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Reflection and transmission of transient acoustic waves with oblique incidence

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The aim of the paper is to determine, within the time domain, the waves produced by an oblique incident wave at the interface between two homogeneous half-spaces. By following the acoustic approximation, the wave solutions for the Fourier transform of the displacement field in a viscous fluid are established in a form which generalizes the concept of plane wave. Next the reflection-transmission problem, associated with the interface between an inviscid fluid and a viscous one, is investigated. The incident wave is supposed to propagate in the inviscid fluid. The reflected and transmitted waves, in the time domain, are eventually determined in two particular cases, namely that of normal incidence on a viscous half-space and that of oblique incidence, beyond the critical angle, on an inviscid half-space. In the first case it follows that, provided an approximation of band-limited data holds for the incident wave, the reflected and the transmitted waves are given by linear combinations of the values of the incident wave and of its time derivative. In the second case, the reflected (transmitted) wave is shown to be the sum of a term proportional to the incident wave and another one, proportional to the Hilbert transform of (a convolution of) the incident wave.

1. Introduction

The aim of this paper is to determine, within the time domain, the waves produced at the interface between two homogeneous half-spaces, by an oblique incident wave. The governing equations are those of the acoustic approximation, a model widely applied, e.g., in seismology and in marine exploration (see [1] and refs. therein). However, for the sake of generality, the underlying body is allowed to be a viscous fluid.

The subject is of interest in many respects. First, there are relatively few results for reflection-transmission (RT) problems in the time domain. Quite often the RT problems are investigated within the frequency domain or, rather, for time-harmonic waves. This is motivated by the relatively simpler calculations and by the observation that, for linear problems, the inverse Fourier transform allows us to obtain the results in the time domain. While conceptually such is the case, in practice the inverse Fourier transformation may be quite involved. Of course, the inverse Fourier transform applies if the solution in the frequency domain is known for every frequency. This requires a detailed analysis of the frequency-dependence of time-harmonic wave solutions. Secondly, RT problems associated with a viscous half-space cannot be solved directly within the time domain. The analysis within the frequency domain shows that the reflection and transmission coefficients are in fact functions of the frequency and this is the main reason why the inverse Fourier transform does not provide a closed-form solution in the time domain. It is then of interest to find closed-form solutions in particular conditions or approximations. Thirdly, there is a renewed attention to direct and inverse problems for wave propagation in dissipative media (see [2]). This gives a further motivation for the investigation of equations and solutions associated with acoustic waves in viscous fluids, perhaps the paradigm of dissipative continua.

By following the acoustic approximation, we first determine the wave solutions, for the Fourier transform of the displacement field in a viscous fluid, in a form which generalizes the concept of plane wave. Next we solve the RT problem associated with the interface between an inviscid fluid and a viscous one. The incident wave is coming from the inviscid fluid. The reflected and transmitted waves, in the time domain, are eventually evaluated in two particular cases, namely that of normal incidence on a viscous half-space and that of oblique incidence, beyond the critical angle, on an inviscid half-space. In the first case we find that, provided the incident wave justifies an approximation of band-limited data, the reflected and the transmitted waves are given by linear combinations of the values of the incident wave and of its time derivative. In the second case, the reflected wave is shown to be the sum of a term proportional to the incident wave and another one proportional to the Hilbert transform of the incident wave. A similar result holds for the transmitted wave with the Hilbert transform of a convolution of the incident wave.

2. Preliminaries and the acoustic approximation

Let $\Omega \subseteq \mathbb{R}^3$ be the region occupied by the fluid under consideration. The symbol $\mathbf{x} \in \Omega$ denotes the position vector relative to a chosen origin, $\mathbf{v}$ is the velocity, $\hat{\rho}$ the mass density, $\hat{p}$ the pressure. Also, ∇ is the gradient operator, Δ the Laplacian.

The mass density $\hat{\rho}$ and the velocity $\mathbf{v}$, on $\Omega \times \mathbb{R}$, are subject to the continuity equation

$$\partial_t \hat{\rho} + \nabla \cdot (\hat{\rho} \mathbf{v}) = 0 \tag{2.1}$$

and the equation of motion

$$\hat{\rho}[\partial_t + (\mathbf{v}\cdot\nabla)]\mathbf{v} = \nabla\cdot\mathbf{T}, \tag{2.2}$$

where $\mathbf{T}$ is the stress tensor and the body force is disregarded. Since we have in mind viscous fluids, we take $\mathbf{T}$ in the form

$$\mathbf{T} = -\hat{p}\mathbf{1} + \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] + \lambda(\nabla\cdot\mathbf{v})\mathbf{1},$$

μ, λ being the viscosity coefficients and the superscript $\dagger$ denoting transpose. Let

$$\hat{\rho} = \rho + \varrho, \qquad \hat{p} = p + \wp$$

and regard ρ, p as the density and pressure at equilibrium. As it is often the case, we let ρ and p be constants. The stress tensor $\mathbf{T}$ can then be written as

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\mathcal{T}}, \qquad \boldsymbol{\mathcal{T}} := -\wp\mathbf{1} + \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] + \lambda(\nabla\cdot\mathbf{v})\mathbf{1}.$$

Henceforth we follow the acoustic approximation. Accordingly, quantities which are nonlinear in $\varrho, \wp$ and $\mathbf{v}$ are disregarded. By (2.1) and (2.2) we have

$$\partial_t\varrho + \rho\nabla\cdot\mathbf{v} = 0, \tag{2.3}$$

$$\rho\partial_t\mathbf{v} = -\nabla\wp + \mu\nabla\cdot[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] + \lambda\nabla(\nabla\cdot\mathbf{v}). \tag{2.4}$$

Let $\mathbf{u}$ be the displacement so that $\mathbf{v} = \partial_t\mathbf{u}$, in the linear approximation. We may then replace (2.3) with

$$\varrho = -\rho\nabla\cdot\mathbf{u}. \tag{2.5}$$

Moreover, the pressure $\hat{p}$ is regarded as a function of $\hat{\rho}$ only and hence, by (2.5),

$$\nabla\wp = -\rho c^2\nabla(\nabla\cdot\mathbf{u}) \tag{2.6}$$

where

$$c^2 = \frac{d\hat{p}}{d\hat{\rho}}(\rho).$$

Eq. (2.4) becomes

$$\rho\partial_t^2\mathbf{u} = \rho c^2\nabla(\nabla\cdot\mathbf{u}) + \partial_t[(\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \mu\Delta\mathbf{u}]. \tag{2.7}$$

By (2.6) we have

$$\wp = -\rho c^2\nabla\cdot\mathbf{u}, \tag{2.8}$$

to within an inessential additive function, of time t, which is set to be zero.

A function f, on the space-time $\Omega \times \mathbb{R}$, is a scalar plane wave propagating in the direction of $\mathbf{q}$, with speed $1/|\mathbf{q}|$, if

$$f(\mathbf{x}, t) = F(t - \mathbf{q} \cdot \mathbf{x})$$

for some function F on $\mathbb{R}$. To look for solutions to (2.7) as plane waves is much too restrictive. Hence we consider generalized plane waves in the form

$$g(\mathbf{x}, t) = G(z, t - \mathbf{m} \cdot \mathbf{x}),$$

where $\mathbf{m}$ is perpendicular to the z-axis, for any function G on $\mathbb{R}^2$. By an appropriate choice of the axes we let $\mathbf{m}$ be directed along the x-axis and write

$$g(\mathbf{x}, t) = G(z, t - \xi x), \qquad \xi \in \mathbb{R}.$$

The parameter ξ is the inverse of what is often called the trace velocity (see [3], p. 124). The Fourier transform $\tilde{g}$, with respect to time t,

$$\tilde{g}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} g(\mathbf{x}, t) \exp(-i\omega t) dt,$$

gives

$$\tilde{g}(\mathbf{x}, \omega) = \exp(-i\omega\xi x) \tilde{G}(z, \omega).$$

By applying the Fourier transform to (2.8) and (2.7) we find that

$$-\rho\omega^2 \tilde{\mathbf{u}} = [\rho c^2 + i\omega(\mu + \lambda)] \nabla(\nabla \cdot \tilde{\mathbf{u}}) + i\omega\mu\Delta\tilde{\mathbf{u}}, \tag{2.9}$$

$$\tilde{\wp} = -\rho c^2 \nabla \cdot \tilde{\mathbf{u}}. \tag{2.10}$$

Equation (2.9) can be viewed as a homogeneous system of second-order linear differential equations for (the components of) $\tilde{\mathbf{u}}$. Once $\tilde{\mathbf{u}}$ is determined, Eq. (2.10) provides $\tilde{\wp}$. In the following analysis, it is understood that $\omega \neq 0$.

Motivated by the concept of generalized plane waves, we look for solutions $\tilde{\mathbf{u}}$, $\tilde{\wp}$ in the form

$$\tilde{\mathbf{u}}(\mathbf{x}, \omega) = \mathbf{d}(\omega) \exp(-i\omega\xi x) U(z, \omega), \qquad \tilde{\wp}(\mathbf{x}, \omega) = \exp(-i\omega\xi x) P(z, \omega).$$

Hence, letting

$$\beta = 1 + i\omega(\mu + \lambda)/\rho c^2$$

we obtain the component form of (2.9) as

$$-\rho\omega^2 d_x U + i\omega\xi\rho c^2 \beta(-i\omega\xi d_x + d_z\partial_z)U - i\omega\mu d_x(-\omega^2\xi^2 + \partial_z^2)U = 0, \tag{2.11}$$

$$[\omega^2(\rho - i\omega\mu\xi^2)U + i\omega\mu\partial_z^2 U] d_y = 0, \tag{2.12}$$

$$-\rho\omega^2 d_z U - \rho c^2 \beta(-i\omega\xi d_x + d_z\partial_z)\partial_z U - i\omega\mu d_z(-\omega^2\xi^2 + \partial_z^2)U = 0. \tag{2.13}$$

2.1. Transverse wave

Let $d_y \neq 0$ and denote by $\boldsymbol{\tau}$ the sought solution for the polarization $\mathbf{d}$. Equation (2.12) implies that

$$U'' = \omega^2 \left(\xi^2 + i \frac{\rho}{\omega\mu} \right) U. \tag{2.14}$$

Hence U takes the form

$$U(z) = U_\tau \exp(-i\omega\sigma_\tau z) \tag{2.15}$$

where U_τ is a constant, parameterized by ω, whereas

$$\sigma_\tau^2 = -i\frac{\rho}{\omega\mu} - \xi^2.$$

Irrespective of the value of ξ, we have $\sigma_\tau^2 \in \mathbb{C}$ and

$$\operatorname{sgn} \Im\sigma_\tau^2 = -\operatorname{sgn}\omega.$$

Since, as it will be the case, $\exp[i\omega(t-\sigma_\tau z)]$ has to represent a forward-propagating wave, we have to require that $\Re\sigma_\tau > 0$ whence we have $\operatorname{sgn}\Im\sigma_\tau = -\operatorname{sgn}\omega$. Otherwise, we might say that the wave must decay from the source whence $\operatorname{sgn}\Im\sigma_\tau = -\operatorname{sgn}\omega$. The explicit form of σ_τ is then given by

$$\sigma_\tau = \frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{\xi^4 + \rho^2/\omega^2\mu^2} - \xi^2} - i\sqrt{\sqrt{\xi^4 + \rho^2/\omega^2\mu^2} + \xi^2}\ \operatorname{sgn}\omega\right].$$

The two Eqs. (2.11) and (2.13) then simplify to

$$\tau_x\xi + \tau_z\sigma_\tau = 0. \tag{2.16}$$

We regard this solution as a transverse wave and say that it is represented by

$$\tilde{\mathbf{u}}(\mathbf{x},\omega) = \boldsymbol{\tau} U_\tau \exp(-i\omega(\xi x + \sigma_\tau z)), \tag{2.17}$$

where the polarization $\boldsymbol{\tau}$ is perpendicular to $\nabla[\exp(-i\omega\xi x)U(z,\omega)]$. It is the superposition of a wave polarized along y,

$$\boldsymbol{\tau}_1 = \tau_y \mathbf{e}_y,$$

and one in the (x,z)-plane,

$$\boldsymbol{\tau}_2 = \tau_x\mathbf{e}_x + \tau_z\mathbf{e}_z,$$

and subject to (2.16).

Formally, the transverse wave solution is characterized by the condition (2.14). We now look for solutions when (2.14) does not hold.

2.2. Longitudinal wave

Let

$$\omega^2(\rho - i\omega\mu\xi^2)U + i\omega\mu U'' \neq 0. \tag{2.18}$$

Hence (2.12) implies that $d_y = 0$. Denote by $\mathbf{l}$ the sought solution for the polarization $\mathbf{d}$. The system (2.11)–(2.13) then provides

$$\{\omega^2[\rho(\xi^2 c^2\beta - 1) + i\omega\mu\xi^2]U - i\omega\mu U''\}l_x + i\omega\xi\rho c^2\beta\, U'\, l_z = 0,$$

$$i\omega\xi\rho c^2\beta\, U'\, l_x + [\omega^2(-\rho + i\omega\mu\xi^2)U - (\rho c^2 + i\omega\mu)U'']l_z = 0.$$

Both equations are linear in U and are parameterized by l_x, l_z. Hence we let

$$U = U_l \exp(-i\omega\sigma_l z),$$

where U_l is a constant, and look for the value of σ_l. It follows at once from (2.18) that

$$\gamma := -\rho + i\omega\mu(\xi^2 + \sigma_l^2) \neq 0. \tag{2.19}$$

Upon substitution we have

$$(\gamma + \xi^2\rho c^2\beta)l_x + \xi\sigma_l\rho c^2\beta\, l_z = 0, \tag{2.20}$$

$$\xi\sigma_l\rho c^2\beta\, l_x + (\gamma + \sigma_l^2\rho c^2\beta)l_z = 0. \tag{2.21}$$

The algebraic system (2.20)-(2.21) has non-trivial solutions for l_x, l_z provided that the determinant vanishes. Since $\gamma \neq 0$, this amounts to

$$(\xi^2 + \sigma_l^2)\left(1 + i\omega\frac{2\mu + \lambda}{\rho c^2}\right) - \frac{1}{c^2} = 0$$

whence

$$\sigma_l^2 = -\xi^2 + \frac{1}{c^2}\frac{1}{1 + i\omega(2\mu + \lambda)/\rho c^2}. \tag{2.22}$$

Irrespective of the value of ξ, we have

$$\operatorname{sgn}\Im\sigma_l^2 = -\operatorname{sgn}\omega.$$

We require that $\exp(i\omega(t - \sigma_l z))$ represents a forward-propagating wave and hence we let $\Re\sigma_l > 0$. By (2.22) we can write

$$\sigma_l^2 = a + ib$$

where

$$a = -\xi^2 + \frac{1}{c^2}\frac{1}{1+\omega^2(2\mu+\lambda)^2/\rho^2c^4}, \qquad b = -\frac{1}{c^2}\frac{\omega(2\mu+\lambda)/\rho c^2}{1+\omega^2(2\mu+\lambda)^2/\rho^2c^4}.$$

The requirement $\Re\sigma_l > 0$ implies that

$$\sigma_l = \frac{1}{\sqrt{2}}\Big(\sqrt{\sqrt{a^2+b^2}+a} - i\sqrt{\sqrt{a^2+b^2}-a}\ \operatorname{sgn}\omega\Big).$$

The components l_x, l_z of the polarization vector ($l_y = 0$) are then given by any of the Eqs. (2.20), (2.21).

The solution

$$\tilde{\mathbf{u}}(\mathbf{x},\omega) = \mathbf{l}U_l \exp(-i\omega(\xi x + \sigma_l z)), \tag{2.23}$$

is regarded as a longitudinal wave in which, by (2.20) or (2.21), $\mathbf{l}$ satisfies

$$l_x\sigma_l - l_z\xi = 0, \tag{2.24}$$

what amounts to

$$\mathbf{l}\times\nabla[\exp(-i\omega\xi x)U(z,\omega)] = 0.$$

This is so because, by (2.22),

$$\sigma_l^2\rho c^2\beta = \rho - i\omega\mu(\xi^2+\sigma_l^2) - \xi^2\rho c^2\beta$$

and hence (2.20) and (2.21) provide (2.24).

Remark. It is a common feature of the transverse and longitudinal waves so determined that

$$\Re\sigma > 0, \qquad \Im\sigma = -|\Im\sigma|\operatorname{sgn}\omega.$$

Hence the dependence on the coordinates is of the form

$$\exp(-i\omega(\xi x + \Re\sigma\ z))\exp(-|\omega\Im\sigma|z).$$

Transverse and longitudinal waves are then inhomogeneous (see [4, 5] and [6]). They propagate in the direction $(\xi, \Re\sigma)$ of the (x,z) plane and decay with z at the rate $|\omega\,\Im\sigma|$. This in turn shows that, in viscous fluids, plane waves are not allowed. If, rather, $\xi = 0$ then both the phase and amplitude are constant at the (same) planes of constant z. Though the $\xi = 0$ solution may be viewed as a plane wave in the frequency domain, the corresponding function in the time domain is not a plane wave.

2.3. Waves in inviscid fluids

Also as a check of consistency, we derive the wave solutions in inviscid fluids by letting $\mu, \lambda = 0$ in (2.11)–(2.13). By (2.12) it follows that $Ud_y = 0$. Equations (2.11) and (2.13) reduce to

$$(\xi^2 c^2 - 1)d_x + \xi\sigma c^2 d_z = 0,$$
$$\xi\sigma c^2 d_x + (\sigma^2 c^2 - 1)d_z = 0.$$

The vanishing of the determinant gives

$$\sigma^2 = \frac{1}{c^2} - \xi^2 =: a_0$$

whence

$$\sigma = \pm\frac{1}{\sqrt{2}}(\sqrt{|a_0| + a_0} - i\sqrt{|a_0| - a_0}\ \operatorname{sgn}\omega), \tag{2.25}$$

$+$ and $-$ being associated to forward- and backward-propagating waves. Moreover,

$$\mathbf{d} \times \nabla[\exp(-i\omega\xi x)U(z,\omega)] = 0.$$

In inviscid fluids only longitudinal waves occur with

$$\tilde{\mathbf{u}} = \mathbf{l}U \exp(-i\omega(\xi x + \sigma z)), \tag{2.26}$$

where $\mathbf{l}$ is subject to

$$\sigma l_x = \xi l_z. \tag{2.27}$$

If $\xi^2 < 1/c^2$, $a_0 > 0$, then

$$\sigma = \sqrt{|a_0|};$$

the solution is an undamped wave which propagates with speed

$$\frac{1}{\sigma} = \frac{c}{\sqrt{1 - \xi^2 c^2}}$$

in the z direction. Meanwhile $\exp(-i\omega(\xi x + \sigma z))$ shows that

$$\frac{1}{\sqrt{\xi^2 + \sigma^2}} = \frac{1}{\sqrt{\xi^2 + (1 - \xi^2 c^2)/c^2}} = c$$

is the wave speed.

If $\xi^2 > 1/c^2$, $a_0 < 0$, then

$$\sigma = -i\sqrt{|a_0|}\ \operatorname{sgn}\omega;$$

the solution is an evanescent wave with decay rate

$$\omega\sigma = \frac{|\omega|}{c}\sqrt{\xi^2 c^2 - 1}.$$

The transition between undamped and evanescent waves occurs when $\xi^2 = 1/c^2$, namely at the critical angle.

Remark. In viscous fluids it is uncommon to deal with critical angles and this is due to the fact that viscosity makes all wave solutions be damped propagating waves for any value of the trace velocity $1/\xi$.

3. Reflection-transmission problem

Denote by $[\![f]\!]$ the jump of a field f across an interface. Let $\mathbf{t}$ be the traction. For fixed interfaces, the balance of linear momentum and energy provides the jump conditions

$$[\![\mathbf{t}]\!] = 0, \qquad [\![\mathbf{v}]\!] \cdot \mathbf{t} = 0. \tag{3.1}$$

Examine the consequences of (3.1) at the interface $z = 0$ between an inviscid fluid ($z < 0$) and a viscous fluid ($z > 0$). Denote by the subscripts $\pm$ the limit values as $z \to 0_\pm$. The normal to the interface is $\mathbf{e}_z$, the unit vector of the z-axis. Hence the jump conditions (3.1) provide

$$-p_-\mathbf{e}_z = \mathbf{T}_+\mathbf{e}_z, \tag{3.2}$$

$$v_z|_- = v_z|_+. \tag{3.3}$$

At equilibrium, i.e. $\hat{p} = p, \mathbf{v} = 0$, Eq. (3.2) implies that

$$p_- = p_+.$$

Hence (3.2) simplifies to

$$-\wp_-\mathbf{e}_z = \boldsymbol{\mathcal{T}}_+\mathbf{e}_z. \tag{3.4}$$

Application of the Fourier transform to (3.4) and (3.3) gives

$$-\tilde{\wp}_-\mathbf{e}_z = \tilde{\boldsymbol{\mathcal{T}}}_+\mathbf{e}_z, \tag{3.5}$$

$$\tilde{u}_z|_- = \tilde{u}_z|_+. \tag{3.6}$$

We now state the RT problem in the frequency domain. The incident wave comes from $z < 0$. Since the half-space $z < 0$ is occupied by an inviscid fluid, the incident and the reflected waves are longitudinal, see (2.26). The known incident wave is taken to be homogeneous, with $\xi_I^2 < 1/c_-^2$ and $\sigma_I = \sqrt{1/c_-^2 - \xi_I^2} > 0$. It is a longitudinal wave, of the form (2.26), as well as the reflected wave, with $\sigma_I > 0$, $\xi_I^2 < 1/c_-^2$ (homogeneous waves). In the half-space $z > 0$ two waves are transmitted of the form (2.17) and (2.23). To simplify the notation we let $\mathbf{l}_T, \boldsymbol{\tau}$ stand for $\mathbf{l}U_l, \boldsymbol{\tau}U_\tau$ and let $\mathbf{l}_T, \boldsymbol{\tau}$ depend on ω. The subscripts, or superscripts, I, R, T indicate quantities pertaining to the incident, reflected, transmitted waves. Hence, by (2.17), (2.23) and (2.26) we can write $\tilde{\mathbf{u}}$ as

$$\tilde{\mathbf{u}}(\mathbf{x}, \omega) = \begin{cases} \mathbf{l}_I \exp(-i\omega(\xi_I x + \sigma_I z)) + \mathbf{l}_R \exp(-i\omega(\xi_R x - \sigma_R z)), & z < 0, \\ \boldsymbol{\tau} \exp(-i\omega(\xi_\tau x + \sigma_\tau z)) + \mathbf{l}_T \exp(-i\omega(\xi_l x + \sigma_l z)), & z > 0. \end{cases}$$

Correspondingly, the stress $\tilde{\boldsymbol{\mathcal{T}}}$ is given by

$$\tilde{\boldsymbol{\mathcal{T}}} = -\tilde{\wp}\mathbf{1} + i\omega\mu[\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^\dagger] + i\omega\lambda(\nabla \cdot \tilde{\mathbf{u}})\mathbf{1}.$$

The RT problem consists in the determination of $\mathbf{l}_R, \boldsymbol{\tau}, \mathbf{l}_T$ in terms of $\mathbf{l}_I$, subject to the continuity conditions (3.5)–(3.6).

By (3.5) we have

$$\tilde{\mathcal{T}}_{xz}|_+ = 0, \qquad \tilde{\mathcal{T}}_{yz}|_+ = 0,$$

whence

$$(\tau_x\sigma_\tau + \tau_z\xi_\tau)\exp(-i\omega\xi_\tau x) + (l_x\sigma_l + l_z\xi_l)\exp(-i\omega\xi_l x) = 0,$$

$$\tau_y \exp(-i\omega\xi_\tau x) = 0.$$

The arbitrariness of x implies that

$$\tau_y = 0, \qquad \xi_\tau = \xi_l,$$

$$\tau_x\sigma_\tau + \tau_z\xi_\tau + l_x\sigma_l + l_z\xi_l = 0. \tag{3.7}$$

Let ξ_T stand for the common value of ξ_τ, ξ_l and $\boldsymbol{\tau}$ for $\boldsymbol{\tau}_2$, i.e. $\boldsymbol{\tau}_1 = 0$, so that

$$\tilde{\mathbf{u}}(\mathbf{x}, \omega) = \begin{cases} \mathbf{l}_I \exp(-i\omega(\xi_I x + \sigma_I z)) + \mathbf{l}_R \exp(-i\omega(\xi_R x - \sigma_R z)), & z < 0, \\ \boldsymbol{\tau} \exp(-i\omega(\xi_T x + \sigma_\tau z)) + \mathbf{l}_T \exp(-i\omega(\xi_T x + \sigma_l z)), & z > 0. \end{cases}$$

The requirement (3.6) results in

$$l_z^I \exp(-i\omega\xi_I x) + l_z^R \exp(-i\omega\xi_R x) = \tau_z \exp(-i\omega\xi_T x) + l_z^T \exp(-i\omega\xi_T x). \tag{3.8}$$

The arbitrariness of x implies that

$$\xi_I = \xi_R = \xi_T =: \xi \tag{3.9}$$

and hence

$$\sigma_R = \sigma_I .$$

As a consequence, (3.8) provides

$$l_z^I + l_z^R = \tau_z + l_z^T, \tag{3.10}$$

whereas (3.7) becomes

$$\tau_x \sigma_\tau + l_x^T \sigma_l + \xi(\tau_z + l_z^T) = 0, \qquad \mu \neq 0. \tag{3.11}$$

The remaining condition of (3.5), namely $-\tilde{\wp}_- = \tilde{\mathcal{T}}_{zz}|_+$, results in

$$\begin{aligned}\rho_- c_-^2 (l_x^I \xi + l_z^I \sigma_I + l_x^R \xi - l_z^R \sigma_I) \\ = (\rho_+ c_+^2 + i\lambda\omega)(l_x^T \xi + l_z^T \sigma_l) + i2\mu\omega(\sigma_\tau \tau_z + \sigma_l l_z^T).\end{aligned} \tag{3.12}$$

The condition (3.9) might have been assumed by invoking Snell's law.

The incident wave is represented by

$$\tilde{\mathbf{u}}_I(\mathbf{x}, \omega) = \mathbf{l}_I \exp[-i\omega(\xi x + \sigma_I z]$$

where $\mathbf{l}_I$ is parameterized by ω and is subject to

$$l_x^I \sigma_I - \xi l_z^I = 0, \qquad \sigma_I = \sqrt{(1/c^2) - \xi^2}.$$

The ratio

$$\frac{l_x^I}{l_z^I} = \frac{\xi}{\sigma_I}$$

is independent of ω.

The RT problem amounts to the determination of the five unknowns τ_x, τ_z, l_x^T, l_z^T, l_z^R by solving the system of five Eqs. (3.10)–(3.12) and (2.16), (2.24) parameterized by l_z^I, ξ, ω. We find that

$$l_z^T = \frac{2i\rho_- \sigma_l \omega \mu (\sigma_\tau^2 - \xi^2)}{\rho_+ \rho_- \sigma_l - \omega^2 \mu^2 \sigma_I [(\sigma_\tau^2 - \xi^2)^2 + 4\sigma_l \sigma_\tau \xi^2]} l_z^I, \tag{3.13}$$

$$l_z^R = \frac{\rho_+ \rho_- \sigma_l + \omega^2 \mu^2 \sigma_I [(\sigma_\tau^2 - \xi^2)^2 + 4\sigma_l \sigma_\tau \xi^2]}{\rho_+ \rho_- \sigma_l - \omega^2 \mu^2 \sigma_I [(\sigma_\tau^2 - \xi^2)^2 + 4\sigma_l \sigma_\tau \xi^2]} l_z^I, \tag{3.14}$$

$$\tau_z = \frac{4i\rho_- \sigma_l \omega \mu \xi^2}{\rho_+ \rho_- \sigma_l - \omega^2 \mu^2 \sigma_I [(\sigma_\tau^2 - \xi^2)^2 + 4\sigma_l \sigma_\tau \xi^2]} l_z^I, \tag{3.15}$$

and by (2.16) and (2.24),

$$l_x^T = \frac{\xi}{\sigma_l} l_z^T, \qquad \tau_x = -\frac{\sigma_\tau}{\xi} \tau_z .$$

Once we know $\mathbf{l}_R$ and $\mathbf{l}_T, \tau$ we determine the reflected and transmitted waves $\mathbf{u}_R, \mathbf{u}_T$ in the time domain. For definiteness we restrict attention to the z components. By the inverse Fourier transform we have

$$u_z^R(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) \tilde{u}_z^R(\mathbf{x}, \omega) d\omega$$

and the like for $\mathbf{u}_z^T$. Because

$$\tilde{u}_z^R(\mathbf{x}, \omega) = l_z^R(\omega) \exp[-i\omega(\xi x - \sigma_I z)],$$

we have

$$u_z^R(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \xi x + \sigma_I z)] l_z^R(\omega) d\omega. \tag{3.16}$$

For the transmitted waves, two modes occur, the longitudinal and the transverse ones. We have

$$u_z^T(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\exp[i\omega(t - \xi x - \sigma_\tau z)] \tau_z(\omega) + \exp[i\omega(t - \xi x - \sigma_l z)] l_z^T(\omega)\} d\omega. \tag{3.17}$$

The general relations so obtained are now applied in two simple cases which allow us to obtain definite results in the time domain.

Remark. We have tacitly assumed that the propagation vectors of the waves lie in the common plane (x, z). We might start without such an assumption and show, by means of (3.1), that the propagation vectors are required to lie in a common plane.

4. Normal incidence on a viscous half-space

The relations for normal incidence follow by letting $\xi = 0$. First we find that $\tau_x, \tau_z = 0$ and hence only longitudinal waves occur. Moreover, $l_x^R, l_x^T = 0$. The relations (3.13) and (3.14) for l_z^T and l_z^R reduce to

$$l_z^T = \frac{2i\rho_-\sigma_l\omega\mu\sigma_\tau^2}{\rho_+\rho_-\sigma_l - \omega^2\mu^2\sigma_I\sigma_\tau^4} l_z^I, \tag{4.1}$$

$$l_z^R = \frac{\rho_+\rho_-\sigma_l + \omega^2\mu^2\sigma_I\sigma_\tau^4}{\rho_+\rho_-\sigma_l - \omega^2\mu^2\sigma_I\sigma_\tau^4} l_z^I. \tag{4.2}$$

We now investigate the form of the reflection and transmission coefficients

$$R(\omega) = \frac{l_z^R}{l_z^I}(\omega), \qquad T(\omega) = \frac{l_z^T}{l_z^I}(\omega).$$

By (4.2) we can write

$$R(\omega) = \frac{1 + \omega^2 \mu^2 \sigma_I \sigma_\tau^4 / \rho_- \rho_+ \sigma_l}{1 - \omega^2 \mu^2 \sigma_I \sigma_\tau^4 / \rho_- \rho_+ \sigma_l}.$$

Moreover,

$$\frac{\sigma_\tau^4}{\sigma_l} = -\frac{\rho_+^2 c_+}{\omega^2 \mu^2} w, \qquad \sigma_I = \frac{1}{c_-},$$

where

$$w = \frac{1}{c_+ \sigma_l}, \qquad w^2 = 1 + i\omega(2\mu + \lambda)/\rho_+ c_+^2.$$

Let w_r, w_i stand for $\Re w, \Im w$. Since $\Re \sigma_l > 0$ and $\operatorname{sgn} \Im \sigma_l = -\operatorname{sgn} \omega$, we have

$$w_r > 0, \qquad \operatorname{sgn} w_i = \operatorname{sgn} \omega.$$

Indeed we obtain

$$w = \frac{1}{\sqrt{2}} \left(\sqrt{\sqrt{1+\alpha^2} + 1} + i \sqrt{\sqrt{1+\alpha^2} - 1} \, \operatorname{sgn} \omega \right),$$

where

$$\alpha = \kappa \omega, \qquad \kappa = \frac{2\mu + \lambda}{\rho_+ c_+^2}.$$

Letting

$$\nu := \frac{\rho_+ c_+}{\rho_- c_-}$$

we can write

$$R(\omega) = \frac{1 - \nu w}{1 + \nu w},$$

whence

$$R(\omega) = \frac{1 - \nu^2 |w|^2}{1 + 2\nu w_r + \nu^2 |w|^2} - 2i\nu \frac{|w_i| \operatorname{sgn} \omega}{1 + 2\nu w_r + \nu^2 |w|^2}. \tag{4.3}$$

Likewise, by

$$T(\omega) = \frac{2\rho_- c_-}{\rho_- c_- + \rho_+ c_+ w}$$

we obtain

$$T(\omega) = \frac{2(1 + \nu w_r)}{(1 + \nu w_r)^2 + \nu^2 w_i^2} + 2i\nu \frac{|w_i| \operatorname{sgn} \omega}{(1 + \nu w_r)^2 + \nu^2 w_i^2}. \tag{4.4}$$

The dependence of R and T on ω, as determined in (4.3) and (4.4), does not allow a closed-form solution for the reflected and transmitted wave in the time domain. Nevertheless, an interesting result follows if the incident wave allows us to work with band-limited data. We assume that

$$\kappa|\omega| \ll 1$$

and hence we find that

$$w_r \simeq 1 + \alpha^2/8 \simeq 1, \qquad w_i \simeq \kappa\omega/2,$$

and

$$\sigma_l \simeq \frac{1}{c_+}\left(1 - i\frac{1}{2}\kappa\omega\right).$$

As a consequence we let

$$R(\omega) = \frac{1-\nu}{1+\nu} + i\frac{\nu\kappa}{(1+\nu)^2}\omega,$$

$$T(\omega) = \frac{2}{1+\nu} + i\frac{2\nu\kappa}{(1+\nu)^2}\omega.$$

We now determine the reflected and transmitted waves $\mathbf{u}_R$, $\mathbf{u}_T$ in the time domain. Look at (3.16) in the case of normal incidence ($\xi = 0$), at $x = 0$. Since

$$u_z^R(0,\omega) = l_z^R(\omega), \qquad l_z^R(\omega) = R(\omega) l_z^I(\omega), \qquad l_z^I(\omega) = u_z^I(0,\omega),$$

we can write

$$u_z^R(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\omega(t + \sigma_I z)] R(\omega) \tilde{u}_z^I(0,\omega) d\omega, \qquad z < 0.$$

Since $i\omega\tilde{u}_z^I(0,\omega)$ is the Fourier transform of $\dot{u}_z^I(0,t)$, we obtain

$$u_z^R(z,t) = \frac{1-\nu}{1+\nu} u_z^I(0, t + \sigma_I z) + \frac{\nu\kappa}{(1+\nu)^2}\dot{u}_z^I(0, t + \sigma_I z), \qquad z < 0. \tag{4.5}$$

Likewise, by (3.17) we can write

$$u_z^T(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\omega(t - z/c_+)] \exp(-\kappa\omega^2 z/2c_+) T(\omega)\tilde{u}_z^I(0,\omega) d\omega, \qquad z > 0.$$

By the convolution theorem we obtain

$$u_z^T(z,t) = \sqrt{\frac{c_+}{2\pi\kappa z}}\int_{-\infty}^{\infty} \exp[-c_+(t - z/c_+ - \eta)^2/2\kappa z] \times \left[\frac{2}{1+\nu}u_z^I(0,\eta) + \frac{2\nu\kappa}{(1+\nu)^2}\dot{u}_z^I(0,\eta)\right] d\eta, \qquad z > 0. \tag{4.6}$$

The reflected wave u_z^R, at (z,t), is a linear combination of the incident wave u_z^I and of the time derivative $\dot{u}_z^I$, at $z=0$ at time $t+z/c_-$. The transmitted wave u_z^T is the convolution of a Gaussian kernel with a linear combination of the incident wave u_z^I and of the time derivative $\dot{u}_z^I$. The result (4.6) shows the dependence of u_z^T on the depth z.

5. Oblique incidence on an inviscid half-space

The limit case where the half-space $z>0$ is occupied by an inviscid fluid cannot be obtained directly from (3.13)–(3.15) by letting $\mu,\lambda=0$. This is so also because σ_T,σ_l are unbounded as $\mu,\lambda\to 0$. In inviscid fluids only longitudinal waves occur. Hence $\boldsymbol{\tau}=0$ and (3.11) does not apply because now $\mu=0$. By (2.24) and (2.27), the vectors $\mathbf{l}^I,\mathbf{l}^R,\mathbf{l}^T$ are subject to

$$(5.1)\qquad l_x^I\sigma_I-\xi l_z^I=0,\qquad l_x^R\sigma_I+\xi l_z^R=0,\qquad l_x^T\sigma_l-\xi l_z^T=0.$$

Upon substitution for l_x^I,l_x^R,l_z^T and letting $\mu,\lambda=0,\boldsymbol{\tau}=0$ we obtain from (3.10), (3.12) that

$$\begin{aligned} l_z^I+l_z^R &= l_z^T,\\ l_z^I-l_z^R &= \frac{\rho_+\sigma_I}{\rho_-\sigma_T}l_z^T. \end{aligned}$$

Hence we find that the reflection and the transmission coefficients, R and T, are given by

$$R=\frac{\rho_-\sigma_T-\rho_+\sigma_I}{\rho_-\sigma_T+\rho_+\sigma_I},\qquad T=\frac{2\rho_-\sigma_T}{\rho_-\sigma_T+\rho_+\sigma_I}.$$

If σ_T is real and positive, then R and T are constants, independent of ω. The passage to the time domain through the inverse Fourier transform is obvious,

$$u_z^R(0,t)=R\,u_z^I(0,t),\qquad u_z^T(0,t)=T\,u_z^I(0,t).$$

5.1. Incidence beyond the critical angle

Letting $c_+>c_-$ we assume that $\xi^2c_+^2>1$, which means that the incidence angle is greater than the critical value. The transmitted wave is evanescent and

$$\sigma_l=\sigma_T=-i\sqrt{\xi^2-1/c_+^2}\ \operatorname{sgn}\omega.$$

As a consequence, R and T depends on ω through the sign. Letting

$$\epsilon=\frac{\rho_+\sigma_I}{\rho_-|\sigma_T|}=\frac{\rho_+c_+}{\rho_-c_-}\sqrt{\frac{1-\xi^2c_-^2}{\xi^2c_+^2-1}}$$

we find that

$$R(\omega) = \frac{1 - i\epsilon \operatorname{sgn}\omega}{1 + i\epsilon \operatorname{sgn}\omega}, \qquad T(\omega) = \frac{2}{1+\epsilon^2}(1 - i\epsilon \operatorname{sgn}\omega).$$

Both R and T are parameterized by ξ through ϵ. By applying the inverse Fourier transform to $\tilde{u}_z^R$ and $\tilde{u}_z^T$ we obtain the reflected wave and the transmitted wave in the time domain, namely

$$u_z^R(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \xi x + \sigma_I z)] R(\omega) \tilde{u}_z^I(0, \omega) d\omega, \tag{5.2}$$

$$u_z^T(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \xi x - \sigma_T z)] T(\omega) \tilde{u}_z^I(0, \omega) d\omega. \tag{5.3}$$

5.2. The reflected wave

Since R can be written as

$$R(\omega) = \frac{1-\epsilon^2}{1+\epsilon^2} - i\frac{2\epsilon}{1+\epsilon^2} \operatorname{sgn}\omega,$$

substitution in (5.2) gives

$$u_z^R(\mathbf{x}, t) = \frac{1-\epsilon^2}{1+\epsilon^2} u_z^I(0, t - \xi x + \sigma_I z) \\ - i\frac{2\epsilon}{1+\epsilon^2} \int_{-\infty}^{\infty} \exp[i\omega(t - \xi x + \sigma_I z)] \tilde{u}_z^I(0, \omega) \operatorname{sgn}\omega \, d\omega.$$

Since

$$i\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{sgn}\omega \ \exp(i\omega t) d\omega = -\frac{1}{\pi t},$$

by the convolution theorem and a change of variable we obtain

$$- i \int_{-\infty}^{\infty} \exp[i\omega(t - \xi x + \sigma_I z)] \tilde{u}_z^I(0, \omega) \operatorname{sgn}\omega \, d\omega \\ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t - \zeta} u_z^I(0, \zeta - \xi x + \sigma_I z) d\zeta \\ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t - t' - \xi x + \sigma_I z} u_z^I(0, t') dt'.$$

As a consequence, the reflected wave, at the place $\mathbf{x}$ and time t, is given by

$$(5.4)\qquad u_z^R(\mathbf{x},t) = \frac{1-\epsilon^2}{1+\epsilon^2} u_z^I(0, t-\xi x+\sigma_I z) - \frac{1}{\pi}\frac{2\epsilon}{1+\epsilon^2}\int_{-\infty}^{\infty} \frac{1}{t'-(t-\xi x+\sigma_I z)} u_z^I(0,t')dt'.$$

The investigation of the reflected wave produced by a dependence on $\operatorname{sgn}\omega$ traces back to ARONS and YENNIE [7] (see also [8]–[10]) who studied the effect of pulse distortion, of

$$F(t) = \begin{cases} 0, & t<0, \\ F_0 \exp(-\lambda t), & t>0, \end{cases}$$

by a constant $\pi/2$ phase shift in each frequency component.

The result (5.4) shows that the reflected wave is plane and homogeneous in that $u_z^R(\mathbf{x},t)$ is a function of $t-\xi x+\sigma_I z$. The first term is merely proportional to u_z^I, evaluated at the retarded time $t-\xi x+\sigma_I z$. Concerning the second term, observe that for a function f on $\mathbb{R}$,

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx'$$

is the Hilbert transform of f at x. Accordingly we can write (5.4) as

$$u_z^R(\mathbf{x},t) = \frac{1-\epsilon^2}{1+\epsilon^2} u_z^I(0,t-\xi x+\sigma_I z) - \frac{2\epsilon}{1+\epsilon^2}(\mathcal{H}u_z^I)(0,t-\xi x+\sigma_I z).$$

By (5.1) we have

$$u_x^R \sigma_I + \xi u_z^R = 0$$

and hence we find u_x^R as

$$u_x^R(\mathbf{x},t) = -\frac{\xi}{\sigma_I} u_z^R(\mathbf{x},t).$$

5.3. The transmitted wave

The transmitted wave is longitudinal ($\boldsymbol{\tau}=0$). By (3.17) and the relation

$$u_z^T(0,\omega) = l_z^T(\omega) = T(\omega) l_z^I(\omega) = T(\omega) u_z^I(0,\omega)$$

we have

$$u_z^T(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\omega(t-\xi x-\sigma_T z)]T(\omega)u_z^I(0,\omega)d\omega, \qquad z>0.$$

Things are now more involved because σ_T is imaginary and

$$\exp(-i\omega\sigma_T z) = \exp(-\sqrt{\xi^2-1/c_+^2}\,|\omega|z) = \exp(-|\sigma_T||\omega|z).$$

Hence, because

$$T(\omega) = \frac{2}{1+\epsilon^2} - i\frac{2\epsilon}{1+\epsilon^2}\operatorname{sgn}\omega,$$

we have

$$u_z^T(\mathbf{x},t) = u_1(\mathbf{x},t)+u_2(\mathbf{x},t), \qquad z>0,$$

where

$$u_1(\mathbf{x},t) = \frac{1}{\pi(1+\epsilon^2)}\int_{-\infty}^{\infty} \exp[i\omega(t-\xi x)]\exp(-|\sigma_T||\omega|z)\,u_z^I(0,\omega)d\omega,$$

$$u_2(\mathbf{x},t) = -i\frac{\epsilon}{\pi(1+\epsilon^2)}\int_{-\infty}^{\infty} \exp[i\omega(t-\xi x)]\exp(-|\sigma_T||\omega|z)\ \operatorname{sgn}\omega\ u_z^I(0,\omega)d\omega.$$

The inverse Fourier transform of $\exp(-|\sigma_T||\omega|z)$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-|\sigma_T||\omega|z)d\omega = \frac{1}{\pi}\frac{|\sigma_T|z}{|\sigma_T|^2z^2+t^2}, \qquad z>0.$$

Hence, by the convolution theorem we find that

$$u_1(\mathbf{x},t) = \frac{1}{\pi(1+\epsilon^2)}\int_{-\infty}^{\infty} G(z,t-\xi x,\zeta)u_z^I(0,\zeta)d\zeta,$$

where

$$G(z,\eta,\zeta) = \frac{|\sigma_T|z}{|\sigma_T|^2z^2+(\eta-\zeta)^2}.$$

To within a factor, u_2 is the inverse Fourier transform, at $t-\xi x$, of the product of $-i\operatorname{sgn}\omega$ and $\exp(-|\sigma_T||\omega|z)u_z^I(0,\omega)$. The inverse Fourier transform of $-i\operatorname{sgn}\omega$

is $1/\pi t$. The inverse Fourier transform of $\exp(-|\sigma_T||\omega|z)u_z^I(0,\omega)$, at time ζ, is given by the convolution

$$\int_{-\infty}^{\infty} G(z,\zeta,\eta)u_z^I(0,\eta)d\eta.$$

Hence, using again the convolution theorem we find that

$$u_2(\mathbf{x},t) = \frac{2\epsilon}{\pi(1+\epsilon^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{t-\xi x-\zeta} G(z,\zeta,\eta)\, u_z^I(0,\eta)d\eta\, d\zeta.$$

The dependence of u_1 and u_2 on the depth z is provided by the kernel G which, though in a different context, appears in [11].

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