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Non-local Korteweg stresses from kinetic theory point of view

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The aim of the paper is to elaborate a kinetic theory having the non-local Korteweg equations for non-isothermal liquid-vapour systems as the hydrodynamic limit. This is the topic of the second part of the paper. The first part of our paper is devoted to presentation of the Korteweg equations. We write the non-local Korteweg equations in a conservative form and we discuss the relations between them and the local ones.

1. Introduction

There coexist two concepts of liquid-vapour systems. The first one assumes that the liquid phase and the vapour phase are separated by a sharp interface endowed with energy and entropy. According to the second concept, the liquid and vapour are separated by a thin layer of finite thickness. Across this layer the flow parameters vary smoothly. J. D. van der Waals [1] was the author who contributed fundamentally to the development of this concept by formulating a pressure formula suitable both for the liquid and gaseous state of the fluid.

The next step was made by J. D. Korteweg [2] who contributed to the dynamics of liquid-vapour systems by modifying the stress tensor. To model the spatial interaction effects in the transition zone he introduced terms containing the first and second order derivatives of the density. But as shown by Dunn and Serrin [3], Korteweg's modification of the Navier-Stokes equations was incompatible with the continuum thermodynamics. In the cited paper, Dunn and Serrin gave correct and rigorous fundamentals of what is now called the Korteweg capillarity theory. Their analysis is based on the principal ideas of modern continuum mechanics and thermodynamics. In a recent study Mehrabadi, Cowin and Massoudi [4] extend the ideas of Dunn and Serrin to dipolar continua.

Gouin [5], and Casal and Gouin [6], [7] derived the Korteweg theory, which they call the thermocapillarity equations, by assuming that the internal energy depends not only on the density and entropy but also on their spatial gradients, up to some final order and by applying the principle of virtual work.

The common thing of all these models is that they are based on phenomenological continuum thermodynamics. The aim of the present paper is to show that the Korteweg equations can be deduced from a kinetic theory. But prior to doing that we present the Korteweg equations. This is done in Sec. 2 for the case of the perfect fluid without dissipative terms. The full system of equations with the dissipative terms included is given at the end of Sec. 3. These equations are of higher order then the Navier-Stokes equations since they contain third order spatial derivatives of the density function. Ch. Rohde [8] is presumably the first author who tried to remove this disadvantage. Using a variational argument he obtained, in the isothermal case, non-local Navier-Stokes-Korteweg equations in the form (2.3), (2.9) with the force term (3.1), (3.2). In Sec. 3, for the first time we formulate the non-local, integral form of the full non-isothermal Korteweg equations and write them in the conservative form. Also we show that in a limit process, the local i. e. differential form of Navier-Stokes-Korteweg can be obtained. The aim of the second part of the paper, Sections 4–6, is derivation of the non-local Korteweg equations via a kinetic approach. In Sec. 4 we formulate the kinetic equations. In Sec. 5 we present briefly the derivation of their hydrodynamic limit, where some theorems concerning the properties of the collision operators are proven. Having determined an approximate form of the distribution function, we evaluate the stresses and the heat flux vector. We also indicate the entropy functional for the obtained equations. The structure of the equations obtained is discussed in Sec. 6.

2. The Korteweg model

We start our presentation of the Korteweg model from the equations given in a paper by Gouin [5]. He assumes that the internal energy per unit mass is a function of the mass density ρ, the entropy s and their spatial gradients up to the order k-1, i.e.

$$\varepsilon = \varepsilon\left(\rho, \rho_i, \rho_{ij} \ldots, \rho_{i_1 i_2 \ldots i_{k-1}}, s, s_i, s_{ij}, \ldots, s_{i_1 i_2 \ldots i_{k-1}}\right), \tag{2.1}$$

where we denote

$$\rho_i = \frac{\partial \rho}{\partial x_i}, \quad \rho_{ij} = \frac{\partial^2 \rho}{\partial x_i \partial x_j}, \quad \ldots \quad, \rho_{i_1 i_2 \ldots i_{k-1}} = \frac{\partial^{n-1} \rho}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_{k-1}}},$$

$$s_i = \frac{\partial s}{\partial x_i}, \quad s_{ij} = \frac{\partial^2 s}{\partial x_i \partial x_j}, \quad \ldots \quad, s_{i_1 i_2 \ldots i_{k-1}} = \frac{\partial^{n-1} s}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_{k-1}}}.$$

In the present paper we are not interested in a such general formulation and limit ourselves to the case of $k = 3$ (fluids of grade 3 or of the third gradient) and confine our considerations to a particular case of (2.1), when the internal energy is of the form

$$\varepsilon = e(\rho, s) + \varsigma(\rho, \rho_i, \rho_{ij}), \tag{2.2}$$

where e is interpreted as the classical internal energy. Applying the principle of the variation of the Hamiltonian action, GOUIN [5] obtained the equations of motion which read

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}, \tag{2.3}$$

$$\frac{D\mathbf{u}}{Dt} = \theta \nabla s - \nabla \left(\epsilon + \frac{\pi}{\rho} \right) \tag{2.4}$$

and

$$\frac{Ds}{Dt} = 0, \tag{2.5}$$

where $\mathbf{u}$ is the velocity vector, $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the substantial derivative, θ is the temperature defined by

$$\theta = \frac{\partial e}{\partial s}, \tag{2.6}$$

and π is defined by

$$\pi = p - \rho\varsigma + \frac{\partial(\rho\varsigma)}{\partial\rho} - \rho \frac{\partial}{\partial x_i}\left(\frac{\partial(\rho\varsigma)}{\partial\rho}\right) + \rho\,\frac{\partial^2}{\partial x_i \partial x_i}\left(\frac{\partial(\rho\varsigma)}{\partial\rho_{ij}}\right), \tag{2.7}$$

where p is the classical pressure

$$p = \rho^2 \frac{\partial e}{\partial \rho}. \tag{2.8}$$

Equation (2.4) can be written in the form:

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla \cdot (p\mathbf{I}) = \rho \mathbf{f}, \tag{2.9}$$

where the components f_i, $i = 1, 2, 3$ of the self-consistent force $\mathbf{f}$ are the following:

$$f_i = -\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial\rho}(\rho\varsigma) - \frac{\partial}{\partial x_j}\left(\rho\frac{\partial\varsigma}{\partial\rho_j}\right) + \frac{\partial^2}{\partial x_j \partial x_k}\left(\rho\frac{\partial\varsigma}{\partial\rho_{jk}}\right)\right], \quad j = 1, 2, 3. \tag{2.10}$$

Next, combining Eqs. (2.3), (2.5), and (2.9) we obtain

$$\rho \frac{D}{Dt}\left(e + \frac{\mathbf{u}^2}{2}\right) + \nabla \cdot (p\mathbf{u}) = \rho \mathbf{f} \cdot \mathbf{u}. \tag{2.11}$$

Equations (2.9), (2.11) have the usual form of the Navier–Stokes equations with a self-consistent force field **f.**

Our purpose is to give a kinetic motivation to the theory of liquid-vapour systems. But every kinetic theory is able to yield only differential equations of the form (2.3), (2.9) and (2.11) (with dissipative terms added) and produces neither variational principles nor integral conservation laws, that is it does not supply any of the starting points of the continuum theory. That is why we have to refer the deduced macroscopic equations to the existing equations derived from continuum thermodynamics, for example we have to refer them to the equations of the form (2.4)–(2.7). To this end we have to find the form of the interfacial internal energy ς compatible with the known form of the force **f**. It means that the problem we face is inverse to that consisting in the passage from (2.4), (2.5) to (2.9), (2.11). However, the inverse problem has infinitely many solutions, i. e. there are infinitely many functionals ς corresponding to the same force **f.** For instance, for

$$\varsigma = \varsigma^{(1)} = \frac{A}{2}\frac{(\nabla\rho)^2}{\rho} \tag{2.12}$$

and

$$\varsigma = \varsigma^{(2)} = -\frac{A}{2}\Delta\rho, \tag{2.13}$$

where A is a positive constant, we obtain the same expression for **f**, namely

$$\mathbf{f} = \mathbf{f}^{(1)} = \mathbf{f}^{(2)} = -A\nabla\Delta\rho. \tag{2.14}$$

2.1. Conservative form of equations

Let us notice that the Eqs. (2.9), (2.11) with the force **f** given by (2.10) can be written in the conservative form. We call the Korteweg tensor every tensor **K** satisfying the relation

$$\frac{\partial}{\partial x_j}K_{ij} = \rho f_i. \tag{2.15}$$

This tensor is not uniquely determined, but it always exists. Using (2.15) we can rewrite the momentum balance equation (2.9) in the conservative form

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla\cdot(p\mathbf{I} - \mathbf{K}) = \mathbf{0}. \tag{2.16}$$

We introduce the following definition: given ς and **K**, every vector **w** such that the relation

$$-\rho\frac{D\varsigma}{Dt} + \frac{\partial}{\partial x_j}\left(K_{ij}u_i + w_j\right) = \rho f_i u_i \tag{2.17}$$

holds, is called the vector of the interstitial working (DUNN and SERRIN [3]). Of course, it is also non-uniquely defined. However, for the chosen form of the interfacial energy ς there exist always $\mathbf{K}$ and $\mathbf{w}$ such that Eqs. (2.16), (2.17) are satisfied. Owing to (2.17), the energy balance equation (2.11) takes the conservative form

$$\rho\frac{D}{Dt}\left(e+\varsigma+\frac{\mathbf{u}^2}{2}\right)+\nabla\cdot(\mathbf{u}\cdot(p\mathbf{I}-\mathbf{K})-\mathbf{w})=0. \tag{2.18}$$

EXAMPLES. *Continued*

The Korteweg stress tensor corresponding to $\varsigma^{(1)}$ is usually taken in the form

$$\mathbf{K}^{(1)}=-\frac{A}{2}\left[\left((\nabla\rho)^2+2\rho\Delta\rho\right)\mathbf{I}-2\nabla\rho\otimes\nabla\rho\right], \tag{2.19}$$

and that corresponding to $\varsigma^{(2)}$ is assumed as

$$\mathbf{K}^{(2)}=\frac{A}{6}\left[\left(2\rho\Delta\rho-(\nabla\rho)^2\right)\mathbf{I}+2\left(2\rho\nabla\nabla\rho-\nabla\rho\otimes\nabla\rho\right)\right], \tag{2.20}$$

although simpler forms are also possible. The vector of the interstitial working compatible with the model (2.12), (2.19) is of the form

$$\mathbf{w}^{(1)}=A\rho(\nabla\cdot\mathbf{u})\nabla\rho, \tag{2.21}$$

whereas the one compatible with (2.13), (2.20) is

$$\mathbf{w}^{(2)}=\frac{A}{6}\rho\nabla\cdot(\rho\mathbf{\Pi}), \tag{2.22}$$

the components Π_{ij} of the tensor $\mathbf{\Pi}$ are given by

$$\Pi_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+(\nabla\cdot\mathbf{u})\,\delta_{ij},\quad i,j=1,2,3. \tag{2.23}$$

3. Non-local theory of capillarity

Now let us consider the case when the force $\mathbf{f}$ is of the form

$$\mathbf{f}=-\nabla\Phi, \tag{3.1}$$

where the potential Φ is taken in the form proposed by VAN DER WAALS [1]

$$\Phi=\frac{v}{\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho\left(\mathbf{y}\right)-\rho\left(\mathbf{x}\right)\right]d\mathbf{y}. \tag{3.2}$$

Here υ is a positive constant and ϕ is the long-range intermolecular potential. We assume that this function satisfies the following conditions:

$$\varphi \in C^1\left([0,\infty[\right), \qquad \int\limits_0^\infty y^4 \left|\varphi(y)\right| dy < \infty, \qquad \int\limits_0^\infty y^3 \left|\varphi'(y)\right| dy < \infty,$$

$$\lim_{y\to 0} y^3 \varphi(y) = 0, \qquad \lim_{y\to\infty} y^4 \varphi(y) = 0.$$

ROHDE [8], for the isothermal case only, derived the momentum conservation Eq. (2.9) with the non-local force term (3.1), (3.2) from a variational principle. We show now that Eq. (2.9), where $\mathbf{f}$ is given by (3.1), (3.2), can be written in the conservative form (2.16). Indeed, we check rather easily that

$$\rho\mathbf{f} = \nabla\cdot\mathbf{K}, \tag{3.3}$$

where

$$\mathbf{K} = \frac{\upsilon}{2\beta^4}\int\limits_0^1 d\lambda \int \frac{\mathbf{y}\mathbf{y}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right)\left[\rho\left(\mathbf{x}+\lambda\mathbf{y}\right)\rho\left(\mathbf{x}+(\lambda-1)\,\mathbf{y}\right) - \rho^2\left(\mathbf{x}\right)\right] d\mathbf{y} \tag{3.4}$$

is the non-local Korteweg stress tensor. Hence we can write Eq. (2.9) in the form (2.16). Next, using the mass conservation equation we have

$$\begin{aligned}
&\rho(\mathbf{x})\,\frac{\upsilon}{2\beta^3}\frac{D}{Dt}\int \varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho(\mathbf{y})-\rho(\mathbf{x})\right] D\mathbf{y} \\
&\qquad\qquad - \frac{\upsilon}{2\beta^3}\nabla\cdot\int\limits_0^1 D\lambda \int \frac{\mathbf{y}\mathbf{y}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right) \\
&\quad\cdot\left[\rho\left(\mathbf{x}+\lambda\mathbf{y}\right)\rho\left(\mathbf{x}+(\lambda-1)\,\mathbf{y}\right)\mathbf{u}\left(\mathbf{x}+\lambda\mathbf{y}\right) - \rho^2(\mathbf{x})\,\mathbf{u}(\mathbf{x})\right] d\mathbf{y} \\
&= \rho(\mathbf{x})\,\frac{\upsilon}{2\beta^3}\frac{D}{Dt}\int \varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho(\mathbf{y})-\rho(\mathbf{x})\right] d\mathbf{y} \\
&+ \rho(\mathbf{x})\,\frac{\upsilon}{2\beta^3}\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho(\mathbf{y})\cdot\mathbf{u}(\mathbf{y})\, d\mathbf{y} \\
&+ \rho(\mathbf{x})\,\mathbf{u}(\mathbf{x})\cdot\frac{\upsilon}{2\beta^3}\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho(\mathbf{y})\, d\mathbf{y}
\end{aligned}$$

$$+\frac{\upsilon}{2\beta^4}\int\limits_0^1 d\lambda\int\frac{\mathbf{yy}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right)d\mathbf{y}:\nabla\left(\rho^2\mathbf{u}\right)$$

$$\begin{aligned}
&=\rho\left(\mathbf{x}\right)\frac{\upsilon}{\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\nabla_y\left(\rho\left(\mathbf{y}\right)\mathbf{u}\left(\mathbf{y}\right)\right)-\nabla_x\left(\rho\left(\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}\right)\right)\right]d\mathbf{y}\\
&\quad+\rho\left(\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}\right)\cdot\frac{\upsilon}{2\beta^4}\int\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho\left(\mathbf{y}\right)d\mathbf{y}\\
&\quad-\rho\left(\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}\right)\nabla\rho\left(\mathbf{x}\right)\frac{\upsilon}{2\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)d\mathbf{y}\\
&\quad+\rho\left(\mathbf{x}\right)\frac{\upsilon}{2\beta^3}\int\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho\left(\mathbf{y}\right)\cdot\mathbf{u}\left(\mathbf{y}\right)d\mathbf{y}\\
&\quad+\rho\left(\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}\right)\cdot\frac{\upsilon}{2\beta^3}\int\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho\left(\mathbf{y}\right)d\mathbf{y}\\
&\quad+\nabla\left(\rho^2\mathbf{u}\right):\frac{\upsilon}{\beta^3}\int\frac{\mathbf{yy}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right)d\mathbf{y}.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
&\rho\left(\mathbf{x}\right)\frac{\upsilon}{2\beta^3}\frac{D}{Dt}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho\left(\mathbf{y}\right)-\rho\left(\mathbf{x}\right)\right]d\mathbf{y}\\
&\quad-\frac{\upsilon}{2\beta^3}\nabla\cdot\int\limits_0^1 d\lambda\int\frac{\mathbf{yy}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right)\\
&\quad\cdot\Big[\rho\left(\mathbf{x}+\lambda\mathbf{y}\right)\rho\left(\mathbf{x}+\left(\lambda-1\right)\mathbf{y}\right)\mathbf{u}\left(\mathbf{x}+\lambda\mathbf{y}\right)-\rho^2\left(\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}\right)\Big]d\mathbf{y}\\
&=\rho\mathbf{u}\cdot\left[\frac{\upsilon}{2\beta^4}\int\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\varphi'\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\rho\left(\mathbf{y}\right)d\mathbf{y}\right.\\
&\quad\left.-\frac{\upsilon}{2\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)d\mathbf{y}\nabla\rho\right]=-\rho\mathbf{f}\cdot\mathbf{u}.
\end{aligned}$$

In this way we have obtained the non-local form of Eq. (2.17), where now the non-local interfacial internal energy ς is

$$\varsigma=\frac{\upsilon}{2\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho\left(\mathbf{y}\right)-\rho\left(\mathbf{x}\right)\right]d\mathbf{y}\tag{3.5}$$

and the non-local vector of the interstitial working is of the form:

$$\mathbf{w} = \frac{\upsilon}{2\beta^4}\int_0^1 d\lambda \int \frac{\mathbf{y}\mathbf{y}}{|\mathbf{y}|}\varphi'\left(\frac{|\mathbf{y}|}{\beta}\right)\rho\left(\mathbf{x}+\lambda\mathbf{y}\right)\rho\left(\mathbf{x}+(\lambda-1)\,\mathbf{y}\right) \cdot \Big[\mathbf{u}\left(\mathbf{x}+\lambda\mathbf{y}\right)-\mathbf{u}\left(\mathbf{x}\right)\Big]d\mathbf{y}. \tag{3.6}$$

As it is clear, Eq. (2.17) is sufficient to write the energy equation (2.11) in the conservative form (2.18) with the interfacial energy ς of the form (3.5); the Korteweg stress tensor is given by (3.4), and the expression for the interstitial working is given by the formula (3.6).

3.1. The local Korteweg equations

We show that the local form of the Korteweg equations can be derived by treating β as a small parameter and by expanding the expressions for the Korteweg stress, the interstitial working and the functional ς in a power series. Let us notice that under certain assumptions, the following approximate formula holds:

$$\frac{\upsilon}{2\beta^3}\int\varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right)\left[\rho\left(\mathbf{y}\right)-\rho\left(\mathbf{x}\right)\right]d\mathbf{y} = \frac{\upsilon}{2}\int\varphi\left(|\mathbf{y}|\right)\left[\rho\left(\mathbf{x}+\beta\mathbf{y}\right)-\rho\left(\mathbf{x}\right)\right]d\mathbf{y}$$
$$= \frac{\pi}{3}\beta^2\upsilon\int_0^\infty y^4\varphi\left(y\right)dy\Delta\rho + O\left(\beta^4\right) \quad \text{as} \quad \beta\to 0.$$

Hence the formula (3.5) for ς takes the same form as $\varsigma^{(2)}$ with the constant A is given by

$$A = -\frac{2\pi}{3}\beta^2\upsilon\int_0^\infty y^4\varphi\left(y\right)dy. \tag{3.7}$$

The approximation of the non-local Korteweg stress tensor (3.4) is the tensor $\mathbf{K}^{(2)}$ of (2.20) with A given above; finally the approximate, valid for small values of β, formula for the interstitial working coincides with the that of (2.22).

3.2. The Korteweg equations for viscous and heat conducting fluids

The presented theory concerns perfect fluids. The dissipative terms are introduced in the theory "manually", i.e. by substituting them at suitable places into the equations. Hence the viscous and heat conducting versions of the equations are the following.

- Equation (2.9) takes the form

$$\rho\frac{D\mathbf{u}}{Dt}+\nabla\cdot\mathbf{P}=\rho\mathbf{f}, \tag{3.8}$$

where $\mathbf{P}$ is the full stress tensor,

$$\mathbf{P}=p\mathbf{I}-\mu\mathbf{D}-\kappa\left(\nabla\cdot\mathbf{u}\right)\mathbf{I}, \tag{3.9}$$

μ and κ are the shear and bulk viscosity coefficients, respectively; they are nonnegative functions of the mass density ρ and temperature θ. Finally in (3.9), the components D_{ij} of the tensor $\mathbf{D}$ are given by

$$D_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\left(\nabla\cdot\mathbf{u}\right)\delta_{ij},\quad i,j=1,2,3. \tag{3.10}$$

- Equation (2.16) is replaced by

$$\rho\frac{D\mathbf{u}}{Dt}+\nabla\cdot\left(\mathbf{P}-\mathbf{K}\right)=\mathbf{0}. \tag{3.11}$$

- The energy equation (2.11) takes the form

$$\rho\frac{D}{Dt}\left(e+\frac{\mathbf{u}^2}{2}\right)+\nabla\cdot\left(\mathbf{P}\cdot\mathbf{u}+\mathbf{q}\right)=\rho\mathbf{f}\cdot\mathbf{u}, \tag{3.12}$$

where $\mathbf{q}$ is the heat flux

$$\mathbf{q}=-\lambda\nabla\theta, \tag{3.13}$$

and λ is the heat conductivity coefficient, which is assumed to be a non-negative function of ρ and θ.

- Equation (2.18) should be replaced by its following one:

$$\rho\frac{D}{Dt}\left(e+\varsigma+\frac{\mathbf{u}^2}{2}\right)+\nabla\cdot\big(\left(\mathbf{P}-\mathbf{K}\right)\cdot\mathbf{u}+\mathbf{q}-\mathbf{w}\big)=0. \tag{3.14}$$

The entropy equation is now

$$\rho\frac{Ds}{Dt}+\nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)=\lambda\left(\frac{\nabla\theta}{\theta}\right)^2+\frac{\mu}{2\theta}\mathbf{D}:\mathbf{D}+\frac{\kappa}{\theta}\left(\nabla\cdot\mathbf{u}\right)^2\geq 0, \tag{3.15}$$

instead of (2.5). Of course, the quantities $\mathbf{K}$, $\mathbf{w}$, $\mathbf{f}$ and p remain unchanged.

4. Kinetic model

We model the fluid as a cloud of hard impermeable spheres of diameter δ. The state of such a set is described by the one-particle distribution function $f = f(t, \mathbf{x}, \mathbf{v})$, which is proportional to the probability density of finding, at time t, a molecule at $\mathbf{x}$, moving with velocity $\mathbf{v}$.

We assume that two molecules with the centres at $\mathbf{x}$ and $\mathbf{y}$ interact, if $|\mathbf{x}-\mathbf{y}| > \delta$, with a force of the long-range potential $\dfrac{\upsilon}{b^3}\varphi\left(\dfrac{|\mathbf{x}-\mathbf{y}|}{b}\right)$, where $\varphi(r)$ is a bounded, smooth non-positive function defined for $r \geq 0$, υ characterizes the intensity of the potential, and b is the range of the potential measured in macroscopic length scale units. The force exerted on the test molecule being at $\mathbf{x}$ by all other molecules is usually taken in the Vlasov form:

$$\mathbf{f}_{\text{Vlasov}} = -\nabla \frac{\upsilon}{b^3} \int\limits_{|\mathbf{x}-\mathbf{y}|>\delta} \varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{b}\right) \rho(t, \mathbf{y})\, d\mathbf{y}, \tag{4.1}$$

where

$$\rho(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$$

is the density. In this paper we take a slightly different form of the force, namely we will use the form (3.1), (3.2) proposed by van der Waals.

The Vlasov or van der Waals force is not the unique factor changing the distribution function. It varies also due to the hard-core elastic collisions when two molecules are in contact, i.e. when the distance between their centres is equal to the molecular diameter, i.e. if $|\mathbf{x}-\mathbf{y}| = \delta$. To describe the changes of the distribution function resulting from this type of molecular interactions, an operator of the Enskog type is used. The dimensionless kinetic equation reads

$$\begin{aligned} \frac{\partial f}{\partial t} &+ \mathbf{v}\cdot\nabla f + \mathbf{f}\cdot\frac{\partial f}{\partial \mathbf{v}} \\ &= \frac{1}{\varepsilon}\int d\mathbf{v}_* \int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\, H(\mathbf{k}\cdot\mathbf{g}) \Big[\chi(\mathbf{x}, \mathbf{x}+\alpha\mathbf{k})\, f(\mathbf{x}, \mathbf{v}')\, f(\mathbf{x}+\alpha\mathbf{k}, \mathbf{v}'_*) \\ &\qquad - \chi(\mathbf{x}, \mathbf{x}-\alpha\mathbf{k})\, f(\mathbf{x}, \mathbf{v})\, f(\mathbf{x}+\alpha\mathbf{k}, \mathbf{v}_*)\Big], \end{aligned} \tag{4.2}$$

where

$$\mathbf{v}' = \mathbf{v} + \mathbf{k}(\mathbf{k}\cdot\mathbf{g}), \quad \mathbf{v}'_* = \mathbf{v}_* - \mathbf{k}(\mathbf{k}\cdot\mathbf{g}) \tag{4.3}$$

are the post-collision velocities, and $\mathbf{g} = \mathbf{v}_* - \mathbf{v}$ is the relative velocity of the colliding particles, $\mathbf{k}$ is a unit vector bisecting the angle between the relative velocities $\mathbf{g} = \mathbf{v}_* - \mathbf{v}$ and $\mathbf{g}' = \mathbf{v}'_* - \mathbf{v}'$; next $H(x)$, $x \in R^1$, is the Heaviside step function, and finally $\chi(\mathbf{x}, \mathbf{y}) = \chi(\mathbf{y}, \mathbf{x}) > 0$ is the local equilibrium pair correlation function.

When reducing t, $\mathbf{x}$, $\mathbf{v}$ to the non-dimensional form, we referred them in the usual way to a macroscopic length-scale L, thermal speed V, and used L/V as the time unit. The distribution function f was scaled by referring it to NV^{-3}, where N is the characteristic density. The parameter ε, defined by

$$\varepsilon = \frac{1}{N\delta^2 L},$$

is the Knudsen number for hard-core collisions. Finally, the dimensionless form of the Vlasov force (4.1) is

$$\mathbf{f}_{\text{Vlasov}} = -\nabla \frac{\upsilon}{\beta^3} \int\limits_{|\mathbf{x}-\mathbf{y}|>\alpha} \varphi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\beta}\right) \rho(t, \mathbf{y})\, d\mathbf{y}, \tag{4.4}$$

where

$$\alpha = \frac{\delta}{L}, \qquad \beta = \frac{b}{L}.$$

A kinetic equation of the form (4.2), called the Enskog–Vlasov equation, was introduced by Grmela [9], [10] to describe the liquid-vapour systems and was derived later by Karkheck and Stell [11], who used the principle of entropy maximization. Unfortunately, the only form of the equation of state consistent with the Enskog–Vlasov kinetic equation is that of Van der Waals [12]. This is due to the fact that in the Enskog–Vlasov equation, the long-range interactions are not fully represented since the Vlasov term is not dissipative and does not take into account the interchange of kinetic and potential energies. When two molecules interact via a potential depending on the distance between them then, as it follows from the energy conservation principle, the velocities of these molecules change with the distance as well. We mimic this phenomenon by assuming that if the centres of two molecules are at a distance $R > \delta$ apart then their velocities change instantaneously as if the attractive potential experienced a jump discontinuity of the value of ϖ or as if this discontinuity were the attractive edge of the square-well potential [11, 13, 14]. It is necessary to distinguish three types of such collisions:

- the entering collisions when the two molecules, one at $\mathbf{x}$ and the other one at $\mathbf{y}$, approach each other, that is when their centres cross the sphere $|\mathbf{x} - \mathbf{y}| = R$ from outside and as the result, they gain some energy;
- the escape collisions when the two molecules successfully become more distant, that is when their centres cross the sphere $|\mathbf{x} - \mathbf{y}| = R$ from inside at the cost of losing some kinetic energy;

- the bound-state collisions when they cannot become more distant than R, because their relative kinetic energy is too weak to overcome the jump of the attractive potential.

This model resembles the square-well potential superposed on the smooth long-range potential, which is similar to the model introduced in [14]. The kinetic equation we consider is

$$
\begin{aligned}
(4.5)\qquad & \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \mathbf{f}\cdot\frac{\partial}{\partial \mathbf{v}} f \\
&= \frac{1}{\varepsilon}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,H(\mathbf{k}\cdot\mathbf{g})\Big[f_2\left(\mathbf{x},\mathbf{v}',\mathbf{x}+\alpha^+\mathbf{k},\mathbf{v}'_*\right) \\
&\qquad - f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}-\alpha^+\mathbf{k},\mathbf{v}_*\right)\Big] \\
&+ \frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\Big[f_2\left(\mathbf{x},\mathbf{v}''',\mathbf{x}-r^+\mathbf{k},\mathbf{v}'''_*\right)H\Big(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\Big) \\
&\qquad - f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}-r^+\mathbf{k},\mathbf{v}_*\right)H(\mathbf{k}\cdot\mathbf{g})\Big] \\
&\frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\Big[f_2\left(\mathbf{x},\mathbf{v}'',\mathbf{x}+r^-\mathbf{k},\mathbf{v}''_*\right)H(\mathbf{k}\cdot\mathbf{g}) \\
&\qquad - f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^-\mathbf{k},\mathbf{v}_*\right)H\Big(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\Big)\Big] \\
&+ \frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,H(\mathbf{k}\cdot\mathbf{g})\,H\Big(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\Big)\Big[f_2\left(\mathbf{x},\mathbf{v}',\mathbf{x}-r^-\mathbf{k},\mathbf{v}'_*\right) \\
&\qquad - f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^-\mathbf{k},\mathbf{v}_*\right)\Big]
\end{aligned}
$$

where

$$f_2\left(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{v}_*\right) = f_2\left(t,\mathbf{y},\mathbf{v}_*,\mathbf{x},\mathbf{v}\right)$$

is a two-particle distribution function, and

$$
(4.6)\qquad
\begin{aligned}
\mathbf{v}'' &= \mathbf{v} + \frac{1}{2}\mathbf{k}\left[\mathbf{k}\cdot\mathbf{g} - \sqrt{(\mathbf{k}\cdot\mathbf{g})^2 + 4\gamma}\right], \\
\mathbf{v}'''_* &= \mathbf{v}_* - \frac{1}{2}\mathbf{k}\left[\mathbf{k}\cdot\mathbf{g} - \sqrt{(\mathbf{k}\cdot\mathbf{g})^2 + 4\gamma}\right],
\end{aligned}
$$

are the velocities assumed by the molecules as a result of the entering collision,

$$\begin{aligned}\mathbf{v}''' &= \mathbf{v} + \frac{1}{2}\mathbf{k}\left[\mathbf{k}\cdot\mathbf{g} - \sqrt{(\mathbf{k}\cdot\mathbf{g})^2 - 4\gamma}\right],\\ \mathbf{v}_*''' &= \mathbf{v}_* - \frac{1}{2}\mathbf{k}\left[\mathbf{k}\cdot\mathbf{g} - \sqrt{(\mathbf{k}\cdot\mathbf{g})^2 - 4\gamma}\right]\end{aligned} \tag{4.7}$$

denote the velocities which the molecules take after the escape collision.

The bound-state collisions are elastic, so after such a collision the post-collision velocities are the same as in (4.3). The nondimensionalisation of Eq. (4.5) was carried out along the same lines as that of Eq. (4.2). The additional new parameters are defined below:

$$r = \frac{R}{L}, \quad \tau = \frac{1}{NR^2L}, \quad \gamma = \frac{\varpi}{V^2}.$$

The force term $\mathbf{f}$ in Eq. (4.5) can be the same as in the Enskog–Vlasov equation (4.2), but it also can be an arbitrary force of a self-consistent or external character. The important assumption concerning the force term is that it does not depend on the molecular velocity $\mathbf{v}$.

Finally, in Eq. (4.5) the expressions of type $f_2(\mathbf{x},\mathbf{v},\mathbf{x}\pm r^+\mathbf{k},\mathbf{v}_*)$ or $f_2(\mathbf{x},\mathbf{v},\mathbf{x}\pm r^-\mathbf{k},\mathbf{v}_*)$ denote the outer or inner limits at the points $\mathbf{x}\pm r\mathbf{k}$, respectively, and are defined as follows:

$$f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^{\pm}\mathbf{k},\mathbf{v}_*\right) = \lim_{h\to\pm 0} f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+(r\pm h)\,\mathbf{k},\mathbf{v}_*\right). \tag{4.8}$$

The distinction between inner and outer limits of $f_2(\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{v}_*)$ at $\mathbf{y}=\mathbf{x}\pm r\mathbf{k}$ means that f_2 has a discontinuity at this point, what is in accordance with the statistical equilibrium mechanics. The magnitude of this jump discontinuity is not arbitrary, and the values of the two-particle distribution function f_2 on both sides of the sphere $|\mathbf{y}-\mathbf{x}| = r$ are related by the formula

$$f_2\left(t,\mathbf{x},\mathbf{v},\mathbf{x}+r^+\mathbf{k},\mathbf{v}_*\right) = \exp\left(-\frac{\gamma}{T\left(t,\mathbf{x}+r\mathbf{k}\right)}\right) f_2\left(t,\mathbf{x},\mathbf{v},\mathbf{x}+r^-\mathbf{k},\mathbf{v}_*\right), \tag{4.9}$$

where $T(t,\mathbf{x})$ is a positive quantity interpreted as the potential temperature. To close the discussion on the kinetic model, we add only that we could presumably improve it by introducing more than one intermolecular distance at which the velocities of the interacting particles vary instantaneously (see [15, 16]). We refrained from doing that since it would raise considerably the level of complexity of the calculations.

4.1. The balance equations

4.1.1. Mass balance equation. We multiply Eq. (4.5) by the function $\Psi(\mathbf{v}) = 1$, integrate over the entire 3-dimensional space of molecular velocities and arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4.10}$$

where the mean velocity $\mathbf{u}$ is defined by

$$\mathbf{u}(t, \mathbf{x}) = \frac{1}{\rho(t, \mathbf{x})} \int \mathbf{v}\, f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}. \tag{4.11}$$

4.1.2. Momentum balance equation. This time we multiply Eq. (4.5) by $\Psi(\mathbf{v}) = v_i$, $i = 1, 2, 3$, integrate over the 3-dimensional velocity space and obtain the momentum balance equation in the form resembling Eq. (3.8)

$$\frac{DU}{Dt} = U_n' \frac{Dn}{Dt} + U_T' \frac{DT}{Dt} \tag{4.12}$$

where now the stress tensor $\mathbf{P}$ consists of several parts:

$$\mathbf{P} = \mathbf{P}^{\mathrm{k}} + \mathbf{P}^{\mathrm{hc}} + \mathbf{P}^{\mathrm{en}} + \mathbf{P}^{\mathrm{es}} + \mathbf{P}^{\mathrm{bs}}, \tag{4.13}$$

and $\mathbf{P}^{\mathrm{k}}$ is the kinetic stress

$$P_{ij}^{\mathrm{k}}(t, \mathbf{x}) = \int c_i c_j f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}, \quad i, j = 1, 2, 3, \tag{4.14}$$

and $\mathbf{c}$ is the velocity of the chaotic motion of molecules: $\mathbf{c} = \mathbf{v} - \mathbf{u}$.

The other components of the stresses result from integrating the collision operator multiplied by the molecular velocity $\mathbf{v}$.

- The hard-core stress $\mathbf{P}^{\mathrm{hc}}$ is

$$P_{ij}^{\mathrm{hc}} = \frac{\alpha}{2\varepsilon} \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k} k_i \left(v_j' - v_j\right) (\mathbf{k} \cdot \mathbf{g})\, H(\mathbf{k} \cdot \mathbf{g}) \cdot f_2\left(\mathbf{x} + \lambda \alpha^+ \mathbf{k}, \mathbf{v}, \mathbf{x} + (\lambda - 1)\, \alpha^+ \mathbf{k}, \mathbf{v}_*\right). \tag{4.15}$$

- The escape stress $\mathbf{P}^{\mathrm{es}}$ is defined by

$$P_{ij}^{\mathrm{es}} = -\frac{r}{2\tau} \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k} k_i (\mathbf{k} \cdot \mathbf{g})\, H\left(\mathbf{k} \cdot \mathbf{g} - \sqrt{4\gamma}\right) \left(v_j''' - v_j\right) \cdot f_2\left(\mathbf{x} + (\lambda - 1)\, r^- \mathbf{k}, \mathbf{v}, \mathbf{x} + \lambda r^- \mathbf{k}, \mathbf{v}_*\right). \tag{4.16}$$

- The formula for the entering stress $\mathbf{P}^{\mathrm{en}}$ reads

$$P_{ij}^{\mathrm{en}} = \frac{r}{2\tau}\int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k} k_i \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right)\left(v_j'' - v_j\right) \cdot f_2\left(\mathbf{x} + \lambda r^+ \mathbf{k}, \mathbf{v}, \mathbf{x} + \left(\lambda - 1\right) r^+ \mathbf{k}, \mathbf{v}_*\right). \tag{4.17}$$

- Finally, we have the following expression for the bound-state stress P^{bs}

$$P_{ij}^{\mathrm{bs}} = -\frac{r}{2\tau}\int_0^1 d\lambda v \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k} k_i \left(v_j' - v_j\right)\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\sqrt{4\gamma} - \mathbf{k}\cdot\mathbf{g}\right) \cdot f_2\left(\mathbf{x} + \left(\lambda - 1\right) r^- \mathbf{k}, \mathbf{v}, \mathbf{x} + \lambda r^- \mathbf{k}, \mathbf{v}_*\right). \tag{4.18}$$

4.1.3. Energy conservation equation. Now we multiply Eq. (4.5) by $\Psi\left(\mathbf{v}\right) = \mathbf{v}^2/2$, integrate over the entire 3-dimensional space of molecular velocities and obtain

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{3}{2}\theta + \frac{\mathbf{u}^2}{2}\right)\right] + \nabla\cdot\left[\rho\mathbf{u}\left(\frac{3}{2}\theta + \frac{\mathbf{u}^2}{2}\right) + \mathbf{P}\cdot\mathbf{u} + \mathbf{q}\right] = \rho\mathbf{f}\cdot\mathbf{u} - \frac{\gamma}{2\tau}\int d\mathbf{v}\int d\mathbf{v}_*\int d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\left[H\left(\mathbf{k}\cdot\mathbf{g} - \sqrt{4\gamma}\right) f_2\left(\mathbf{x}, \mathbf{v}, \mathbf{x} + r^- \mathbf{k}, \mathbf{v}_*\right) - H\left(\mathbf{k}\cdot\mathbf{g}\right) f_2\left(\mathbf{x}, \mathbf{v}, \mathbf{x} - r^+ \mathbf{k}, \mathbf{v}_*\right)\right], \tag{4.19}$$

where the kinetic temperature θ is defined by

$$\frac{3}{2}\rho\theta = \frac{1}{2}\int \left(\mathbf{v} - \mathbf{u}\right)^2 f\left(t, \mathbf{x}, \mathbf{v}\right) d\mathbf{v}. \tag{4.20}$$

Next, the heat flux is also a sum of the following components

$$\mathbf{q} = \mathbf{q}^{\mathrm{k}} + \mathbf{q}^{\mathrm{hc}} + \mathbf{q}^{\mathrm{es}} + \mathbf{q}^{\mathrm{en}} + \mathbf{q}^{\mathrm{bs}}, \tag{4.21}$$

where:

- The kinetic part of the heat flux is defined as usually

$$\mathbf{q}^{\mathrm{k}} = \frac{1}{2}\int \left(\mathbf{v} - \mathbf{u}\right)\left(\mathbf{v} - \mathbf{u}\right)^2 f\left(t, \mathbf{x}, \mathbf{v}\right) d\mathbf{v}. \tag{4.22}$$

- The hard-core heat flux is given by

$$\mathbf{q}^{\mathrm{hc}} = \frac{1}{2}\sigma^3 \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k}\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\mathbf{v}'-\mathbf{v}\right) \cdot \left(\frac{\mathbf{v}'+\mathbf{v}}{2}-\mathbf{u}\right) f\left(\mathbf{x}+\lambda r^+\mathbf{k},\mathbf{v},\mathbf{x}+\left(\lambda-1\right)r^+\mathbf{k},\mathbf{v}_*\right). \tag{4.23}$$

- The escape heat flux is evaluated from

$$\mathbf{q}^{\mathrm{es}} = -\frac{r}{2\tau} \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k}\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\left(\mathbf{v}'''-\mathbf{v}\right) \cdot \left(\frac{\mathbf{v}''+\mathbf{v}}{2}-\mathbf{u}\right) f_2\left(\mathbf{x}+\left(\lambda-1\right)r^-\mathbf{k},\mathbf{v},\mathbf{x}+\lambda r^-\mathbf{k},\mathbf{v}_*\right), \tag{4.24}$$

- The formula for the entering heat flux $\mathbf{q}^{\mathrm{en}}$ reads

$$\mathbf{q}^{\mathrm{en}} = \frac{r}{2\tau} \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k}\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\mathbf{v}''-\mathbf{v}\right) \cdot \left(\frac{\mathbf{v}''+\mathbf{v}}{2}-\mathbf{u}\right) f_2\left(\mathbf{x}+\lambda r^+\mathbf{k},\mathbf{v},\mathbf{x}+\left(\lambda-1\right)r^+\mathbf{k},\mathbf{v}\right). \tag{4.25}$$

- Finally, we have the following expression for the bound-state heat flux $\mathbf{q}^{\mathrm{bs}}$:

$$\mathbf{q}^{\mathrm{bs}} = -\frac{r}{2\tau} \int_0^1 d\lambda \int d\mathbf{v} \int d\mathbf{v}_* \int d\mathbf{k}\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) \cdot H\left(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\right)\left(\mathbf{v}'-\mathbf{v}\right)\left(\frac{\mathbf{v}'+\mathbf{v}}{2}-\mathbf{u}\right) \times f_2\left(\mathbf{x}+\lambda r^-\mathbf{k},\mathbf{v},\mathbf{x}+\left(\lambda-1\right)r^-\mathbf{k},\mathbf{v}_*\right). \tag{4.26}$$

We impose the following condition: for any continuously differentiable function $U = U(t,\mathbf{x})$ the distribution function f_2 satisfies also the following equation:

$$\frac{\partial}{\partial t}(\rho U) + \nabla\cdot(\rho\mathbf{u}U) = \frac{\gamma}{2\tau}\int d\mathbf{v}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\Big[H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^-\mathbf{k},\mathbf{v}_*\right) - H(\mathbf{k}\cdot\mathbf{g})\,f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}-r^+\mathbf{k},\mathbf{v}_*\right)\Big]. \tag{4.27}$$

Under this assumption, the energy balance equation (4.18) takes the usual form (cf. (3.12))

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{3}{2}\theta+U+\frac{\mathbf{u}^2}{2}\right)\right]+\nabla\cdot\left[\rho\mathbf{u}\left(\frac{3}{2}\theta+U+\frac{\mathbf{u}^2}{2}\right)+\mathbf{P}\cdot\mathbf{u}+\mathbf{q}\right]=\rho\mathbf{f}\cdot\mathbf{u}. \tag{4.28}$$

The function U will be chosen later since most of our considerations do not depend on the specific form of it.

The system of two kinetic equations (4.5), (4.27) is not closed since the two-particle distribution function f_2 is present in it. We have to assume a relation between it and the one-particle distribution function f.We make the fundamental assumptions:

$$\begin{aligned}
f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+\alpha^+\mathbf{k},\mathbf{v}_*\right) &= \chi\left(\mathbf{x}+\frac{1}{2}\alpha\mathbf{k}\right)f(t,\mathbf{x},\mathbf{v})\,f(t,\mathbf{x}+\alpha\mathbf{k},\mathbf{v}_*),\\
f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^-\mathbf{k},\mathbf{v}_*\right) &= \eta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right)f(t,\mathbf{x},\mathbf{v})\,f(t,\mathbf{x}+r\mathbf{k},\mathbf{v}_*),\\
f_2\left(\mathbf{x},\mathbf{v},\mathbf{x}+r^+\mathbf{k},\mathbf{v}_*\right) &= \zeta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right)f(t,\mathbf{x},\mathbf{v})\,f(t,\mathbf{x}+r\mathbf{k},\mathbf{v}_*).
\end{aligned} \tag{4.29}$$

Functions χ η, and ζ are the local equilibrium pair correlation functions. In (4.29) we have assumed that the correlation functions are evaluated at the middle point of the vector connecting the centers of the colliding molecules. This is an assumption simplifying a little bit the calculations but with some additional effort it can be omitted. All the correlation functions are assumed to be given, strictly positive and twice continuously differentiable functions of the density ρ and potential temperature T.

It follows from the continuity of the one-particle distribution function f and (4.9) that the functions ζ and η are related by

$$\eta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right)=\exp\left(\frac{\gamma}{T\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right)}\right)\zeta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right). \tag{4.30}$$

Under assumptions (4.29), Eqs. (4.5) and (4.27) take the form

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{f}\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{1}{\varepsilon}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,H(\mathbf{k}\cdot\mathbf{g})& \\
\left[\chi\left(\mathbf{x}+\frac{1}{2}\alpha\mathbf{k}\right) f\left(\mathbf{x},\mathbf{v}'\right) f\left(\mathbf{x}+\alpha\mathbf{k},\mathbf{v}'_*\right)\right.& \\
\left.-\,\chi\left(\mathbf{x}-\frac{1}{2}\alpha\mathbf{k}\right) f(\mathbf{x},\mathbf{v})\, f(\mathbf{x}+\alpha\mathbf{k},\mathbf{v}_*)\right]& \\
+\frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,\eta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right)\left[f\left(\mathbf{x},\mathbf{v}''\right) f\left(\mathbf{x}+r\mathbf{k},\mathbf{v}''_*\right) H(\mathbf{k}\cdot\mathbf{g})\right.& \\
\left.-\,f(\mathbf{x},\mathbf{v})\, f(\mathbf{x}+r\mathbf{k},\mathbf{v}_*)\, H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\right]& \\
+\frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,\varsigma\left(\mathbf{x}-\frac{1}{2}r\mathbf{k}\right)\left[f\left(\mathbf{x},\mathbf{v}'''\right) f\left(\mathbf{x}-r\mathbf{k},\mathbf{v}'''_*\right) H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\right.& \\
\left.-\,f(\mathbf{x},\mathbf{v})\, f(\mathbf{x}+r\mathbf{k},\mathbf{v}_*)\, H(\mathbf{k}\cdot\mathbf{g})\right]& \\
+\frac{1}{\tau}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\,H(\mathbf{k}\cdot\mathbf{g})\,H\left(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\right)& \\
\times\left[\eta(\mathbf{x}-r\mathbf{k})\, f\left(\mathbf{x},\mathbf{v}'\right) f\left(\mathbf{x}-r\mathbf{k},\mathbf{v}'_*\right) - \eta(\mathbf{x}+r\mathbf{k})\, f(\mathbf{x},\mathbf{v})\, f(\mathbf{x}+r\mathbf{k},\mathbf{v}_*)\right]&
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho U) + \nabla\cdot(\rho U\mathbf{u}) = \frac{\gamma}{2\tau}\int d\mathbf{v}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g}) f(\mathbf{x},\mathbf{v})& \\
\left[H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\eta\left(\mathbf{x}+\frac{1}{2}r\mathbf{k}\right) f(\mathbf{x}+r\mathbf{k},\mathbf{v}_*)\right.& \\
\left.-\,\zeta\left(\mathbf{x}-\frac{1}{2}r\mathbf{k}\right) H(\mathbf{k}\cdot\mathbf{g})\, f(\mathbf{x}-r\mathbf{k},\mathbf{v}_*)\right]&
\end{aligned}
\tag{4.32}
$$

5. The hydrodynamic limit equations

To evaluate the stresses and heat fluxes in Eqs. (4.11), (4.28) we need to know the one-particle distribution function f. We determine them only approximately. To this end we assume that the parameter ε is a small quantity and that

$$
\alpha = O(\varepsilon), \quad \tau = O(\varepsilon), \quad r = O(\varepsilon) \quad as \quad \varepsilon \to 0. \tag{5.1}
$$

We will employ the Chapman-Enskog asymptotic procedure and look for solutions of the system (4.31), (4.32) in the form of the formal power series expansion

$$f\left(t,\mathbf{x},\mathbf{v},\varepsilon\right)=f_0\left(t,\mathbf{x},\mathbf{v}\right)+\varepsilon f_1\left(t,\mathbf{x},\mathbf{v}\right)+\varepsilon^2 f_2\left(t,\mathbf{x},\mathbf{v}\right)+\ldots \tag{5.2}$$

Classical version of the Chapman–Enskog method [17] assumes that the distribution function f depends on t and $\mathbf{x}$ only via the five lowest-order moments of f, which are: ρ, $\mathbf{u}$, the kinetic temperature θ and their gradients. However, for the case under consideration one has to change a little the classical procedure. Namely we assume that f depends on the density ρ, the mean velocity $\mathbf{u}$ and the potential energy T, which is not a moment of f. It is assumed that ρ and $\mathbf{u}$ are given by the lowest order approximation to f and the higer order approximations do not contribute to the first two quantities. Precisely it is assumed that

$$\begin{gathered}\int f_0 d\mathbf{v}=\rho, \qquad \int f_k d\mathbf{v}=0, \quad k\geq 1,\\ \text{and}\\ \int \mathbf{v} f_0 d\mathbf{v}=\rho\mathbf{u}, \qquad \int \mathbf{v} f_k d\mathbf{v}=\mathbf{0}, \quad k\geq 1,\end{gathered} \tag{5.3}$$

and

$$f_k=f_k\left(\rho,T,\mathbf{u},\rho_i,T_i,\mathbf{u}_i,\ldots,\rho_{i_1 i_2\ldots i_k},T_{i_1 i_2\ldots i_k},\mathbf{u}_{i_1 i_2\ldots i_k}\right), \quad k\geq 0, \tag{5.4}$$

where we have used the notation analogs to the one introduced in Sec. 2. We have to stress that we do not assume any condition of the type (5.3) related either the temperature T or any sort of energy. Our next assumptions are the following: the force term $\mathbf{f}$, the function U of Eqs. (4.28), (4.32), the pair correlation functions χ, ζ, η; all these quantities depend on ρ and T only. As these two are not expanded, the quantities $\mathbf{f}$, T, U, χ, ζ, and η are not expanded either. Under our assumptions Eq. (4.31) becomes

$$\frac{\partial f_0}{\partial t}+\mathbf{v}\cdot\nabla f_0+\mathbf{f}\cdot\frac{\partial f_0}{\partial \mathbf{v}}\cdots=\frac{1}{\varepsilon}Q_0\left(f_0\right)+\left(Q_1\left(f_0\right)+L\left(\varPhi\right)\right)\cdots, \tag{5.5}$$

where we have substituted $f_1=f_0\varPhi$ and adopted the following notations

$$\begin{aligned}Q_0\left(f_0\right)=\chi\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\left(f_0' f_{0*}'-f_0 f_{0*}\right)&\\ +\frac{\varepsilon}{\tau}\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\left(\eta f_0'' f_{0*}''-\eta f_0 f_{0*}\right)&\\ +\frac{\varepsilon}{\tau}\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\zeta f_0''' f_{0*}'''-\eta f_0 f_{0*}\right)&\\ +\frac{\varepsilon}{\tau}\eta\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\right)\left(f_0' f_{0*}'-f_0 f_{0*}\right),&\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
Q_1\left(f_0\right) = \frac{\alpha}{\varepsilon} \int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right) \mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathbf{k} & \\
\cdot\left[\chi\left(f_0'\nabla f_{0*}' + f_0\nabla f_{0*}\right) + \frac{1}{2}\left(f_0' f_{0*}' + f_0 f_{0*}\right)\nabla\chi\right] & \\
\frac{r}{\tau}\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathbf{k} & \\
\cdot\left[\left(\eta\, f_0''\nabla f_{0*}'' + \zeta\, f_0\nabla f_{0*}\right) + \frac{1}{2}\left(f_0'' f_{0*}''\nabla\eta + f_0 f_{0*}\nabla\zeta\right)\right] & \\
-\frac{r}{\tau}\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g} - \sqrt{4\gamma}\right)\mathbf{k} & \\
\cdot\left[\left(\zeta f_0'''\nabla f_{0*}''' + \eta f_0\nabla f_{0*}\right) + \frac{1}{2}\left(f_0''' f_{0*}'''\nabla\zeta + f_0 f_{0*}\nabla\eta\right)\right] & \\
-\frac{r}{\tau}\int d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\sqrt{4\gamma} - \mathbf{k}\cdot\mathbf{g}\right)\mathbf{k} & \\
\cdot\left[\eta\left(f_0'\nabla f_{0*}' + f_0\nabla f_{0*}\right) + \frac{1}{2}\left(f_0' f_{0*}' + f_0 f_{0*}\right)\nabla\eta\right], &
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
L\left(\varPhi\right) = \chi\int f_0 f_{0*} d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\varPhi' + \varPhi_*' - \varPhi - \varPhi_*\right) & \\
+\frac{\varepsilon}{\tau}\zeta\int f_0 f_{0*} d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\varPhi'' + \varPhi_*'' - \varPhi - \varPhi_*\right) & \\
+\frac{\varepsilon}{\tau}\eta\int f_0 f_{0*} d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g} - \sqrt{4\gamma}\right)\left(\varPhi''' + \varPhi_*''' - \varPhi - \upsilon_*\right) & \\
+\frac{\varepsilon}{\tau}\eta\int f_0 f_{0*} d\mathbf{v}_* d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\mathbf{k}\cdot\mathbf{g}\right)\mathrm{H}\left(\sqrt{4\gamma} - \mathbf{k}\cdot\mathbf{g}\right)\left(\varPhi' + \varPhi_*' - \varPhi - \varPhi_*\right). &
\end{aligned}
\tag{5.8}
$$

In (5.6)–(5.8) as well as in the future we use the following, typical in kinetic theory, abbreviated notation: if F is a function of the molecular velocity $\mathbf{v}$, $F = F\left(\mathbf{v}\right)$, then $F_* = F\left(\mathbf{v}_*\right)$, $F' = F\left(\mathbf{v}'\right), F_*' = F\left(\mathbf{v}_*'\right)$, where $\mathbf{v}', \mathbf{v}_*'$ are given by (4.3). The meaning of F'', F_*'', etc. is similar.

The operator L defined by (5.8) is called the linearized collision operator.

The Eq. (4.32) under the above assumption is expanded as follows

$$
\frac{\partial}{\partial t}\left(\rho U\right) + \frac{\partial}{\partial x_i}\left(\rho U u_i\right) = \frac{1}{\varepsilon}\mathrm{R}_0\left(f_0\right) + \mathrm{R}_1 + \cdots, \tag{5.9}
$$

where

$$R_0\left(f_0\right)=\int d\mathbf{v}\int d\mathbf{v}_*\int d\mathbf{k}\left(\mathbf{k}\cdot\mathbf{g}\right)f_0\left(\mathbf{x},\mathbf{v}\right)f_0\left(\mathbf{x},\mathbf{v}_*\right) \cdot\left[H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\eta\left(\mathbf{x}\right)-H\left(\mathbf{k}\cdot\mathbf{g}\right)\zeta\left(\mathbf{x}\right)\right], \tag{5.10}$$

and

$$R_1=\frac{\gamma r}{12\tau}\frac{\zeta}{T}\int f_0f_{0*}\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}\right) \cdot\left(\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2+4\gamma}\right)d\mathbf{v}d\mathbf{v}_*d\mathbf{k}\nabla\cdot\mathbf{u} +\frac{\varepsilon\gamma}{\tau}\int f_0f_{0*}\Phi\left(\mathbf{k}\cdot\mathbf{g}\right)\left[H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)\eta-\zeta H\left(\mathbf{k}\cdot\mathbf{g}\right)\right]d\mathbf{v}d\mathbf{v}_*d\mathbf{k}.$$

Order ε^1

In the lowest order of approximation, Eqs. (5.6) and (5.10) reduce to

$$Q_0(f_0)=0,\quad R_0(f_0)=0. \tag{5.11}$$

The relation (4.29) takes in this approximation the following form:

$$\eta(\mathbf{x})=\exp\left(\frac{\gamma}{T}\right)\zeta(\mathbf{x}). \tag{5.12}$$

THEOREM 1. *Let the relation (5.12) hold. Then the only solution of system (5.11) is the local Maxwellian*

$$f_0=\frac{\rho}{\left(2\pi T\right)^{3/2}}\exp\left(-\frac{\left(\mathbf{v}-\mathbf{u}\right)^2}{2T}\right). \tag{5.13}$$

Proof. Using (5.12), (5.13) and the definitions of the post-collisional velocities (4.3), (4.6), (4.7), we verify easily that the Maxwellian (5.13) satisfies both equations of the system (5.11).

Inversely, let f_0, maybe different from (5.13), satisfy Eqs. (5.11). We multiply the first one by $\log f_0$ and integrate over the velocity space. We have

$$\begin{aligned}
(5.14)\qquad 0 = &-\frac{1}{4}\int \chi f_0 f_{0*}\left(\frac{f_0' f_{0*}'}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0' f_{0*}'}{f_0 f_{0*}}\right)(\mathbf{k}\cdot\mathbf{g})\, H(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v}\\
&-\frac{\varepsilon}{4\tau}\int \zeta f_0 f_{0*}\left(\frac{f_0'' f_{0*}''}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0'' f_{0*}''}{f_0 f_{0*}}\right) H(\mathbf{k}\cdot\mathbf{g})\,(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v}\\
&-\frac{\varepsilon}{4\tau}\int \eta f_0 f_{0*}\left(\frac{f_0''' f_{0*}'''}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0''' f_{0*}'''}{f_0 f_{0*}}\right) H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v}\\
&-\frac{\varepsilon}{4\tau}\int \eta f_0 f_{0*}\left(\frac{f_0' f_{0*}'}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0' f_{0*}'}{f_0 f_{0*}}\right)(\mathbf{k}.\mathbf{g})\, H(\mathbf{k}\cdot\mathbf{g})\, H\left(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{k}d\mathbf{v}_*\\
&+\frac{\gamma}{2T}\int d\mathbf{v}\int d\mathbf{v}_*\int d\mathbf{k}\,(\mathbf{k}\cdot\mathbf{g})\left[\eta H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)-\zeta H(\mathbf{k}\cdot\mathbf{g})\right] f_0 f_{0*}.
\end{aligned}$$

The last term disappears because f_0 satisfies also the second equation of the system (5.11). Hence (5.14) says that the sum of four non-positive terms is equal to zero. Consequently, each of the terms equals zero, i.e.

$$\begin{aligned}
&\int \chi f_0 f_{0*}\left(\frac{f_0' f_{0*}'}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0' f_{0*}'}{f_0 f_{0*}}\right)(\mathbf{k}\cdot\mathbf{g})\, H(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v} = 0,\\
&\int \zeta f_0 f_{0*}\left(\frac{f_0'' f_{0*}''}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0'' f_{0*}''}{f_0 f_{0*}}\right) H(\mathbf{k}\cdot\mathbf{g})\,(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v} = 0,\\
&\int \eta f_0 f_{0*}\left(\frac{f_0''' f_{0*}'''}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0''' f_{0*}'''}{f_0 f_{0*}}\right) H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)(\mathbf{k}\cdot\mathbf{g})\, d\mathbf{k}d\mathbf{v}_* d\mathbf{v} = 0,\\
&\int \eta f_0 f_{0*}\left(\frac{f_0' f_{0*}'}{f_0 f_{0*}}-1\right)\left(\log\frac{f_0' f_{0*}'}{f_0 f_{0*}}\right)(\mathbf{k}.\mathbf{g})\, H(\mathbf{k}\cdot\mathbf{g}) H\left(\sqrt{4\gamma}-\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{k}d\mathbf{v}_* d\mathbf{v} = 0.
\end{aligned}$$

The first equality is satisfied by the local Maxwellian only [17]. In such a case the other equalities are satisfied as well. The same holds true for Eq. (5.11). The proof is complete.

Order ε^0

In this order of approximation to Eq. (5.11) we have to solve the following linear equation:

$$\frac{\partial f_0}{\partial t}+\mathbf{v}\cdot\nabla f_0=\frac{1}{T}f_0\mathbf{c}\cdot\mathbf{f}+Q_1(f_0)+L(\varPhi).$$

Performing the indicated calculations we rewrite this equation as follows:

$$
\begin{aligned}
(5.15)\qquad & f_0\left[\left(\frac{D}{Dt}+\mathbf{c}\cdot\nabla\right)\log\rho - S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right)\left(\frac{D}{Dt}+\mathbf{c}\cdot\nabla\right)\log T \right.\\
&\qquad\left. +\frac{\mathbf{c}}{T}\left(\frac{D}{Dt}+\mathbf{c}\cdot\nabla\right)\mathbf{u}\right]\\
&+ f_0\left\{\varLambda\left[\mathbf{c}\cdot\nabla\log\left(\rho\varLambda T\right) - \frac{3}{5}S^{(1)}_{\frac{3}{2}}\left(\varOmega^2\right)\mathbf{c}\cdot\nabla\log T\right.\right.\\
&\qquad\left. + \frac{4}{5}\nabla\mathbf{u} : \left(\mathbf{\Omega\Omega} - \frac{\varOmega^2}{3}\mathbf{E}\right) - \frac{2}{3}S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right)\nabla\cdot\mathbf{u}\right]\\
&\qquad - \frac{r}{2\tau}\frac{\zeta}{T}\int f_{0*}\left[\frac{\mathbf{G}\nabla T}{T}+\nabla\mathbf{u}\right]\\
&\qquad : \mathbf{kk}\left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2+4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}\right)d\mathbf{v}_*d\mathbf{k}\\
&\qquad + \frac{r}{2\tau}\frac{\eta}{T}\int f_{0*}\left[\frac{\mathbf{G}\nabla T}{T}+\nabla\mathbf{u}\right]\\
&\qquad\left. : \mathbf{kk}\left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2-4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)d\mathbf{v}_*d\mathbf{k}\right\}\\
&\qquad - \frac{1}{T}f_0\mathbf{c}\cdot\mathbf{f} = L\left(\varPhi\right),
\end{aligned}
$$

where

$$
(5.16)\qquad \varLambda = \frac{2\pi}{3}\rho\left(\frac{\alpha}{\varepsilon}\chi + \frac{r}{\tau}\left(\zeta-\eta\right)\right),
$$

and

$$
S^{(1)}_m\left(x\right) = m+1-x, \quad x\in R^1,
$$

is the first degree Sonine polynomial, and

$$
\mathbf{c} = \mathbf{v}-\mathbf{u}, \quad \mathbf{G} = \frac{1}{2}\left(\mathbf{c}+\mathbf{c}_*\right), \quad \mathbf{\Omega} = \frac{\mathbf{c}}{\sqrt{2T}}
$$

Finally, the linearized form of Eq. (5.9) is

$$
(5.17)\qquad \frac{\partial}{\partial t}\left(\rho U\right) + \nabla\cdot\left(\rho U\mathbf{u}\right) = R_1,
$$

where

$$R_1 = \frac{\gamma r}{12\tau}\frac{\zeta}{T}\int f_0 f_{0*}\left(\mathbf{k}\cdot\mathbf{g}\right) H$$

$$\cdot\left(\mathbf{k}\cdot\mathbf{g}\right)\left(\mathbf{k}\cdot\mathbf{g} + \sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2 + 4\gamma}\right) d\mathbf{v}d\mathbf{v}_* d\mathbf{k}\nabla\cdot\mathbf{u} + T\int L\left(\varPhi\right) S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c}.$$

5.1. The linearised collision operator

In this subsection we establish three fundamental properties of the linearised collision operator defined in (5.9).

Lemma 1. *For any functions $\varPhi$, $\varPsi$ such that the corresponding integrals exist, the following equality holds:*

$$\begin{aligned}
\int \varPsi L\varPhi d\mathbf{v} = -\frac{1}{4}\chi\int & f_0 f_{0*}\left(\varPsi' + \varPsi'_* - \varPsi - \varPsi\right) \\
&\cdot\left(\varPhi' + \varPhi'_* - \varPhi - \varPhi_*\right)\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v}d\mathbf{v}_* d\mathbf{k} \\
&-\frac{1}{4}\frac{\varepsilon}{\tau}\eta\int f_0 f_{0*}\left(\varPsi''' + \varPsi'''_* - \varPsi - \varPsi_*\right) \\
&\cdot\left(\varPhi''' + \varPhi'''_* - \varPhi - \varPhi_*\right)\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g} - \sqrt{4\gamma}\right) d\mathbf{v}d\mathbf{v}_* d\mathbf{k} \\
&+\frac{1}{4}\frac{\varepsilon}{\tau}\zeta\int f_0 f_{0*}\left(\varPsi'' + \varPsi''_* - \varPsi - \varPsi_*\right) \\
&\cdot\left(\varPhi'' + \varPhi''_* - \varPhi - \varPhi_*\right)\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v}d\mathbf{v}_* d\mathbf{k} \\
&-\frac{1}{4}\frac{\varepsilon}{\tau}\eta\int f_0 f_{0*}\left(\varPsi' + \varPsi'_* - \varPsi - \varPsi\right) \\
&\left(\varPhi' + \varPhi'_* - \varPhi - \varPhi_*\right)\left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\sqrt{4\gamma} - \mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v}d\mathbf{v}_* d\mathbf{k}
\end{aligned} \tag{5.18}$$

Proof. The above formula is similar to the one known from the theory of the Boltzmann equation [17], hence following these lines, the present equality can be easily proven.

Lemma 2. *For any functions $\varPhi$ and $\varPsi$ as in Lemma 1, the following identity holds*

$$\int \varPsi L\varPhi dv = \int \varPhi L\varPsi dv. \tag{5.19}$$

Proof. The above relation is an immediate consequence of (5.18). The proof is complete.

Lemma 3. *For any function Φ, the following inequality is true:*

$$\int \Phi L \Phi d\mathbf{v} \leq 0, \tag{5.20}$$

and the equality sign takes place if and only if

$$\Phi\left(\mathbf{v}\right) = a + \mathbf{b} \cdot \mathbf{v}, \tag{5.21}$$

where a is an arbitrary scalar independent of the velocity $\boldsymbol{v}$, and $\boldsymbol{b}$ is an arbitrary vector neither independent of $\boldsymbol{v}$.

Proof. Setting $\Phi = \Psi$ in (5.18) we immediately obtain the inequality (5.20). To prove the second part of the thesis let us assume that

$$\int \Phi L \Phi d\mathbf{v} = 0. \tag{5.22}$$

As it follows from (5.18) function Φ satisfies this equation only if it satisfies the following four equations:

$$\int f_0 f_{0*} \left(\Phi' + \Phi'_* - \Phi - \Phi_*\right)^2 \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v} d\mathbf{v}_* d\mathbf{k} = 0, \tag{5.23}$$

$$\int f_0 f_{0*} \left(\Phi''' + \Phi'''_* - \Phi - \Phi_*\right)^2 \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g} - \sqrt{4\gamma}\right) d\mathbf{v} d\mathbf{v}_* d\mathbf{k} = 0, \tag{5.24}$$

$$\int f_0 f_{0*} \left(\Phi'' + \Phi''_* - \Phi - \Phi_*\right)^2 \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v} d\mathbf{v}_* d\mathbf{k} = 0, \tag{5.25}$$

$$\int f_0 f_{0*} \left(\Phi' + \Phi'_* - \Phi - \Phi_*\right)^2 \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\sqrt{4\gamma} - \mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v} d\mathbf{v}_* d\mathbf{k} = 0. \tag{5.26}$$

It is well known that the general solution of Eq. (5.23) is of the form

$$\Phi\left(\mathbf{v}\right) = a + \mathbf{b} \cdot \mathbf{v} + c v^2, \tag{5.27}$$

where a, $\mathbf{b}$ and c do not depend on $\mathbf{v}$. Since such a function satisfies also Eq. (5.26) it remains to check whether it is a solution of Eqs. (5.24) and (5.25). Inserting (5.27) into (5.24) or (5.25), we obtain

$$c^2 \gamma^2 \int f_0 f_{0*} \left(\mathbf{k}\cdot\mathbf{g}\right) H\left(\mathbf{k}\cdot\mathbf{g}\right) d\mathbf{v} d\mathbf{v}_* d\mathbf{k} = 0.$$

This equation is satisfied only for $c = 0$. Hence (5.27) reduces to (5.21). The proof is complete.

5.2. Solution of the Equation (5.15)

Equation (5.15) is solvable if the left-hand side is orthogonal to the kernel of the operator L which consists of functions of the form (5.21). Integrating Eq. (5.15) with respect to $\mathbf{c}$ we obtain

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0. \tag{5.28}$$

Multiplying both sides of Eq. (5.15) by the thermal velocity $\mathbf{c}$ and integrating the result with respect to this variable, we obtain the next solvability condition

$$\rho\frac{D\mathbf{u}}{Dt} + \nabla\left[\rho T\left(1+\Lambda\right)\right] = \rho\mathbf{f}. \tag{5.29}$$

These two equations define the time derivatives of density ρ and velocity $\mathbf{u}$, but in Eq. (5.15) we have also the time derivative of temperature T. To find an equation for it we proceed as follows. We multiply Eq. (5.15) by c^2 and integrate over the velocity space. The result is

$$\frac{3}{2}\rho\frac{DT}{Dt} + \rho T\left(1+\Lambda\right)\nabla\cdot\mathbf{u} = -R_1. \tag{5.30}$$

Combining (5.30) and (5.17) we obtain

$$\left(\frac{3}{2} + U'_T\right)\frac{DT}{Dt} = -\left(T\left(1+\Lambda\right) - \rho U'_\rho\right)\nabla\cdot\mathbf{u}. \tag{5.31}$$

Comment. We can solve this equation for $\dfrac{DT}{Dt}$ provided that the following obvious condition

$$\frac{3}{2} + U'_T \neq 0 \tag{5.32}$$

is satisfied. On the other hand, the energy conservation equation (4.28) suggests that the quantity $\frac{3}{2}T + U\left(\rho, T\right)$ can be treated as the classical internal energy e considered in the first four sections of the paper, provided that we approximate the kinetic temperature θ by the potential temperature T. If so, then condition (5.32) means that the equation

$$\frac{3}{2}T + U\left(\rho, T\right) = e \tag{5.33}$$

can be uniquely solved for T. Hence it is possible to use in the expansion (5.2), the internal energy e instead of the temperature T . Both theories will be equivalent, provided that (5.32) is satisfied. This type of approach was chosen in the reference [18] (see also [19]), The present choice was taken due to its simplicity.

We use (5.28), (5.29) and (5.31) in Eq. (5.15) and conclude that the function $\varPhi$ can be represented in the form

$$\varPhi = -\frac{1}{\rho}\mathbf{a}\cdot\frac{\nabla T}{T} - \frac{1}{\rho}\mathbf{B} : (\nabla\mathbf{u}) - \frac{1}{3\rho}C\nabla\cdot\mathbf{u}, \tag{5.34}$$

where $\mathbf{a}$, $\mathbf{B}$ and C are solutions of the equations listed below: the equation for $\mathbf{a}$

$$\begin{aligned} f_0\Bigg[-\mathbf{c}\left(1+\frac{3}{5}\varLambda\right)S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) - \frac{r}{2\tau}\frac{\zeta}{T}\int f_{0*}\mathbf{k}\left(\mathbf{Gk}\right)\\ \left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2+4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}\right)d\mathbf{v}_*d\mathbf{k} + \frac{r}{2\tau}\frac{\eta}{T}\int f_{0*}\mathbf{k}\left(\mathbf{G}\cdot\mathbf{k}\right)\\ \left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2-4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{kg}-\sqrt{4\gamma}\right)d\mathbf{v}_*d\mathbf{k}\Bigg] = -\frac{1}{\rho}L\left(\mathbf{a}\right), \end{aligned} \tag{5.35}$$

Next, the equation for $\mathbf{B}$:

$$\begin{aligned} f_0\Bigg[2\left(1+\frac{2}{5}\varLambda\right)\left(\varOmega\varOmega-\frac{\varOmega^2}{3}\mathbf{I}\right) - \frac{r}{2\tau}\frac{\zeta}{T}\int f_{0*}\left(\mathbf{kk}-\frac{1}{3}\mathbf{I}\right)\\ \left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2+4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}\right)d\mathbf{v}_*d\mathbf{k}\\ +\frac{r}{2\tau}\frac{\eta}{T}\int f_{0*}\left(\mathbf{kk}-\frac{1}{3}\mathbf{I}\right)\left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2-4\gamma}\right]\\ \left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)d\mathbf{v}_*d\mathbf{k}\Bigg] = -\frac{1}{\rho}L\left(\mathbf{B}\right), \end{aligned} \tag{5.36}$$

and that for C

$$\begin{aligned} f_0\Bigg[-2S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right)\frac{\frac{2}{3}TU'_T\left(1+\varLambda\right)+\rho U'_\rho}{T\left(1+\frac{2}{3}U'_T\right)} - \frac{r}{2\tau}\frac{\zeta}{T}\int f_{0*}\\ \left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2+4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}\right)d\mathbf{v}_*d\mathbf{k}\\ \frac{r}{2\tau}\frac{\eta}{T}\int f_{0*}\left[\mathbf{k}\cdot\mathbf{g}+\sqrt{\left(\mathbf{k}\cdot\mathbf{g}\right)^2-4\gamma}\right]\left(\mathbf{k}\cdot\mathbf{g}\right)H\left(\mathbf{k}\cdot\mathbf{g}-\sqrt{4\gamma}\right)d\mathbf{v}_*d\mathbf{k}\Bigg]\\ = -\frac{1}{\rho}L\left(C\right). \end{aligned} \tag{5.37}$$

The vector field $\mathbf{a}$ can be taken in the form

$$\mathbf{a} = A\left(\mathit{\Omega}\right)\boldsymbol{\Omega}, \tag{5.38}$$

whereas $\mathbf{B}$ can be assumed to be

$$\mathbf{B} = B\left(\mathit{\Omega}\right)\left(\boldsymbol{\Omega}\boldsymbol{\Omega} - \frac{\mathit{\Omega}^2}{3}\mathbf{I}\right). \tag{5.39}$$

The function $\mathit{\Phi}$ should satisfy the conditions resulting from assumptions (5.3). These implies

$$\int f_0 A\left(\Omega\right)\mathit{\Omega}^2 d\mathbf{c} = 0, \qquad \int f_0 C\left(\mathit{\Omega}\right) d\mathbf{c} = 0. \tag{5.40}$$

6. Equations of hydrodynamics

First, let us substitute

$$f = f_0\left(1 + \varepsilon\mathit{\Phi}\right), \tag{6.1}$$

where f_0 is given by (5.13) and $\mathit{\Phi}$ by (5.34), into the definition (4.20) of the kinetic temperature θ and obtain

$$\theta = T\left(1 + \frac{2\varepsilon}{9\rho^2}\int f_0 C S_{\frac{1}{2}}^{(1)}\left(\mathit{\Omega}^2\right) d\mathbf{c}\nabla\cdot\mathbf{u}\right). \tag{6.2}$$

Next, we insert (6.1) into the definitions (4.14)–(4.18) of the component stresses and sum the results to obtain the total stress tensor, as indicated by (4.13). Very heavy calculations yield the following approximate expression for this tensor:

$$\mathbf{P} = \rho\theta\left(1 + \Lambda\right)\mathbf{I} - \mu\mathbf{D} - \tilde{\kappa}\nabla\cdot\mathbf{u}\mathbf{I} + \frac{2\varepsilon}{9\rho}\frac{DU}{Dt}\int f_0 C S_{\frac{1}{2}}^{(1)}\left(\mathit{\Omega}^2\right) d\mathbf{c}\mathbf{I}, \tag{6.3}$$

where

$$\begin{aligned}\mu = &-\varepsilon\frac{T}{10\rho^2}\int \mathbf{B}L\left(\mathbf{B}\right)d\mathbf{c}\\ &+\frac{\varepsilon}{15}\rho^2\sqrt{\pi T}\left[\frac{\alpha^2}{\varepsilon^2}\chi + \frac{r^2}{\varepsilon\tau}\eta - \frac{r^2}{\varepsilon\tau}\zeta\left(\frac{1}{2}\int\limits_0^\infty e^{-\frac{y}{2}}y^2\sqrt{y^2 + \frac{4\gamma}{T}}dy + \frac{\gamma}{T}\right)\right]\end{aligned} \tag{6.4}$$

is the shear viscosity, and

$$\tilde{\kappa} = -\varepsilon \frac{T}{9\rho^2} \int C L\left(C\right) d\mathbf{c} + \frac{\varepsilon}{9}\rho^2\sqrt{\pi T}\left[\frac{\alpha^2}{\varepsilon^2}\chi + \frac{r^2}{\varepsilon\tau}\eta - \frac{r^2}{\varepsilon\tau}\zeta\left(\frac{1}{2}\int_0^\infty e^{-\frac{y}{2}}y^2\sqrt{y^2+\frac{4\gamma}{T}}dy + \frac{\gamma}{T}\right)\right] \tag{6.5}$$

is the bulk viscosity, and the components of tensor $\mathbf{D}$ are defined in (3.10).

To find the expression for the heat flux we proceed in a very similar way. We use (6.1) in the definitions (4.22)–(4.26) and after tedious calculations (as in the case of stresses) we obtain the Fourier law

$$\mathbf{q} = -\lambda \nabla T, \tag{6.6}$$

with the following formula for the heat conductivity coefficient λ:

$$\lambda = \frac{2}{3}\varepsilon\left[-\frac{T}{\rho^2}\int \mathbf{a}\cdot L\left(\mathbf{a}\right) d\mathbf{c} + \rho^2\sqrt{\pi T}\left(\frac{\alpha^2}{\varepsilon^2}\chi + \frac{r^2}{\varepsilon\tau}\eta - \frac{r^2}{\varepsilon\tau}\zeta\left(\frac{1}{2}\int_0^\infty e^{-\frac{y}{2}}y^2\sqrt{y^2+\frac{2\gamma}{T}}dy + \frac{\gamma}{T}\right)\right)\right]. \tag{6.7}$$

Our expressions for the transport coefficients μ, $\tilde{\kappa}$, and λ are similar to those obtained by previous authors and discussed in [13, 14] and [18].

6.1. The entropy problem

We have formally derived the system of hydrodynamic limit equations under the only assumption imposed on the function U that it is continuously differentiable with respect to its only arguments ρ and T. This is however insufficient to claim that the obtained system of equations is satisfactory from the physical point of view, since we should not expect that, under such a weak assumption, the entropy functional exists. On the contrary, it can exist only for a special form of the function U. An example is presented below. We take

$$U = -T^2\frac{\partial}{\partial T}\int_{\rho_0}^{\rho}\frac{\Lambda\left(\nu, T\right)}{\nu}d\nu + U_0\left(T\right), \tag{6.8}$$

where $U_0(T)$ is an arbitrary differentiable function, and ρ_0 is a positive number. We define

$$V(\rho,T) = -\frac{\partial}{\partial T}\left(T\int_{\rho_0}^{\rho}\frac{\Lambda(\nu,T)}{\nu}d\nu\right) + \int_{T_0}^{T}\frac{U_0(\tau)}{\tau}d\tau. \tag{6.9}$$

As the entropy function we take

$$s(\rho,T) = -\log\frac{\rho}{\theta^{\frac{3}{2}}} + V(\rho,T), \tag{6.10}$$

Combining the energy equation (4.27) with (6.6) and the momentum equation (4.10) with the equation of state (6.3) we obtain, owing to the relation (6.2),

$$\rho\frac{DS}{Dt} = \frac{1}{\theta}\left[-\frac{\rho\theta}{T}\frac{DU}{Dt} - \rho\theta\mathit{\Lambda}\nabla\cdot\mathbf{u} + \frac{\mu}{2}\mathbf{D}:\mathbf{D} + \kappa(\nabla\cdot\mathbf{u})^2 - \nabla\cdot\mathbf{q}\right]$$

$$+\rho\frac{DV}{Dt} = -\frac{1}{\theta}\nabla\cdot\mathbf{q} + \frac{\mu}{2\theta}\mathbf{D}:\mathbf{D} + \frac{\kappa}{\theta}(\nabla\cdot\mathbf{u})^2 + \rho\left(\frac{DV}{Dt} - \mathit{\Lambda}\nabla\cdot\mathbf{u} - \frac{1}{T}\frac{DU}{Dt}\right).$$

The functions U and V are related by

$$\mathit{\Lambda}\nabla\cdot\mathbf{u} + \frac{1}{T}\frac{DU}{Dt} = \frac{DV}{Dt},$$

hence finally we obtain

$$\theta\left[\frac{\partial}{\partial t}(\rho s) + \nabla\cdot(\rho s\mathbf{u})\right] + \theta\nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) = -\mathbf{q}\cdot\frac{\nabla\theta}{\theta} + \frac{\mu}{2}\mathbf{D}:\mathbf{D} + \kappa(\nabla\cdot\mathbf{u})^2, \tag{6.11}$$

what has the form of the classical entropy equation.

6.2. Modified hydrodynamic limit equations

The hydrodynamic limit equations differ in two aspects from the classical equations of fluid dynamics. The first difference is the presence of two different quantities θ and T of the character of temperature. For the first time this rather unusual fact was obtained in [14] and [18] (see also [19]). Secondly, another unusual thing is the presence of the term

$$\frac{2\varepsilon}{9\rho}\frac{DU}{Dt}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\mathit{\Omega}^2\right)d\mathbf{c}. \tag{6.12}$$

H. van Beijeren [19] gave two proposals of how to introduce only one temperature. Our idea is simpler and, at least in our opinion, more natural. It consists in using the kinetic temperature θ, because it enters the energy balance equation in a very natural way. Let us notice that using the implicit function theorem, we can invert (6.2) and obtain

$$T = \theta\left(1 - \frac{2\varepsilon}{9\rho^2}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c}\nabla\cdot\mathbf{u} + O\left(\varepsilon^2\right)\right). \tag{6.13}$$

Using this formula we can write

$$\begin{aligned}
&\rho\theta\left(1+\varLambda\left(\rho,T\right)\right) + \frac{2\varepsilon}{9\rho}\frac{DU}{Dt}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c} \\
&\qquad = \rho\theta\left(1+\varLambda\left(\rho,\theta\right)\right) - \rho\theta\left(\varLambda\left(\rho,\theta\right)-\varLambda\left(\rho,T\right)\right) \\
&\qquad - \frac{2\varepsilon}{9\rho}\frac{\frac{2}{3}TU'_T\left(1+\varLambda\right)+\rho U'_\rho}{\left(1+\frac{2}{3}U'_T\right)}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c}\nabla\cdot\mathbf{u} + O\left(\varepsilon^2\right) = p\left(\rho,\theta\right) \\
&\qquad - \frac{2\varepsilon}{9\rho}\frac{T^2\varLambda'_T+\rho U'_\rho+\frac{2}{3}TU'_T\left(T\left(1+\varLambda\right)\right)'_T}{\left(1+\frac{2}{3}U'_T\right)}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c}\nabla\cdot\mathbf{u} + O\left(\varepsilon^2\right).
\end{aligned} \tag{6.14}$$

where we have introduced the following notation:

$$p = p\left(\rho,\theta\right) = \rho\theta\left(1+\varLambda\left(\rho,\theta\right)\right). \tag{6.15}$$

This quantity has obviously the meaning of hydrostatic pressure. Under the assumption of (6.8), we obtain from (6.14)

$$\begin{aligned}
&\rho\theta\left(1+\varLambda\left(\rho,T\right)\right) + \frac{2\varepsilon}{9\rho}\frac{DU}{Dt}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c} \\
&\qquad = p\left(\rho,\theta\right) - \frac{4}{27}\varepsilon\frac{\theta U'_\theta p'_\theta}{1+\frac{2}{3}U'_\theta}\int f_0 C S^{(1)}_{\frac{1}{2}}\left(\varOmega^2\right) d\mathbf{c}\nabla\cdot\mathbf{u} + O\left(\varepsilon^2\right),
\end{aligned}$$

where we have replaced T with θ. Since the error we make by applying such a procedure is of higher order of magnitude, we can write

$$\mathbf{P} = p\left(\rho,\theta\right)\mathbf{I} - \mu\mathbf{D} - \kappa\left(\nabla\cdot\mathbf{u}\right)\mathbf{I}, \tag{6.16}$$

where the present viscosity coefficients are

$$\begin{aligned}\mu = {} & -\varepsilon\frac{\theta}{10\rho^2}\int \mathbf{B} : L\left(\mathbf{B}\right)d\mathbf{c} \\ & + \frac{\varepsilon}{15}\rho^2\sqrt{\pi\theta}\left[\frac{\alpha^2}{\varepsilon^2}\chi + \frac{r^2}{\varepsilon\tau}\eta - \frac{r^2}{\varepsilon\tau}\zeta\left(\frac{1}{2}\int\limits_0^\infty e^{-\frac{y}{2}}y^2\sqrt{y^2+\frac{4\gamma}{\theta}}dy + \frac{\gamma}{\theta}\right)\right]\end{aligned} \tag{6.17}$$

and

$$\kappa = \tilde{\kappa} + \frac{4}{27}\varepsilon\frac{\theta U_\theta' p_\theta'}{1+\frac{2}{3}U_\theta'}\int f_0 C S_{\frac{1}{2}}^{(1)}\left(\Omega^2\right)d\mathbf{c}. \tag{6.18}$$

Let us notice that Eq. (4.12), with the approximate expression (6.16) for the stress tensor **P** coincides in form with the phenomenological momentum conservation equations (3.8), (3.9), but now we have deduced the expressions for the pressure and the transport coefficients instead of postulating them. In addition, if the force term in the kinetic equation (4.5) is taken, for instance, in the form (2.14) or (3.1), (3.2), then we can give Eq. (4.12) the form (3.11) with the corresponding Korteweg tensor **K** since now the considerations of Secs. 2 and 3 can be applied.

The heat flux vector **q** given by (6.6) takes now the form

$$\mathbf{q} = -\lambda\nabla\theta, \tag{6.19}$$

where now the heat conductivity coefficient λ is

$$\begin{aligned}\lambda = \frac{2}{3}\varepsilon\Bigg[& -\frac{\theta}{\rho^2}\int \mathbf{a}\cdot L\left(\mathbf{a}\right)d\mathbf{c} \\ & + \rho^2\sqrt{\pi\theta}\left(\frac{\alpha^2}{\varepsilon^2}\chi + \frac{r^2}{\varepsilon\tau}\eta - \frac{r^2}{\varepsilon\tau}\zeta\left(\frac{1}{2}\int\limits_0^\infty e^{-\frac{y}{2}}y^2\sqrt{y^2+\frac{2\gamma}{\theta}}dy + \frac{\gamma}{\theta}\right)\right)\Bigg],\end{aligned} \tag{6.20}$$

The functions U and V are

$$U = -\theta^2\frac{\partial}{\partial\theta}\frac{1}{\theta}\int\limits_{\rho_0}^{\rho}\frac{p\left(\nu,\theta\right)}{\nu^2}d\nu + U_0\left(\theta\right), \tag{6.21}$$

$$V = -\frac{\partial}{\partial\theta}\left(\theta\int\limits_{\rho_0}^{\rho}\frac{\Lambda\left(\nu,\theta\right)}{\nu}d\nu\right) + \int\limits_{\theta_0}^{\theta}\frac{U_0\left(\tau\right)}{\tau}d\tau. \tag{6.22}$$

Let us notice again that Eq. (4.28) with the approximate expressions (6.16) for the stress tensor $\mathbf{P}$, (6.19) for the heat flux $\mathbf{q}$ and the internal energy $e = \frac{3}{2}\,\theta + U$ coincides with the phenomenological energy conservation equations (3.12), (3.13). Similarly to the momentum case, if the force term in the kinetic equation (4.5) has the form (2.14) or (3.1), (3.2) then we can give Eq. (4.28) the conservative form (3.14) since the considerations of Secs. 2 and 3 apply to the present case as well.

6.3. Correlation functions for some specific equations of state

At the end of our considerations let us to discuss the problem of the pair correlation functions χ and ζ present in the kinetic equation (4.5) (function η is known by (5.12) once we know ζ). Up to now we have assumed that they are given, known functions of the density ρ and the potential temperature T, but the question arises how to determine them? As they cannot be determined from the presented kinetic theory they must be determined from some other considerations. Our proposal is to determine these functions by fitting the obtained expression (6.15) for the hydrostatic pressure to the existing in the literature, phenomenological equations of state. Below we give some examples of the choice of the correlation functions corresponding to some specific phenomenological equations of state, which are reviewed for example in [20]. We begin with

- The van der Waals equation of state:

$$p = \frac{\rho\,\theta}{1 - b\rho} - a\rho^2, \tag{6.23}$$

where a, b are positive constants. A comparison of (6.23) with Eqs. (5.12), (5.16) and (6.15) suggests the following choice:

$$a = \frac{2\pi}{3}\gamma\frac{r}{\tau}, \quad b = \frac{2\pi}{3}\frac{\alpha}{\varepsilon}, \quad \chi = \frac{1}{1 - b\rho}, \quad \zeta = \frac{\gamma}{\theta}\left(e^{\frac{\gamma}{\theta}} - 1\right)^{-1}. \tag{6.24}$$

- More accurate is the Redlich–Kwong equation of state [21]

$$p = \frac{\rho\theta}{1 - b\rho} - \frac{a\rho^2}{\sqrt{\theta}\,(1 + b\rho)}, \tag{6.25}$$

where a, b are positive constants. This time we take a, b and χ as in the previous case and

$$\zeta = \frac{1}{\sqrt{\theta}\,(1 + b\rho)}\frac{\gamma}{\theta}\left(e^{\frac{\gamma}{\theta}} - 1\right)^{-1}. \tag{6.26}$$

Let us notice that now the correlation function depends on both density and temperature.

- The Carnaham–Starling models [22, 23], (see also [20]). In paper [23] Carnaham and Starling modified their model of the equation of state formulated in [22] for repelling hard spheres by adding to it an attractive part as in the van der Waals equation or else as in the Redlich-Kwong equation. The Carnaham–Starling–van der Waals model is

$$p = \rho\,\theta \left[\frac{1 + \dfrac{b\rho}{4} + \left(\dfrac{b\rho}{4}\right)^2 - \left(\dfrac{b\rho}{4}\right)^3}{\left(1 - \dfrac{b\rho}{4}\right)^3} \right] - a\rho^2, \tag{6.27}$$

where a, b are positive constants. We take them and the function ζ such in (6.24), whereas the formula for χ is obtained in the following form:

$$\chi = \frac{1 - \dfrac{b\rho}{8}}{\left(1 - \dfrac{b\rho}{4}\right)^3}. \tag{6.28}$$

The Carnaham-Starling-Redlich-Kwong equation reads

$$p = \rho\,\theta \left[\frac{1 + \dfrac{b\rho}{4} + \left(\dfrac{b\rho}{4}\right)^2 - \left(\dfrac{b\rho}{4}\right)^3}{\left(1 - v\dfrac{b\rho}{4}\right)^3} \right] - \frac{a\rho^2}{\sqrt{\theta}\,(1 + b\rho)}, \tag{6.29}$$

where a, b are as previously. This time the function χ is given by Eq. (6.28), whereas ζ is defined by Eq. (6.26).

The model of CHRISTOFORAKOS and FRANCK [24] is a combination of the Carnaham–Starling repulsive model with the attractive part in the form of the square-well potential. The latter fact is essential since our kinetic model contains square-well-like terms. The CF equation of state is of the form

$$p = \rho\,\theta \left[\frac{1 + \left(\dfrac{b\,\rho}{\theta^m}\right) + \left(\dfrac{b\rho}{\theta^m}\right)^2 - \left(\dfrac{b\rho}{\theta^m}\right)^3}{\left(1 - \dfrac{b\rho}{\theta^m}\right)^3} \right] - \frac{\rho^2 ab}{\theta^{m-1}} \left(e^{\frac{\gamma}{\theta}} - 1\right), \tag{6.30}$$

where b is a positive constant characterizing the medium, m is usually equal to 0.3, a characterizes the width of the well, and γ is the depth of the well. The formula (6.15) coincides with that of (6.30) if we assume

$$a = \frac{\varepsilon}{\alpha}\frac{r}{\tau}, \quad b = \frac{2\pi}{3}\frac{\alpha}{\varepsilon}, \quad \chi = \frac{4}{\theta^m}\left(1 - \frac{1}{2}\frac{b\,\rho}{\theta^m}\right)\left(1 - \frac{b\,\rho}{\theta^m}\right)^{-3}, \quad \zeta = \frac{1}{\theta^m}. \tag{6.31}$$

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