_{2,2}\bar{w}_{2,2} + (w_2 + w_{3,2})(\bar{w}_2 + \bar{w}_{3,2})] dx_2. \tag{3.18}$$

In view of the hypothesis $\omega < \omega_1$, if we set in (3.17) $a = \sqrt{k_m/\mu}h$, from Eq. (3.15) we obtain

$$\begin{aligned} -K'(x_1) \geq \int_0^l [&M_1 h^2 w_{\alpha,1}\bar{w}_{\alpha,1} + M_2(w_1 + w_{3,1})(\bar{w}_1 + \bar{w}_{3,1}) \\ &+ M_3 h^2 w_{1,2}\bar{w}_{1,2} + M_4 h^2 w_{2,2}\bar{w}_{2,2} + M_5 w_{3,2}\bar{w}_{3,2} \\ &+ M_6 h^2 \phi\bar{\phi} + M_7 h^4 \phi_{,1}\bar{\phi}_{,1}] dx_2, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} &M_1 = M_2 = 2k_m, \qquad M_3 = 2k_m\left(1 - \frac{\omega^2}{\omega_1^2}\right), \\ &M_4 = k_m\left(1 - \frac{\omega^2}{\omega_1^2}\right), \qquad M_5 = \frac{k_m \mu h^2 \pi^2}{2(k_m h^2 \pi^2 + \mu l^2)}\left(1 - \frac{\omega^2}{\omega_1^2}\right), \\ &M_6 = 2\left[k_m + \frac{\alpha(l^2 + \pi^2 h^2)}{l^2 h^2}\right]\left(1 - \frac{\omega^2}{\omega_1^2}\right), \qquad M_7 = 2\frac{\alpha}{h^2}. \end{aligned} \tag{3.20}$$

Moreover, by means of the Schwarz inequality and the arithmetic-geometric inequality, we obtain

$$\begin{aligned} |K(x_1)| \leq \int_0^l \bigg(&\frac{1}{\varepsilon_\alpha} M_{1\alpha}\bar{M}_{1\alpha} + \frac{1}{\varepsilon_3} S_{13}\bar{S}_{13} + \frac{1}{\varepsilon_4} T_1\bar{T}_1 \\ &+ \varepsilon_\alpha w_\alpha \bar{w}_\alpha + \varepsilon_3 w_3\bar{w}_3 + \varepsilon_4 \phi\bar{\phi} \bigg) dx_2. \end{aligned} \tag{3.21}$$

Furthermore, by using the relation (3.3), the Schwarz inequality for the bilinear form $\mathcal{F}$ in terms of $(w_{1,1}, w_{2,2}, \phi)$ and $(S_{11}, 0, 0)$ and the inequality (3.14), we obtain

$$M_{11}\bar{M}_{11} \leq h^4 k_M^2 (w_{1,1}\bar{w}_{1,1} + w_{2,2}\bar{w}_{2,2} + \phi\bar{\phi}). \tag{3.22}$$

Thus, by setting $\varepsilon_1 = \varepsilon_2 = hk_M$, $\varepsilon_3 = \mu/h$ and $\varepsilon_4 = \alpha h$, in view of relations $(2.11)_{2,3}$ and (3.3), we deduce

$$|K(x_1)| \leq \int_0^l [M_1^* h^2 w_{\alpha,1}\bar{w}_{\alpha,1} + M_2^*(w_1 + w_{3,1})(\bar{w}_1 + \bar{w}_{3,1}) + M_3^* h^2 w_{1,2}\bar{w}_{1,2} + M_4^* h^2 w_{2,2}\bar{w}_{2,2} + M_5^* w_{3,2}\bar{w}_{3,2} + M_6^* h^2 \phi\bar{\phi} + M_7^* h^4 \phi_{,1}\bar{\phi}_{,1}]dx_2, \tag{3.23}$$

where

$$\begin{aligned} M_1^* &= M_2^* = hk_M, & M_3^* &= M_5^* = k_M \frac{l^2}{h\pi^2}, \\ M_4^* &= k_M\left(h + \frac{l^2}{h\pi^2}\right), & M_6^* &= hk_M + \frac{\alpha}{h}, \qquad M_7^* = \frac{\alpha}{h}. \end{aligned} \tag{3.24}$$

From the relations (3.19) and (3.23) we have the following first-order differential inequality

$$\sigma|K(x_1)| + K'(x_1) \leq 0 \qquad \text{for all } x_1 \in [0, L], \tag{3.25}$$

where

$$\frac{1}{\sigma} = \max_{i=1,2,\ldots,7}\left(\frac{M_i^*}{M_i}\right). \tag{3.26}$$

We now proceed to integrate this inequality. To this end we note, from the relation (3.19), that the function K is decreasing on $[0, L]$ and moreover, from the boundary conditions we have $K(L) = 0$ and hence $K(x_1)$ is an acceptable measure of the amplitude of the steady-state vibration.

By integration we obtain the estimate (3.7) and the proof of the Theorem 1 is complete.

Let us discuss further the case of a semi-infinite rectangular plate (the case when $L \to \infty$).

For this purpose we define the energetic measure

$$\mathcal{K}(x_1') = \int_{x_1'}^{\infty}\int_0^l [h^2 w_{\alpha,\beta}\bar{w}_{\alpha,\beta} + (w_1 + w_{3,1})(\bar{w}_1 + \bar{w}_{3,1}) + h^2\phi\bar{\phi} + h^4\phi_{,1}\bar{\phi}_{,1}]dx_1 dx_2. \tag{3.27}$$

If $K(x_1) \geq 0$ for all $x_1 \in [0, \infty)$, then we have the spatial decay estimate (3.7). From this estimate and the inequality (3.19), we can say that the energetic measure $\mathcal{K}(x_1)$ exists and is finite.

If there exists such $x_1^* \in [0, \infty)$ that $K(x_1^*) < 0$, since $K(\cdot)$ is a decreasing function on $[0, \infty)$, then we have

$$K(x_1) < 0 \qquad \text{for all } x_1 \in [x_1^*, \infty). \tag{3.28}$$

In consequence, the differential inequality (3.25) leads to the inequality

$$K'(x_1) - \sigma K(x_1) \leq 0 \qquad \text{for all } x_1 \in [x_1^*, \infty), \tag{3.29}$$

and in consequence we have the estimate

$$-K(x_1) \geq -K(x_1^*) \exp[\sigma(x_1 - x_1^*)] > 0. \tag{3.30}$$

Using this estimate and the inequality (3.23) we observe that, in this case, $\mathcal{K}(x_1)$ is infinite.

The above results are embodied in the following Phragmèn–Lindelöf alternative result.

Theorem 2. *In the context of a semi-infinite rectangular plate we have the following alternative: a) for the amplitudes having a finite volume energetic measure* $\mathcal{K}(x_1)$*, the* $K(x_1)$*, as given by* (3.4)*, is an acceptable measure which decays spatially faster than the exponential* $\exp(-\sigma x_1)$*, or b) for the amplitudes having an infinite volume energetic measure* $\mathcal{K}(x_1)$*, the* $-K(x_1)$ *grows spatially faster than the exponential* $\exp[\sigma(x_1 - x_1^*)]$.

4. Second estimate

In this section we describe a second method for discussing the spatial behaviour of the amplitudes which allows us to extend the class of materials by relaxing the range of elastic moduli.

Throughout this section we will assume the following inequalities:

$$\mu > 0, \quad \lambda < 0, \quad \alpha > 0, \quad \lambda + 2\mu > 0, \quad \xi > \frac{2\beta^2}{\lambda + 2\mu}. \tag{4.1}$$

It is easy to see that the basic equations $(2.13)_1$ can be written in the following form:

$$m_{\beta\alpha,\beta} - S_{\alpha 3} + \rho h^2 \omega^2 w_\alpha = 0, \tag{4.2}$$

where

$$m_{\beta\alpha} = h^2[\mu w_{\alpha,\beta} + (\lambda + \mu) w_{\beta,\alpha} + \beta\phi\delta_{\alpha\beta}]. \tag{4.3}$$

We associate with the problem defined by the relation (2.13) and the boundary condition (2.12), the function

$$I(x_1) = -\int_0^l (\bar{m}_{1\alpha} w_\alpha + m_{1\alpha}\bar{w}_\alpha + \bar{S}_{13} w_3 + S_{13}\bar{w}_3 + \bar{T}_1\phi + T_1\bar{\phi}) dx_2, \tag{4.4}$$

for all $x_1 \in [0, L]$, and we introduce the quantities

$$\begin{aligned} \mathfrak{c}_m &= \min\left\{\lambda+2\mu, \frac{1}{2}\left(\xi+\lambda+2\mu-\sqrt{[\xi-(\lambda+2\mu)]^2+8\beta^2}\right)\right\}, \\ \mathfrak{c}_M &= \max\left\{\lambda+2\mu, \frac{1}{2}\left(\xi+\lambda+2\mu+\sqrt{[\xi-(\lambda+2\mu)]^2+8\beta^2}\right)\right\}, \\ \mathfrak{b}_m &= \min\{-\lambda, \lambda+2\mu\}, \qquad \mathfrak{b}_M = \max\{-\lambda, \lambda+2\mu\}. \end{aligned} \tag{4.5}$$

In what follows we will suppose that ω satisfies the inequality

$$\omega < \omega_2, \tag{4.6}$$

where

$$\omega_2 = \min\left\{\frac{\pi}{l}\sqrt{\frac{\mathfrak{b}_m}{2\rho}}, \frac{\pi}{l}\sqrt{\frac{\mathfrak{c}_m}{2\rho}}, \frac{\pi^2 h}{2l}\sqrt{\frac{\mathfrak{c}_m\mu}{\rho(\mathfrak{c}_m h^2\pi^2+\mu l^2)}}, \frac{1}{lh}\sqrt{\frac{l^2h^2\mathfrak{c}_m+\alpha(l^2+h^2\pi^2)}{\rho\chi}}\right\}. \tag{4.7}$$

Theorem 3. *Suppose that the hypotheses* (4.1) *and* (4.6) *hold true. The functional* $I(x_1)$ *represents an acceptable measure of the solution that satisfies the following exponential decay estimate*

$$0 \le I(x_1) \le I(0)\exp(-\tilde{\sigma}x_1), \tag{4.8}$$

for every $x_1 \in [0, L]$, *where the positive constant* $\tilde{\sigma}$ *depends on the constitutive constants* $\lambda, \mu, \xi, \alpha, \beta$, *the thickness* h_0 *and the width* l *of the plate.*

P r o o f. For the proof of this Theorem we use a method similar to the one used in the proof of Theorem 1.

Thus, by direct differentiation and then using integration by parts, the relations (2.10), (2.11), (4.2) and (4.3) and the boundary conditions (2.12), we have

$$\begin{aligned} I'(x_1) = -2\int_0^l \{&h^2[\mu w_{\alpha,\beta}\bar{w}_{\alpha,\beta}+(\lambda+\mu)w_{\alpha,\beta}\bar{w}_{\beta,\alpha}+\beta(\phi\bar{w}_{\rho,\rho}+\bar{\phi}w_{\rho,\rho}) \\ &+\xi\phi\bar{\phi}]+\alpha h^2\phi_{,\alpha}\bar{\phi}_{,\alpha}+\alpha\phi\bar{\phi}+\mu(w_\alpha+w_{3,\alpha})(\bar{w}_\alpha+\bar{w}_{3,\alpha}) \\ &-\rho\omega^2h^2w_\alpha\bar{w}_\alpha-\rho\omega^2w_3\bar{w}_3-\rho\omega^2h^2\chi\phi\bar{\phi}\}dx_2. \end{aligned} \tag{4.9}$$

We proceed now to obtain the estimates for the $-I'(x_1)$ and $|I(x_1)|$. To this end we define the bilinear forms

$$\begin{aligned}
\Omega(\mathbf{a},\mathbf{b}) = \frac{1}{2}\big\{&(\lambda+2\mu)(a_1\bar{b}_1+\bar{a}_1b_1+a_2\bar{b}_2+\bar{a}_2b_2)\\
&+\beta[\bar{a}_3(b_1+b_2)+a_3(\bar{b}_1+\bar{b}_2)+\bar{b}_3(a_1+a_2)\\
&+b_3(\bar{a}_1+\bar{a}_2)]+\xi(a_3\bar{b}_3+\bar{a}_3b_3)\big\},\\
\Pi(\mathbf{c},\mathbf{d}) = \frac{1}{2}[&\ \mu(c_1\bar{d}_1+\bar{c}_1d_1+c_2\bar{d}_2+\bar{c}_2d_2)\\
&+(\lambda+\mu)(c_1\bar{d}_2+\bar{c}_1d_2+c_2\bar{d}_1+\bar{c}_2d_1)],
\end{aligned} \tag{4.10}$$

for all $\mathbf{a}=\{a_1,a_2,a_3\}$, $\mathbf{b}=\{b_1,b_2,b_3\}$ and for all $\mathbf{c}=\{c_1,c_2\}$, $\mathbf{d}=\{d_1,d_2\}$. Moreover, we introduce the following quadratic forms:

$$\begin{aligned}
\Omega^* &= (\lambda+2\mu)(w_{1,1}\bar{w}_{1,1}+w_{2,2}\bar{w}_{2,2})+\beta(\bar{\phi}w_{\rho,\rho}+\phi\bar{w}_{\rho,\rho})+\xi\phi\bar{\phi},\\
\Pi^* &= \mu(w_{1,2}\bar{w}_{1,2}+w_{2,1}\bar{w}_{2,1})+(\lambda+\mu)(w_{1,2}\bar{w}_{2,1}+\bar{w}_{1,2}w_{2,1}),
\end{aligned} \tag{4.11}$$

in terms of $(w_{1,1},\ w_{2,2},\ \phi)$ and $(w_{1,2},\ w_{2,1})$, respectively.

In view of the assumption (4.1), these two quadratic forms are positive definite. The eigenvalue of the matrix associated with the quadratic forms Π^* are

$$\mathfrak{b}_1=-\lambda,\qquad \mathfrak{b}_2=\lambda+2\mu, \tag{4.12}$$

and the eigenvalues of the matrix associated to the quadratic form Ω^* are

$$\begin{aligned}
\mathfrak{c}_1 &= \lambda+2\mu,\\
\mathfrak{c}_2 &= \frac{1}{2}\left(\xi+\lambda+2\mu-\sqrt{[\xi-(\lambda+2\mu)]^2+8\beta^2}\right),\\
\mathfrak{c}_3 &= \frac{1}{2}\left(\xi+\lambda+2\mu+\sqrt{[\xi-(\lambda+2\mu)]^2+8\beta^2}\right).
\end{aligned} \tag{4.13}$$

The following two inequalities hold:

$$\begin{aligned}
\mathfrak{c}_m(w_{1,1}\bar{w}_{1,1}+w_{2,2}\bar{w}_{2,2}+\phi\bar{\phi}) &\le \Omega^* \le \mathfrak{c}_M(w_{1,1}\bar{w}_{1,1}+w_{2,2}\bar{w}_{2,2}+\phi\bar{\phi}),\\
\mathfrak{b}_m(w_{1,2}\bar{w}_{1,2}+w_{2,1}\bar{w}_{2,1}) &\le \Pi^* \le \mathfrak{b}_M(w_{1,2}\bar{w}_{1,2}+w_{2,1}\bar{w}_{2,1}),
\end{aligned} \tag{4.14}$$

where $\mathfrak{c}_m$, $\mathfrak{c}_M$, $\mathfrak{b}_m$, $\mathfrak{b}_M$ are the quantities defined by the relations (4.5).

Under the hypothesis of Theorem 3, if we use the inequality (3.17) for $a=\sqrt{\mathfrak{c}_m/\mu}h$, we obtain

$$\begin{aligned}
-I'(x_1) \ge \int_0^l [&\tilde{M}_1h^2w_{\alpha,1}\bar{w}_{\alpha,1}+\tilde{M}_2(w_1+w_{3,1})(\bar{w}_1+\bar{w}_{3,1})\\
&+\tilde{M}_3h^2w_{1,2}\bar{w}_{1,2}+\tilde{M}_4h^2w_{2,2}\bar{w}_{2,2}+\tilde{M}_5w_{3,2}\bar{w}_{3,2}\\
&+\tilde{M}_6h^2\phi\bar{\phi}+\tilde{M}_7h^4\phi_{,1}\bar{\phi}_{,1}]dx_2,
\end{aligned} \tag{4.15}$$

where

$$\tilde{M}_1 = 2\mathfrak{c}_m, \qquad \tilde{M}_2 = 2\mathfrak{b}_m, \qquad \tilde{M}_3 = 2\mathfrak{b}_m\left(1-\frac{\omega^2}{\omega_2^2}\right),$$

$$\tilde{M}_4 = \mathfrak{c}_m\left(1-\frac{\omega^2}{\omega_2^2}\right), \qquad \tilde{M}_5 = \frac{\mathfrak{c}_m\mu h^2\pi^2}{2(\mathfrak{c}_m h^2\pi^2+\mu l^2)}\left(1-\frac{\omega^2}{\omega_2^2}\right), \tag{4.16}$$

$$\tilde{M}_6 = 2\left[\mathfrak{c}_m + \frac{\alpha(l^2+\pi^2h^2)}{l^2h^2}\right]\left(1-\frac{\omega^2}{\omega_2^2}\right), \qquad \tilde{M}_7 = 2\frac{\alpha}{h^2}.$$

It is easy to see that, from (4.2), we have

$$\begin{aligned} m_{11}\bar{m}_{11} &= h^2\Omega((s_{11},0,0),(w_{1,1},w_{2,2},\phi)),\\ m_{12}\bar{m}_{12} &= h^2\Pi((w_{2,1},w_{1,2}),(s_{12},0)), \end{aligned} \tag{4.17}$$

which, in view of the Schwarz inequality and the inequalities (4.11), lead to the estimates

$$\begin{aligned} m_{11}\bar{m}_{11} &\le h^4 c_M(w_{1,1}\bar{w}_{1,1}+w_{2,2}\bar{w}_{2,2}+\phi\bar{\phi}),\\ m_{12}\bar{m}_{12} &\le h^4 b_M(w_{1,2}\bar{w}_{1,2}+w_{1,2}\bar{w}_{1,2}). \end{aligned} \tag{4.18}$$

By using the relations $(2.11)_{2,3}$, the above inequalities and the arithmetic-geometric inequality, we deduce

$$\begin{aligned} |I(x_1)| \le \int_0^l [&\tilde{M}_1^* h^2 w_{\alpha,1}\bar{w}_{\alpha,1} + \tilde{M}_2^*(w_1+w_{3,1})(\bar{w}_1+\bar{w}_{3,1})\\ &+ \tilde{M}_3^* h^2 w_{1,2}\bar{w}_{1,2} + \tilde{M}_4^* h^2 w_{2,2}\bar{w}_{2,2} + \tilde{M}_5^* w_{3,2}\bar{w}_{3,2}\\ &+ \tilde{M}_6^* h^2\phi\bar{\phi} + \tilde{M}_7^* h^4\phi_{,1}\bar{\phi}_{,1}]dx_2, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} &\tilde{M}_1^* = c_M h, \qquad \tilde{M}_2^* = b_M h, \qquad \tilde{M}_3^* = hb_M + c_M\frac{l^2}{h\pi},\\ &\tilde{M}_4^* = hc_M + b_M\frac{l^2}{h\pi}, \quad \tilde{M}_5^* = \mu\frac{l^2}{h\pi^2}, \quad \tilde{M}_6^* = hc_M + \frac{\alpha}{h}, \quad \tilde{M}_7^* = \frac{\alpha}{h}. \end{aligned} \tag{4.20}$$

Thus, using the relations (4.12) and (4.15) we obtain the following first-order differential inequality

$$\tilde{\sigma} I(x_1) + I'(x_1) \le 0, \qquad x_1 \in [0,L], \tag{4.21}$$

where

$$\frac{1}{\tilde{\sigma}} = \max_{i=1,2,\dots,7}\left(\frac{\tilde{M}_i^*}{\tilde{M}_i}\right). \tag{4.22}$$

By direct integration we obtain the estimate (4.8) and the proof of Theorem 3 is complete.

Following a procedure with that above Section, the results may be easy extended to a semi-infinite rectangular plate.

Theorem 4. *In the context of a semi-infinite rectangular plate we have the following alternative: a) for the amplitudes having a finite volume energetic measure* $\mathcal{K}(x_1)$, *the* $I(x_1)$, *as given by* (4.3), *is an acceptable measure which decays spatially faster than the exponential* $\exp(-\tilde{\sigma}x_1)$, *or b) for the amplitudes having an infinite volume energetic measure* $\mathcal{K}(x_1)$, *the* $-I(x_1)$ *grows spatially faster than the exponential* $\exp[\tilde{\sigma}(x_1 - x_1^*)]$, *where* x_1^* *is lower so that* $I(x_1^*) < 0$.

5. Concluding remarks

In the present paper we have introduce two measures, (3.4) and (4.4), to study the spatial behaviour of the amplitude of harmonic vibration in a rectangular Mindlin-type plate filled by an isotropic, homogeneous elastic material with voids.

Our purpose was to describe a method which allows us to study the spatial behaviour of a large class of materials. We note that:

a) For the class of materials characterized by the inequalities

$$\mu > 0, \quad \alpha > 0, \quad -2\mu < \lambda \leq -\mu, \quad \xi > \frac{2\beta^2}{\lambda + 2\mu}, \tag{5.1}$$

we have the measure $I(\cdot)$;

b) For the class of materials characterized by the inequalities

$$\mu > 0, \quad \alpha > 0, \quad -\mu < \lambda, \quad \xi > \frac{2\beta^2}{2\lambda + 3\mu}, \tag{5.2}$$

we have the measure $K(\cdot)$.

Thus, the class covered by our study is the class of elastic materials with voids, for which the constitutive constants satisfies the assumptions (2.14). The method presented here is believed to be used successfully for the study of materials with negative Poisson's ratio which are most useful in biomechanics (porous implants for example) [27], [29].

If we use the common writing $(2.11)_1$ of the equations $(2.13)_1$ and the method described in the Sections 3 and 4, for the function defined by

$$J(x_1) = -\int_0^l (\bar{S}_{1i} w_i + S_{1i} \bar{w}_i + \bar{H}_1 \phi + H_1 \bar{\phi}) dx_2, \tag{5.3}$$

for all $x_1 \in [0, L]$, we can establish the following theorem.

THEOREM 5. *For the class of materials characterized by the inequalities* (2.5), *the functional* $J(x_1)$ *represents an acceptable measure of the solution that satisfies the following exponential decay estimate*

$$0 \le J(x_1) \le J(0)\exp(-\hat{\sigma}x_1), \tag{5.4}$$

for every $x_1 \in [0, L]$ *and* $\omega < \omega_3$, *where the positive constants* $\hat{\sigma}$ *and* ω_3 *are given by*

$$\begin{aligned} &\frac{1}{\hat{\sigma}} = \max_{i=1,2,\ldots,6}\left(\frac{\hat{M}_i^*}{\hat{M}_i}\right), \\ &\omega_3 = \min\left\{\frac{\pi}{l}\sqrt{\frac{\varkappa_m}{2\rho}}, \frac{\pi^2 h}{2l}\sqrt{\frac{\varkappa_m\mu}{\rho(\varkappa_m h^2\pi^2+\mu l^2)}}, \frac{1}{lh}\sqrt{\frac{l^2h^2\varkappa_m+\alpha(l^2+h^2\pi^2)}{\rho\chi}}\right\}, \end{aligned} \tag{5.5}$$

with

$$\begin{aligned} \varkappa_m &= \min\left\{2\mu, \frac{1}{2}\left(\xi+\lambda+\mu-\sqrt{[\xi-(\lambda+\mu)]^2+4\beta^2}\right)\right\}, \\ \varkappa_M &= \max\left\{2\mu, \frac{1}{2}\left(\xi+\lambda+\mu+\sqrt{[\xi-(\lambda+\mu)]^2+4\beta^2}\right)\right\}, \\ \hat{M}_1 &= 2\varkappa_m, \quad \hat{M}_2 = 2\mu\left(1-\frac{\omega^2}{\omega_3^2}\right), \quad \hat{M}_3 = \varkappa_m\left(1-\frac{\omega^2}{\omega_3^2}\right), \\ \hat{M}_4 &= \frac{\varkappa_m\mu h^2\pi^2}{2(\varkappa_m h^2\pi^2+\mu l^2)}\left(1-\frac{\omega^2}{\omega_2^2}\right), \\ \hat{M}_5 &= 2\left[\varkappa_m+\frac{\alpha(l^2+\pi^2h^2)}{l^2h^2}\right]\left(1-\frac{\omega^2}{\omega_3^2}\right), \qquad \hat{M}_6 = 2\frac{\alpha}{h^2}, \\ \hat{M}_1^* &= h\varkappa_M, \qquad \hat{M}_2^* = h\mu+\varkappa_M\frac{l^2}{h\pi}, \\ \hat{M}_3^* &= \varkappa_M\mu+\mu\frac{l^2}{h\pi}, \quad \hat{M}_4^* = \mu\frac{l^2}{h\pi^2}, \quad \hat{M}_5^* = h\varkappa_M+\frac{\alpha}{h}, \quad \hat{M}_6^* = \frac{\alpha}{h}. \end{aligned} \tag{5.6}$$

The class of materials discussed in the above theorem is more restrictive and it is included in the class of materials considered in the Secs. 3 and 4.

For a fixed type of material, other criteria to choose the most appropriate measure can be represented by the critical frequencies ω_i or by the speeds of decay $\sigma, \tilde{\sigma}, \hat{\sigma}$.

The results are extended to a semi–infinite rectangular plate to obtain appropriate alternatives of the Phragmèn–Lindelöf type.

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References

1. J.W. Nunziato and S.C. Cowin, *A non-linear theory of elastic materials with voids*, Arch. Rational Mech. Anal., **72**, 175–201, 1979.
2. M.A. Goodman and S.C. Cowin, *A continuum theory for granular materials*, Arch. Rational Mech. Anal., **44**, 249–266, 1972.
3. S.C. Cowin and J.W. Nunziato, *Linear elastic materials with voids*, J. Elasticity, **13**, 125–147, 1983.
4. D. Ieşan, *A theory of thermoelastic materials with voids*, Acta Mechanica, **60**, 67–89, 1986.
5. D. Ieşan and M. Ciarletta, *Non-classical elastic solids, Longman Scientific and Technical*, Harlow, Essex, UK and John Wiley & Sons, Inc., New York 1993.
6. D. Ieşan, *Thermoelastic models of continua*, Kluwer Academic Publishers, Boston/Dordrecht/London 2004.
7. R.D. Mindlin, *Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates*, J. Appl. Mech., **18**, 31–38, 1951.
8. E. Reissner, *On the theory of bending of elastic plates*, J. Math. Phys., **23**, 184–191, 1944.
9. C. Constanda, *A mathematical analysis of bending of plates with transverse shear deformation*, Langman Scientific&Technical, 1990.
10. P. Schiavone and C. Constanda, *Oscillation problems in thin plates with transverse shear deformation*, SIAM J. Appl. Math., **53**, 5, 1253–1263, 1993.
11. P. Schiavone and R.J. Tait, *Steady time-harmonic oscillations in a linear thermoelastic plate model*, Quart. Appl. Math., **LIII**, 215–223, 1995.
12. M. Ciarletta, *On the spatial behaviour of the transient and steady-state solutions in thin plates with transverse shear deformation*, Int. J. Engn. Sci., **40**, 485–498, 2002.
13. C. D'Apice and S. Chiriţă, *Spatial behaviour in a Mindlin-type thermoelastic plate*, Quarterly of Applied Mathematics, **61**, 783–796, 2003.
14. C. D'Apice, *A mathematical study of the spatial behaviour of the time-harmonic ascillations in a thermoelastic rectangular plate*, An. Şt. Univ. Al. I. Cuza Iaşi, ser. Matematica, **51**, 9–23, 2005.
15. E. Scarpetta, *Minimum principle for the bending problem of elastic plates with voids*, Int. J. Engn. Sci., **40**, 1317–1327, 2002.
16. M. Birsan, *A bending theory of porous thermoelastic plates*, Journal of Thermal Stresses, **26**, 67–90, 2003.

17. M. BIRSAN, *On the bending equations for elastic plates with voids*, Math. Mech. Solids, **12**, 40–57, 2007.

18. M. BIRSAN, *Transient and steady-state solutions for porous thermoelastic plates*, An. Şt. Univ A.I. Cuza Iasi, ser. Matematica, **52**, 159–176, 2006.

19. S. CHIRIŢĂ and M. CIARLETTA, *Time-weighted surface power function method for the study of spatial behaviour in dynamics of continua*, European Journal of Mechanics, A/Solids, **18**, 915–933, 1999.

20. S. CHIRIŢĂ, *Further results on the spatial behavior in linear elastodynamics*, An. Şt. Univ. Iasi, ser. Matematica, **50**, 289–304, 2004.

21. S. CHIRIŢĂ, *Some growth and decay estimates for a cylinder made of an elastic material with voids*, Rev. Roum. Math. Pures et Appl., **39**, 17–26, 1994.

22. S. CHIRIŢĂ, *On some exponential decay estimates for porous elastic cylinder*, Arch. Mechanics, **56**, 199–212, 2004.

23. D. IEŞAN and R. QUINTANILLA, *Decay estimates and energy bounds for porous elastic cylinders*, Z. angew. Math. Phys., **46**, 268–281, 1995.

24. M. CIARLETTA, S. CHIRIŢĂ and F. PASSARELLA, *Some results on the spatial behaviour in linear porous elasticity*, Arch. Mechanics, **57**, 43–65, 2005.

25. R. LAKES, *Foam structure with negative Poisson's ratio*, Science, **235**, 1038–1040, 1987.

26. R.S. LAKES, *No contractile obligations*, Nature, **358**, 713–714, 1992.

27. R. LAKES, *A broader view of membranes*, Nature, **414**, 503–504, 2001.

28. N.K. MCGUIRE, *Negative-coefficient materials can point the way to positive value in the right matrixes*, Today's chemist at work, American Chemical Society, Nov., 24–28, 2002.

29. L.B. PARK and R.S. LAKES, *Biomaterials*, third edition, Springer, 2007.

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