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Slow motion of a rotating circular cylinder through a micropolar fluid

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Presented is an analytical solution to creeping flow of a micropolar fluid past a rotating circular cylinder of infinite length in spanwise direction. The solution is decomposed into two parts; first, the flow past a stationary circular cylinder is solved by the use of matched asymptotic expansions method. Afterwards, the rotation of a circular cylinder in a stationary ocean of a micropolar fluid is investigated. Due to linearity of the governing equations, the principle of superposition is then recalled to construct the desired flow field. Ultimately, several kinematic and kinetic quantities of the flow are studied by the use of the obtained closed-form analytical solution.

Key words: micropolar fluid, creeping flow, low-Reynolds number flow, rotating circular cylinder.

1. Introduction

An analytical solution of the so-called Navier–Stokes differential equations can be obtained by the assumptions of creeping flow, potential flow and a boundary-layer flow. Under the assumption of small Reynolds numbers (Re), i.e. creeping flow, the nonlinear terms of Navier–Stokes equations, which are inertial terms, become negligible and the flow is viscous dominant. Stokes obtained a closed-form analytical solution to the problem of very slow motion of a sphere through a viscous, Newtonian, incompressible fluid in 1851 [1]. The solution is only valid for $\mathrm{Re} \ll 1$ and for higher values of Re, the nonlinear terms become significant and a general analytical solution is impossible. The solution of Stokes' equation in 2D, e.g. for an infinite cylinder in a cross-stream, cannot fulfill the boundary conditions far from the body. This mismatch is called the Stokes' paradox. Oseen solved the paradox by a linear approximation of the nonlinear term [2].

Lamb solved the problem of slow motion of a sphere through a viscous fluid [3]. Faxen solved the Oseen's differential equation for slow flow past a circular cylinder [4]. Tomotika and Aoi investigated the steady low-Reynolds number flow of a viscous fluid past a sphere and a circular cylinder [5]. PROUDMAN and PEARSON considered the flow about spheres and cylinders at small Reynolds numbers using asymptotic expansions [6]. KAPLUN and LAGERSTROM also investigated creeping flow past a circular cylinder by utilizing asymptotic expansions [7–9]. ATEFI solved the Oseen's differential equation for flow past stationary and rotating circular cylinders at small Reynolds numbers with mixed stick-slip boundary conditions, analytically [10–12]. PADMAVATHI *et al.* investigated the Stokes flow past a sphere with mixed stick-slip boundary conditions [13].

Micropolar fluids are fluids with microstructures. They belong to a class of fluids with a non-symmetric stress tensor. Micropolar fluids consist of rigid, randomly oriented (or spherical) particles with their own spins and microrotations, suspended in a viscous medium. The concept of microrotation was proposed by COSSERAT and COSSERAT in the theory of elasticity [14, 15]. CONDIFF and DAHLER [16], and ERINGEN [17] applied the concept to describe fluids with microstructures in the middle of the 1960s. TROSTEL investigated local and non-local Cosserat-type fluids [18]. ALEXANDRU developed a second-grade Cosserat-type fluid theory based on the generalized continuum conceptions of TROSTEL [19]. Moreover, comprehensive textbooks on micropolar fluids have been published [20–22]. More recently, MOOSAIE and ATEFI have analytically investigated the turbulent flows as well as a flow of complex fluids by means of Cosserat-type (micropolar) and microstretch fluids [23–26].

There are some relevant previous investigations on creeping flows in micropolar fluid mechanics [27–30]. In particular, RAMKISSOON [31] has obtained the solution of a micropolar fluid flow around a sphere and the drag force exerted on the sphere. Later, POWER and RAMKISSOON [32] presented a fundamental solution, i.e., the Green function, etc., for the Stokesian micropolar flow. BUCHUKURI and CHICHINADZE [33] obtained the fundamental solution and predicted the fluid flow around a cylinder as an integral form, but they could not present the explicit velocity field and the drag force. HAYAKAWA solved the problems of axisymmetric slow viscous flow of a micropolar fluid past a stationary sphere and a stationary cylinder explicitly, and computed the drag force in each case [34].

Micropolar fluids are of interest in the general context of non-Newtonian fluid mechanics. Some application ranges from flow of blood and blood-like fluids [24], suspensions of rigid particles in Newtonian fluids [25, 26], liquid crystals, granular fluids [34] and hydrodynamic turbulence [23].

As mentioned above, the published materials on creeping flow of micropolar fluids are related to some flow situations around fixed bodies. Thus, the important issue of flow past rotating bodies in micropolar fluid mechanics, due to the author's knowledge, is missing. In this paper, an analytical closed-form solution to the problem of slow flow of a micropolar fluid around a rotating circular cylinder is obtained. This paper is an extension of the work done by HAYAKAWA [34] in which the solution for slow flow past a stationary cylinder is given. At first, a brief review of governing equations is presented. Afterwards, thanks to the linearity of governing equations of creeping flow, the flow of micropolar fluid past a stationary circular cylinder is superposed on the flow induced by rotation of a circular cylinder in a stationary ocean of micropolar fluid to obtain the desired solution. Finally, several aspects of the flow field are studied.

2. Governing equations and boundary conditions

In this section, the general governing equations for the calculation of slow micropolar fluid flows are presented. Let's restrict our interest to steady viscous flow in which the partial derivatives with respect to time are assumed to be zero. For simplicity, the dimensionless quantities are used for later discussion, which are normalized by the velocity far from the cylinder and the radius of the cylinder. Thus, we start from the following set of equations. The continuity equation for an incompressible micropolar fluid is identical with that of a classical fluid. The incompressibility condition is given by

$$\nabla \cdot \mathbf{v} = \operatorname{div} \mathbf{v} = 0, \tag{2.1}$$

where $\mathbf{v}$ is the velocity field. The equation of linear momentum is [21]

$$\mathrm{Re}\, \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Delta \mathbf{v} + \mu_r \operatorname{rot} \boldsymbol{\omega}, \tag{2.2}$$

where Δ is the Laplacian, Re is the effective Reynolds number, p is the pressure, μ_r is the dimensionless viscosity of microrotation field which is assumed to be less than 2. Note that the Reynolds number Re and μ_r are represented by quantities with physical units as $\mathrm{Re} = \rho U a/(\eta + \eta_r)$ and $\mu_r = 2\eta_r/(\eta + \eta_r)$, where ρ, U, a, η and η_r are the density, the magnitude of the characteristic flow, i.e., the free-stream velocity far from the cylinder, the radius of the cylinder, the conventional viscosity and the viscosity for microrotation, respectively. The flow of microrotation is governed by the equation of angular momentum

$$\frac{\mathrm{Re} \cdot I}{\mu_r} \mathbf{v} \cdot \nabla \boldsymbol{\omega} = \operatorname{rot} \mathbf{v} - 2\boldsymbol{\omega} + \mu_A \nabla \operatorname{div} \boldsymbol{\omega} + \mu_B \Delta \boldsymbol{\omega}, \tag{2.3}$$

where I is the dimensionless microinertia coefficient and μ_A and μ_B are dimensionless rotational viscosities [21]. Similar to Newtonian fluids, μ_A and μ_B are dimensionless bulk and shear viscosities with respect to the microrotation field, respectively. In general, $\operatorname{div} \boldsymbol{\omega}$ is not equal to zero, but it is easy to show that

$$\operatorname{div} \boldsymbol{\omega} = 0, \tag{2.4}$$

for axisymmetric flows. For the case of flow around a rotating cylinder we have

$$
\begin{aligned}
v_z &= \omega_r = \omega_\theta = 0,\\
v_r &= v_r(r,\theta),\\
v_\theta &= v_\theta\,(r,\theta)\,,\\
\omega_z &= \omega_z\,(r,\theta)\,.
\end{aligned}
\tag{2.5}
$$

Equations $(2.5)_1$ and $(2.5)_4$ imply that $\operatorname{div}\boldsymbol{\omega} = 0$. Thus, the microrotation field is regarded as a solenoidal field, where the term proportional to μ_A in Eq. (2.3) is zero in later discussion. Let us remark on the micropolar fluid model. It is obvious that the model is reduced to the Navier–Stokes equation for $\mu_r \to 0$.

We assume the boundary condition outside the cylinder as

$$
\begin{aligned}
&\mathbf{v} = \mathbf{v}_{\text{boundary}}, \qquad \boldsymbol{\omega} = \mathbf{0} \qquad \text{at } r = 1,\\
&\mathbf{v} = \mathbf{e}_x, \qquad \boldsymbol{\omega} \to \frac{1}{2}\operatorname{rot}\mathbf{v} \qquad \text{as } r \to \infty,
\end{aligned}
\tag{2.6}
$$

where r is the distance from the center of the cylinder whose radius is unity, and $\mathbf{e}_x$ is the unit vector along the x axis. The above boundary conditions are not always valid in micropolar fluids. Here, we assume the no-slip boundary condition on the boundary surface. Effective slip of particles is included as microrotation. The microrotation on the surface is assumed to be zero because the center of rotation cannot exist on the surface, but it exists at a position removed by the particle radius. The microrotation coincides with the rotation of the flow velocity $\mathbf{v}$ if the place is far away enough from the cylinder. Regarding the first condition in $(2.6)_1$, if a stationary cylinder is considered then $\mathbf{v}_{\text{boundary}} = \mathbf{0}$, but when a rotating cylinder is going to be investigated, the velocity boundary condition becomes $\mathbf{v}_{\text{boundary}} = a\Omega\mathbf{e}_\theta$, where Ω is the angular velocity of the cylinder. The problem geometry and coordinates are depicted in Fig. 1.

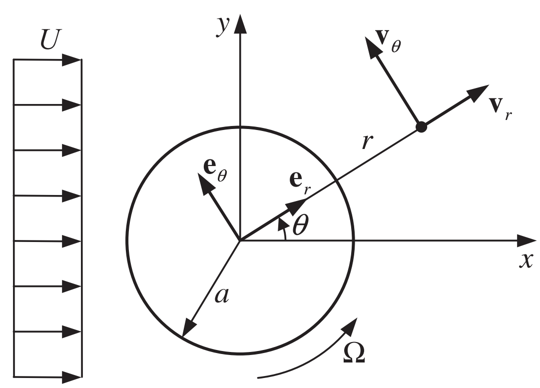


FIG. 1. Problem geometry and the cylindrical coordinate system used.

Now, we are going to carry out a systematic calculation of the flow field around a rotating cylinder. For this purpose, we adopt the matched asymptotic method developed by Kaplun and Lagerstrom [7–9] for the 2D-problem, explained in [35,36] and later used by Hayakawa [34]. It is well known that the Stokes approximation ($\mathrm{Re} \to 0$ and $\mu_r \to 0$ in Eq. (2.2)) is invalid far from the cylinder. Therefore, in micropolar fluids, we need careful treatments to calculate the flow around a cylinder. To remove such difficulties we introduce an appropriate contracted coordinate as

$$\widetilde{x} = \mathrm{Re} \cdot x, \qquad \widetilde{y} = \mathrm{Re} \cdot y, \tag{2.7}$$

and the scaled variables

$$\begin{aligned} \mathbf{v} &= \mathbf{e}_x + \alpha(\mathrm{Re})\mathbf{u}(\widetilde{\mathbf{r}}), \\ \boldsymbol{\omega} &= \alpha(\mathrm{Re})\,\mathrm{Re} \cdot \widetilde{\omega}\mathbf{e}_z, \\ p &= \alpha(\mathrm{Re})\,\mathrm{Re} \cdot \widetilde{p}. \end{aligned} \tag{2.8}$$

The function $\alpha(\mathrm{Re})$ will be determined by the matching procedure. Thus, for the 2D-case Eqs. (2.2) and (2.3) are reduced to

$$\frac{\partial \mathbf{u}}{\partial \widetilde{x}} = -\widetilde{\nabla}\widetilde{p} + \widetilde{\Delta}\mathbf{u} + \mu_r \widetilde{\mathrm{rot}}(\widetilde{\omega}\mathbf{e}_z), \tag{2.9}$$

$$\frac{\mathrm{Re} \cdot I}{\mu_r} \frac{\partial \widetilde{\omega}}{\partial \widetilde{x}} = \frac{\partial u_y}{\partial \widetilde{x}} - \frac{\partial u_x}{\partial \widetilde{y}} - 2\widetilde{\omega} + \mu_B \widetilde{\Delta}\widetilde{\omega}, \tag{2.10}$$

where Eq. (2.4) is assumed to be valid for the case of flow past a rotating cylinder. One can conclude from Eq. (2.10) that for the limiting case of $\mathrm{Re} \to 0$, the relations

$$\begin{aligned} \widetilde{\omega} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial \widetilde{x}} - \frac{\partial u_x}{\partial \widetilde{y}}\right), \\ \widetilde{\mathrm{rot}}(\widetilde{\omega}\mathbf{e}_z) &= -\frac{1}{2}\widetilde{\Delta}\mathbf{u}, \end{aligned} \tag{2.11}$$

are held. The outer equations of the micropolar fluid are thus reduced to

$$\frac{\partial \mathbf{u}}{\partial \widetilde{x}} = -\widetilde{\nabla}\widetilde{p} + \left(1 - \frac{\mu_r}{2}\right)\widetilde{\Delta}\mathbf{u} + O(\mathrm{Re}). \tag{2.12}$$

The solution of Eq. (2.12) is regular even far from the cylinder. As a result, we do not have to solve the Oseen approximation of Eqs. (2.2) and (2.3), which cannot be represented by an explicit form [33]. Eq. (2.12) supports the validity of the boundary condition $\boldsymbol{\omega} \to \mathrm{rot}\,\mathbf{v}/2$ as $r \to \infty$ in Eq. $(2.6)_2$.

Since we know the solution of Eq. (2.12) for 2D and 3D, what we need to solve is the Stokes approximation of Eqs. (2.2) and (2.3) in an axisymmetric flow as

$$-\nabla p + \Delta \mathbf{v} + \mu_r \operatorname{rot}(\omega_z \mathbf{e}_z) = \mathbf{0}, \tag{2.13}$$

$$(\operatorname{rot} \mathbf{v})_z - 2\omega_z + \mu_B \Delta\omega_z = 0, \tag{2.14}$$

near the cylinder. The solutions of Eq. (2.12) and Eqs. (2.13) and (2.14) will be connected with the aid of the matching asymptotic technique. Moreover, from the divergence of Eq. (2.13) we have

$$\Delta p = 0, \tag{2.15}$$

which implies that pressure is a harmonic function. On the other hand, the rotation of Eq. (2.13) yields

$$\Delta \operatorname{rot} \mathbf{v} + \mu_r \operatorname{rot}\operatorname{rot}(\omega_z \mathbf{e}_z) = \mathbf{0}. \tag{2.16}$$

3. Flow past a rotating cylinder

So far, we derived equations governing the slow flow of micropolar fluids around a rotating cylinder. In this section, we are going to solve these differential equations. It is obvious that all the equations and the boundary conditions involved are linear. The linearity implies the applicability of the superposition principle. We can split our problem into two parts by utilizing the principle of superposition: a stationary cylinder in a cross-stream and a rotating cylinder in a stationary ocean of a micropolar fluid. The solution of our problem will be the sum of these two partial solutions. In the following subsections we will investigate each of the partial problems.

3.1. Stationary cylinder in a cross-stream

This problem has been investigated by Buchukuri and Chichinadze [33] and Hayakawa [34]. The stream function for a two-dimensional problem in cylindrical coordinate system is defined through

$$\begin{aligned} v_r &= \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \\ v_\theta &= -\frac{\partial \Psi}{\partial r}. \end{aligned} \tag{3.1}$$

The mass balance equation (2.1), i.e. the continuity equation, is fulfilled automatically by this choice of stream function. The procedure is as follows: first we will give the explicit calculation of the Stokes approximation as an inner solution. Then, we will obtain an outer solution and use the matched asymptotic method.

At the first step, let's obtain the solution of Eqs. (2.13) and (2.14) based on the Stokes approximation. From Eqs. (3.1) and (2.16) we obtain an equation for ω_z and Ψ,

$$\Delta\Delta\Psi + \mu_r \Delta\omega_z = 0. \tag{3.2}$$

On the other hand, from Eq. (2.14) we obtain

$$\Delta\Psi + 2\omega_z - \mu_B \Delta\omega_z = 0. \tag{3.3}$$

From the operation of the Laplacian operator on Eq. (3.3) and with the help of Eq. (3.2), we obtain

$$\Delta(\Delta - \xi^{-2})\omega_z = 0, \tag{3.4}$$

where

$$\xi = \sqrt{\frac{\mu_B}{2 - \mu_r}}. \tag{3.5}$$

The problem will be solved under the boundary conditions on the cylinder surface. Although we cannot adopt the boundary condition far from the cylinder, we know that the leading singularity comes from a logarithmic divergent term which will be regularized by matching with the outer solution. Thus, higher divergent terms which obey power laws will be omitted in discussion in this subsection. The boundary condition far from the cylinder is

$$\omega_z = -\frac{1}{2}\Delta\Psi, \tag{3.6}$$

which is equivalent to $\frac{1}{2}\operatorname{rot}\mathbf{v}$ according to the last equation of $(2.6)_2$.

Taking into account the boundary condition on the cylinder, namely

$$\begin{aligned} &\frac{\partial\Psi}{\partial\theta}(1,\theta) = 0,\\ &\frac{\partial\Psi}{\partial r}(1,\theta) = 0,\\ &\omega_z(1,\theta) = 0, \end{aligned} \tag{3.7}$$

we obtain

$$\Delta\omega_z(r,\theta) = \sum_{n=1}^{\infty} C_n^{(2)} K_n\left(\frac{r}{\xi}\right)\sin n\theta, \tag{3.8}$$

where $C_n^{(2)}$ is a constant and K_n denotes the modified Bessel function of the second kind of order n. Thus, it is easy to obtain the general form of ω_z as

$$\omega_z(r,\theta) = \sum_{n=1}^{\infty}\left[a_{\omega,n}r^n + \frac{b_{\omega,n}}{r^n} + C_n^{(2)}\xi^2 K_n\left(\frac{r}{\xi}\right)\right]\sin n\theta. \tag{3.9}$$

It is obvious that $a_{\omega,n} = 0$ for all n to satisfy the boundary condition far from the cylinder (the microrotation must be finite at infinity). From Eq. $(3.7)_3$ we have

$$b_{\omega,n} = -C_n^{(2)}\xi^2 K_n\left(\frac{1}{\xi}\right). \tag{3.10}$$

Similarly, we can obtain Ψ as

$$\begin{aligned}\Psi(r,\theta) = \sin\theta\left[a_\Psi^{(2)} r + \frac{b_\Psi^{(2)}}{r} + \alpha r\ln r - \mu_r\xi^4 C_1^{(2)} K_1\left(\frac{r}{\xi}\right)\right] \\ + \sum_{n=2}^{\infty}\left[\frac{b_{\Psi,n}^{(2)}}{r^n} + \frac{d_{\Psi,n}^{(2)}}{r^{n-2}} - \mu_r\xi^4 C_n^{(2)} K_n\left(\frac{r}{\xi}\right)\right]\sin n\theta.\end{aligned} \tag{3.11}$$

It is shown by Hayakawa in [34] that the mode $n \geq 2$ will become zero. Thus, the problem can be simplified. From Eq. (3.6) and the first two relations of Eq. (3.7) we obtain

$$\begin{aligned} b_{\omega,1} &= -\alpha, \\ a_\Psi^{(2)} &= \alpha\widehat{a}, \\ b_\Psi^{(2)} &= \alpha\widehat{b}, \\ C_1^{(2)} &= \frac{\alpha}{\xi^2 K_1(1/\xi)}\widehat{a}, \end{aligned} \tag{3.12}$$

where

$$\widehat{a} = \frac{1}{2}\left\{\mu_r\xi\left[\xi + \frac{K_1'(1/\xi)}{K_1(1/\xi)}\right] - 1\right\}, \tag{3.12_5}$$

$$\widehat{b} = \frac{1}{2}\left\{\mu_r\xi\left[\xi - \frac{K_1'(1/\xi)}{K_1(1/\xi)}\right] + 1\right\}, \tag{3.12_6}$$

in which the prime denotes the ordinary derivative of the modified Bessel function of the second kind, which is given by

$$K_1'(x) = -\frac{1}{2}[K_0(x) + K_2(x)]. \tag{3.12_7}$$

Thus, we obtain

$$(3.13)\qquad \begin{aligned} \Psi(r,\theta) &= \alpha \sin\theta \left[\widehat{a} r + \frac{\widehat{b}}{r} + r\ln r - \mu_r \xi^2 \frac{K_1(r/\xi)}{K_1(1/\xi)}\right], \\ \omega(r,\theta) &= \alpha \sin\theta \left[-\frac{1}{r} + \xi^2 \frac{K_1(r/\xi)}{K_1(1/\xi)}\right]. \end{aligned}$$

The flow field following from Eqs. (3.13) has a logarithmic singularity in the limit of $r \to \infty$. The obtained Stokes' solution is treated as an inner solution for our matching procedure.

To resolve Stokes' paradox for $r \to \infty$ we adopt the matched asymptotic method developed by KAPLUN and LAGERSTROM [8] and used by HAYAKAWA [34] successfully. Now, we are going to obtain the outer solution and to this end, we reconsider Eq. $(3.13)_1$,

$$(3.14)\qquad \Psi(r,\theta) \sim \alpha(\mathrm{Re}) \left[\widehat{a} r + \frac{\widehat{b}}{r} + r\ln r - \mu_r \xi^2 \frac{K_1(r/\xi)}{K_1(1/\xi)}\right] \sin\theta,$$

where α is replaced by a multiplier $\alpha(\mathrm{Re})$ which is allowed to depend upon the Reynolds number, because our asymptotic sequence is unspecified. Although this approximation cannot satisfy the condition $\mathbf{v} = \mathbf{e}_x$ in Eq. $(2.6)_2$ or $\Psi \to r\sin\theta$ as $r \to \infty$, it can be matched to the uniform stream, regarded as the first term of an Oseen expansion.

Now, introducing a new variable

$$(3.15)_1\qquad \rho = \frac{\mathrm{Re}}{1-\mu_r/2} r = \varepsilon r,$$

in which

$$(3.15)_2\qquad \varepsilon = \frac{\mathrm{Re}}{1-\mu_r/2},$$

the outer Eq. (2.12) becomes

$$(3.16)_1\qquad \frac{\partial \mathbf{u}}{\partial \widehat{x}} = -\widehat{\nabla}\widetilde{p} + \widehat{\Delta}\mathbf{u},$$

where

$$(3.16)_2\qquad \widehat{\nabla} = \frac{1}{1-\mu_r/2}\widetilde{\nabla}.$$

Then the Oseen expansion begins with

$$(3.17)\qquad \Psi \sim \frac{1}{\varepsilon}\rho\sin\theta + \cdots \qquad \text{as } \varepsilon \to 0.$$

Writing the Stokes expression (3.14) with the Oseen variables (3.15), the leading term is now

$$\Psi \sim \frac{\alpha(\varepsilon)}{\varepsilon} \ln\left(\frac{1}{\varepsilon}\right) \rho \sin\theta, \tag{3.18}$$

where $\alpha(\varepsilon) = \alpha(\mathrm{Re})$ in the limit of $\varepsilon \to 0$. This matches Eq. (3.17) if

$$\alpha(\varepsilon) = \frac{1}{\ln(1/\varepsilon) + k}, \tag{3.19}$$

where k is a constant to be determined later.

Expansion of the Stokes approximation (3.14) further by ρ and $\alpha(\varepsilon)$ leads to

$$\Psi \sim \frac{1}{\varepsilon}[1 + \alpha(\varepsilon)(\ln\rho - k + \widehat{a})]\rho\sin\theta. \tag{3.20}$$

This requires the Oseen expansion (3.17) to continue as

$$\Psi \sim \frac{1}{\varepsilon}[\rho\sin\theta + \alpha(\varepsilon)\psi(\rho,\theta) + \cdots]. \tag{3.21}$$

Substituting this into the full equation, ψ satisfies the linearized Oseen equation

$$\left(\widehat{\Delta} - \frac{\partial}{\partial \widehat{x}}\right)\widehat{\Delta}\psi = 0. \tag{3.22}$$

The appropriate solution for the stream function can be found as an infinite series [6,10,12]. The fundamental solution due to Oseen gives as the Cartesian velocity components

$$\begin{aligned} u_x &= \frac{\partial\psi}{\partial(\rho\sin\theta)} \\ &= 2c_2\left\{\frac{\partial}{\partial(\rho\cos\theta)}\left[\ln\rho + e^{(\rho\cos\theta)/2}K_0\left(\frac{\rho}{2}\right)\right] - e^{(\rho\cos\theta)/2}K_0\left(\frac{\rho}{2}\right)\right\}, \\ u_y &= -\frac{\partial\psi}{\partial(\rho\cos\theta)} = 2c_2\frac{\partial}{\partial\,(\rho\sin\theta)}\left[\ln\rho + e^{(\rho\cos\theta)/2}K_0\left(\frac{\rho}{2}\right)\right], \end{aligned} \tag{3.23}$$

where c_2 is a constant which will be determined by the matching procedure. The term in $\ln\rho$ at the origin cancels the term involving $K_0(\rho/2)$. For small ρ we obtain the integrated form of ψ as

$$\psi \sim -c_2\left(\ln\frac{4}{\rho} + 1 - \gamma\right)\rho\sin\theta + O(\rho^2\ln\rho), \tag{3.24}$$

where γ is Euler's constant $\gamma = 0.5772\dots$. Using this we find that the Oseen expansion (3.21) behaves near the cylinder as

$$\Psi \sim \frac{1}{\varepsilon}\rho\sin\theta\left[1 + c_2\alpha(\varepsilon)\left(\ln\frac{\rho}{4} + \gamma - 1\right)\right]. \tag{3.25}$$

This can match Eq. (3.20) if we choose

$$\begin{aligned} c_2 &= 1, \\ k &= \widehat{a} - \gamma + 1 + \ln 4. \end{aligned} \tag{3.26}$$

Thus, we obtain

$$\alpha(\varepsilon) = \left(\ln\frac{4}{\varepsilon} - \gamma + 1 + \widehat{a}\right)^{-1}. \tag{3.27}$$

This vanishes for $\varepsilon \to 0$. The explicit expression near the cylinder is thus given by

$$\Psi(r,\theta) = \alpha(\varepsilon)\left[\widehat{a}r + \frac{\widehat{b}}{r} + r\ln r - \mu_r\xi^2\frac{K_1(r/\xi)}{K_1(1/\xi)}\right]\sin\theta. \tag{3.28}$$

This expression satisfies all the boundary conditions.

Utilizing Eqs. (3.1), the explicit expression of the inner solution is thus given by

$$\begin{aligned} v_r &= \alpha(\varepsilon)\left[\widehat{a} + \frac{\widehat{b}}{r^2} + \ln r - \mu_r\xi^2\frac{K_1(r/\xi)}{rK_1(1/\xi)}\right]\cos\theta, \\ v_\theta &= \alpha(\varepsilon)\left[-\widehat{a} + \frac{\widehat{b}}{r^2} - \ln r - 1 + \mu_r\xi\frac{K_1'(r/\xi)}{K_1(1/\xi)}\right]\sin\theta. \end{aligned} \tag{3.29}$$

The pressure is similarly determined from Eqs. (2.13) and (3.1) as

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial\theta}(\Delta\Psi + \mu_r\omega_z), \\ \frac{1}{r}\frac{\partial p}{\partial\theta} &= -\frac{\partial}{\partial r}(\Delta\Psi + \mu_r\omega_z). \end{aligned} \tag{3.30}$$

As stated in [34], the pressure is determined by the Stokes pole $\Psi \sim r\ln r$. The result is

$$p(r,\theta) = p_0 - \frac{2-\mu_r}{r}\alpha(\varepsilon)\cos\theta, \tag{3.31}$$

where p_0 is an unimportant constant (the free-stream pressure).

3.2. Rotating cylinder in a stationary micropolar fluid

The first partial problem was solved. In this subsection, we want to obtain the solution of the second partial problem, i.e. a rotating cylinder with the angular velocity Ω in a stationary micropolar fluid. Since all equations and results have been presented in non-dimensional form, the angular velocity Ω is non-dimensionalized as $\overline{\Omega} = a\Omega/U$. The flow field is characterized by

$$
\begin{aligned}
v_r &= v_z = 0,\\
\omega_r &= \omega_\theta = 0,\\
v_\theta &= v_\theta(r),\\
\omega_z &= \omega_z(r).
\end{aligned}
\tag{3.32}
$$

The mass balance equation (2.1) is automatically satisfied in this case. Introducing conditions (3.32) to the general governing equations of micropolar fluid mechanics (2.2) and (2.3), we have

$$
\begin{aligned}
\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} - \mu_r \frac{d\omega_z}{dr} &= 0,\\
\mu_B \frac{d^2\omega_z}{dr^2} + \frac{\mu_B}{r}\frac{d\omega_z}{dr} - 2\omega_z + \frac{dv_\theta}{dr} + \frac{v_\theta}{r} &= 0.
\end{aligned}
\tag{3.33}
$$

The general solution of these two coupled ordinary differential equations is

$$
\begin{aligned}
v_\theta(r) &= A_1 r + \frac{A_2}{r} + A_3 J_1\Big(\frac{r}{\xi}\Big) + A_4 Y_1\Big(\frac{r}{\xi}\Big),\\
\omega_z(r) &= A_1 + \frac{1}{\mu_r \xi}\Big[A_3 J_0\Big(\frac{r}{\xi}\Big) + A_4 Y_0\Big(\frac{r}{\xi}\Big)\Big],
\end{aligned}
\tag{3.34}
$$

where A_i $(i = 1, \dots, 4)$ are integration constants and, J_n and Y_n are the Bessel functions of the first and the second kind of order n, respectively.

The following boundary conditions which are derived from the general boundary conditions (2.6) must be enforced to the general solutions (3.34) in order to obtain the integration constants explicitly.

$$
\begin{aligned}
v_\theta(r = 1) &= \overline{\Omega},\\
\omega_z(r = 1) &= 0,\\
v_\theta(r \to \infty) &= 0,\\
\omega_z(r \to \infty) &= 0.
\end{aligned}
\tag{3.35}
$$

Linear and angular velocities must be finite at infinity and hence, one has $A_1 = 0$. On the other hand, this solution based on micropolar fluid mechanics should reduce to classical solution derived from the Navier–Stokes equations which reveals $A_2 = \overline{\Omega}$. From the boundary conditions on angular velocity we conclude that $A_3 = A_4 = 0$. Finally, the particular solution becomes

$$v_\theta(r) = \frac{\overline{\Omega}}{r}, \qquad \omega_z = 0, \tag{3.36}$$

which is identical with the classical solution. We could expect this result, because this flow situation is essentially an inviscid (irrotational) flow and the theory of micropolar fluids is concerned with the viscosity effects and viscous flows. The stream function is obtained by employing Eq. (3.1)

$$\overline{\Psi}(r) = -\overline{\Omega}\ln r = \overline{\Omega}\ln\frac{1}{r}. \tag{3.37}$$

4. Flow of micropolar fluid past a rotating cylinder

So far, we solved our two partial problems and now we want to combine them and obtain the solution of our problem, i.e. the flow of micropolar fluid past a rotating cylinder, by utilizing the principle of superposition.

4.1. Stream function

The stream function is obtained simply by summing up the stream functions (3.28) and (3.37) which is

$$\begin{aligned}\Psi_{\text{total}}(r,\theta) &= \Psi(r,\theta) + \overline{\Psi}(r)\\ &= \alpha(\varepsilon)\left[\widehat{a}r + \frac{\widehat{b}}{r} + r\ln r - \mu_r\xi^2\frac{K_1(r/\xi)}{K_1(1/\xi)}\right]\sin\theta - \overline{\Omega}\ln r.\end{aligned} \tag{4.1}$$

Now, we are able to depict streamline contours which are simply curves of constant $\Psi_{\text{total}}(r,\theta)$. The streamline contours for a flow past a stationary cylinder, i.e. for $\overline{\Omega} = 0$, are shown in Fig. 2. The obtained streamline contours for the

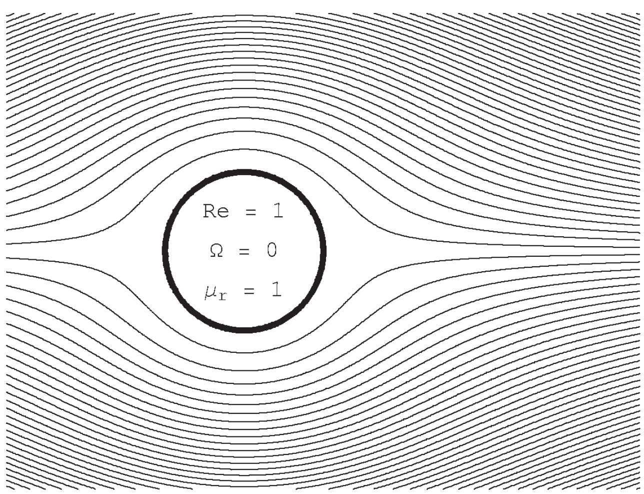
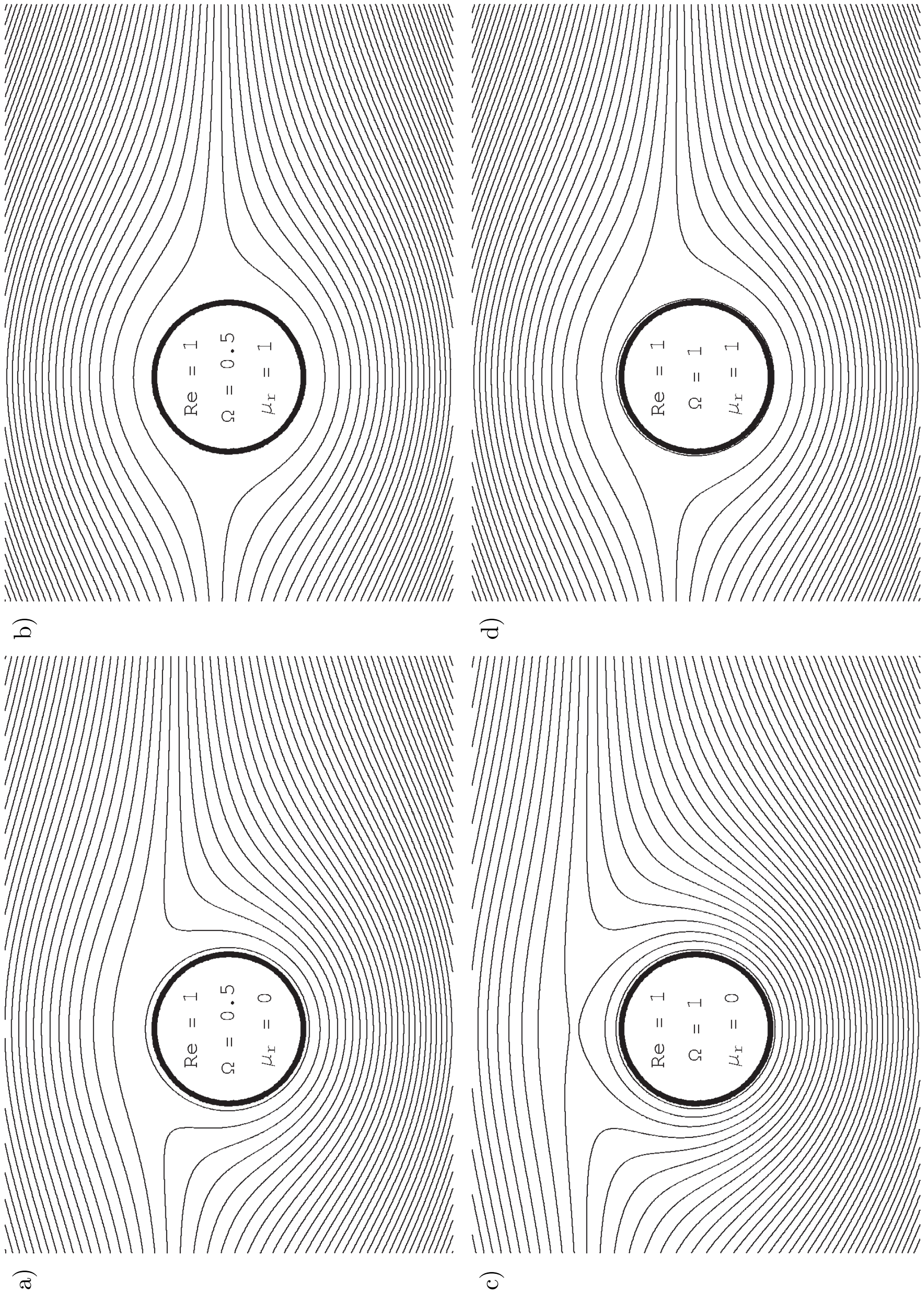


FIG. 2. Streamline contour for flow past a stationary cylinder



[Fig. 3a–d]

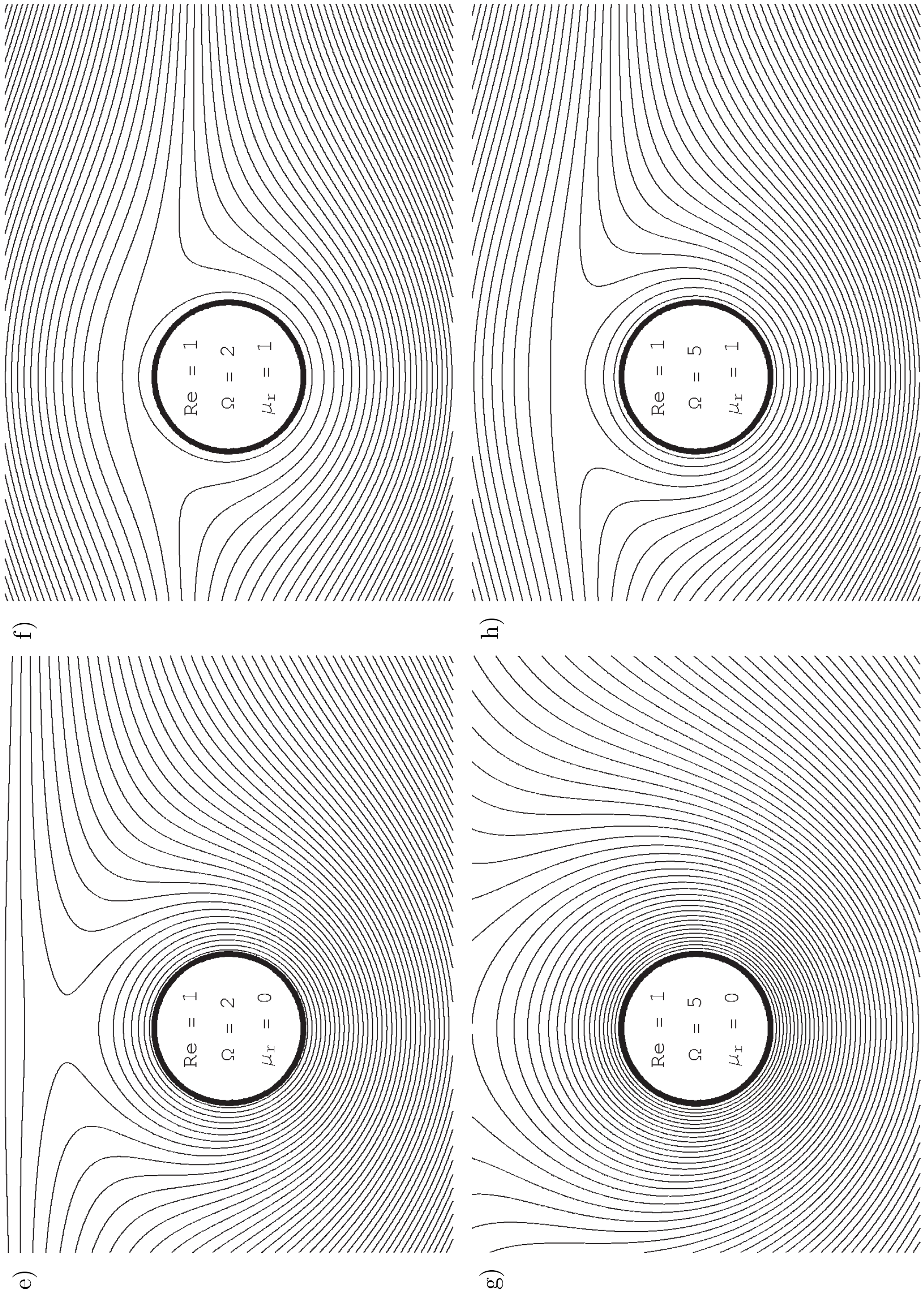


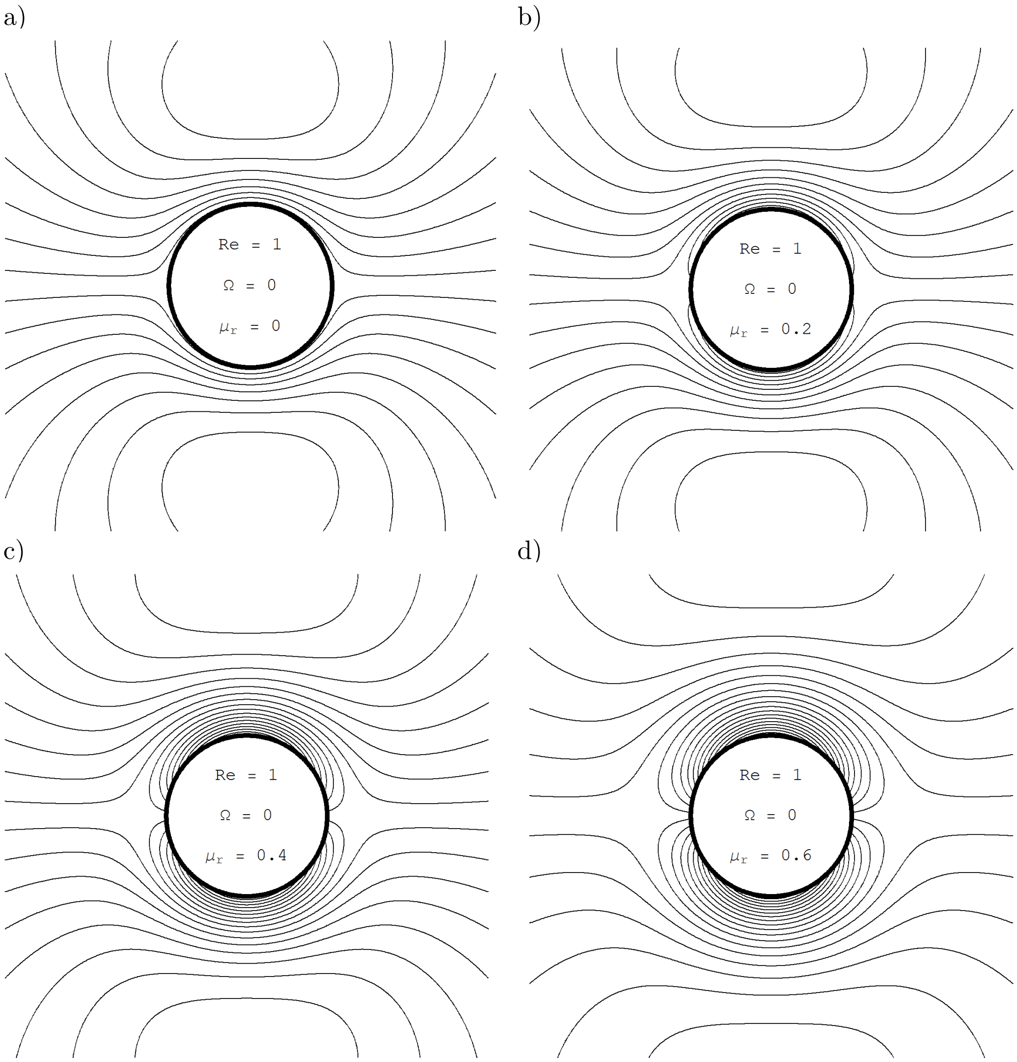
FIG. 3. Streamline contours for flow past a rotating cylinder.

flow past a rotating cylinder are also depicted in Fig. 3 for Re $= 1$ and different values of $\overline{\Omega}$. In order to demonstrate the effect of micropolarity of fluid on the flow pattern, the streamlines for the cases $\mu_r = 0$ and $\mu_r = 1$ are shown in Fig. 3.

As it is obvious from Fig. 3, the micropolarity of the fluid severely affects the flow field around the rotating cylinder. The influence of the cylinder rotation on the flow field is less pronounced with increase in the fluid micropolarity.

4.2. Microrotation field

Microrotation field is an important quantity in micropolar fluid dynamics which is given by Eq. $(3.13)_2$. Therefore, it is worthwhile to take a look at



[Fig. 4a–d]

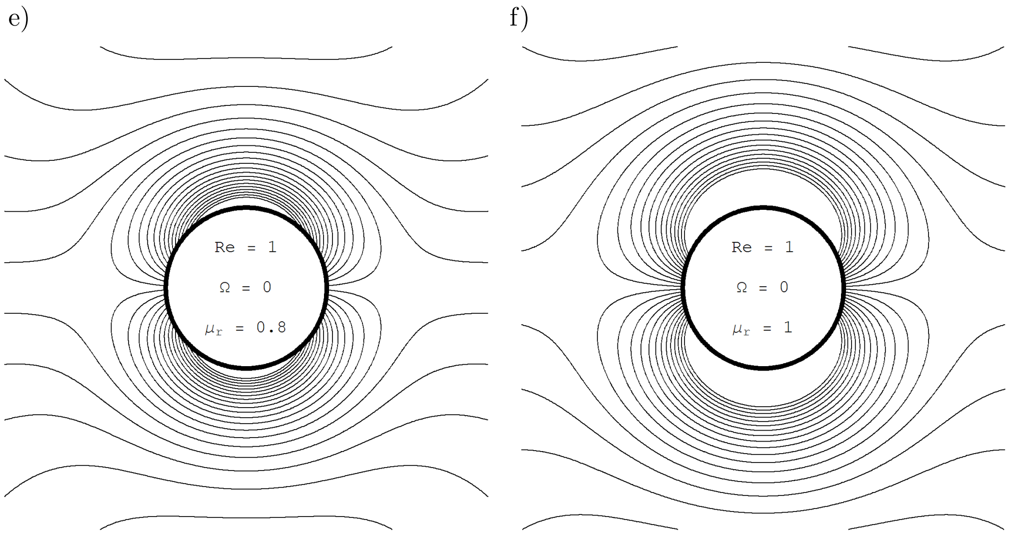


FIG. 4. Microrotation contours around the cylinder.

its behavior. The microrotation contours at Re $= 1$ and $\overline{\Omega} = 0$ are depicted in Fig. 4 for various values of μ_r. The most significant difference between the conventional Stokes flow ($\mu_r = 0$) and the micropolar Stokes flow ($\mu_r > 0$) appears as a localized microrotation near the cylinder surface. The transition between these regimes is apparently shown in Fig. 4.

4.3. Velocity components

In order to compute the velocity components, we have two ways: employing the obtained stream function (4.1) and then computing the velocities by the use of Eqs. (3.1). Alternatively, we can make use of the linearity of the mathematical system and then, employing the principle of superposition to sum up the velocity fields (3.29) and (3.36). Ultimately, the velocity components are obtained as

$$\begin{aligned} v_r &= \alpha(\varepsilon)\left[\widehat{a} + \frac{\widehat{b}}{r^2} + \ln r - \mu_r \xi^2 \frac{K_1(r/\xi)}{r K_1(1/\xi)}\right]\cos\theta, \\ v_\theta &= \alpha(\varepsilon)\left[-\widehat{a} + \frac{\widehat{b}}{r^2} - \ln r - 1 - \frac{\mu_r \xi}{2}\frac{K_0(r/\xi) + K_2(r/\xi)}{K_1(1/\xi)}\right]\sin\theta + \frac{\overline{\Omega}}{r}. \end{aligned} \tag{4.2}$$

Figure 5 shows the radial and tangential velocity profiles at $\theta = \pi/4$ and $\theta = \pi/2$. The results of the Atefi's analytical solution to Oseen's differential equation for creeping flow of Newtonian fluid past a circular cylinder [10] and the results obtained from the potential flow theory are depicted as well. As it is obvious from the diagrams, our solution reduces to the flow of a Newtonian fluid

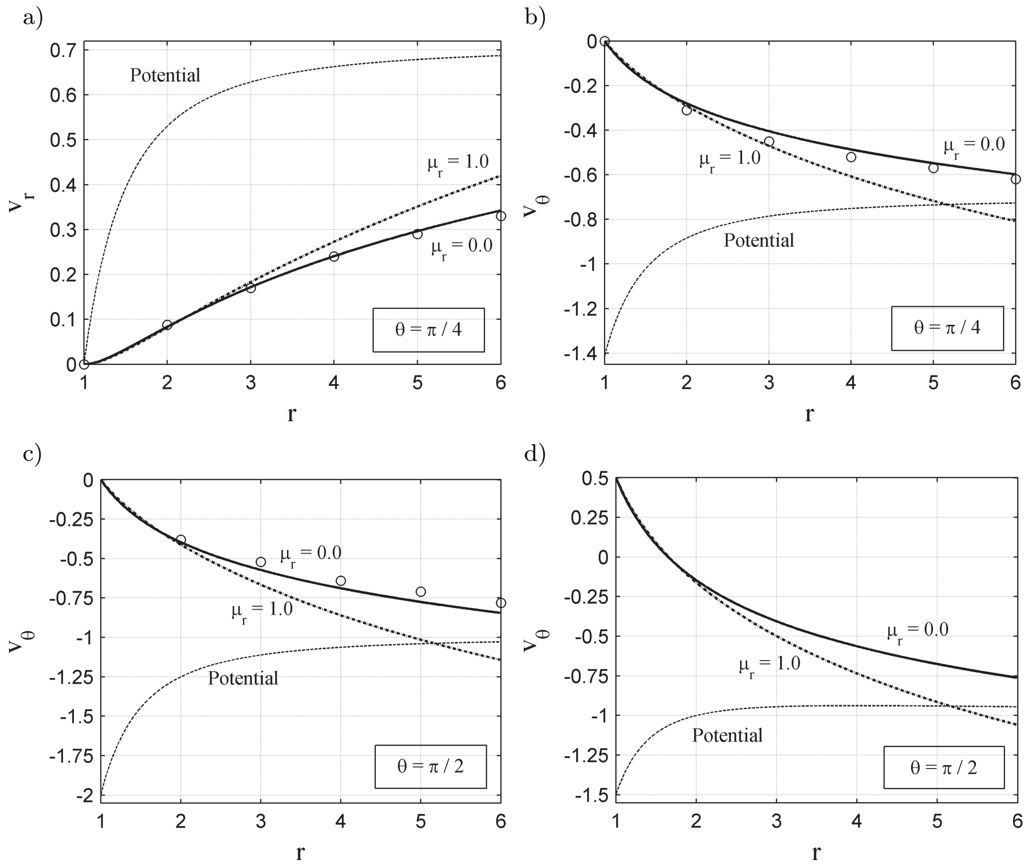


Fig. 5. Velocity profiles for flow past a stationary and rotating cylinder: a) radial velocity profiles, Re = 0.25 and $\overline{\Omega} = 0.0$; b) tangential velocity profiles, Re = 0.25 and $\overline{\Omega} = 0.0$; c) tangential velocity profiles, Re = 0.25 and $\overline{\Omega} = 0.0$; d) tangential velocity profiles, Re = 0.25 and $\overline{\Omega} = 0.5$ (∘ Atefi's analytical solution [10]).

past a circular cylinder for $\mu_r = 0$. The trends of velocity profiles for both $\mu_r = 0$ and $\mu_r = 1$ are similar, while there is a significant difference with potential flow pattern. Furthermore, the magnitudes of both radial and tangential velocities for the case of $\mu_r = 1$ are greater than those of $\mu_r = 0$, and the differences between them gradually grow with increase in r.

4.4. Drag and lift forces

In the motion of a rotating cylinder through a viscous fluid, drag and lift forces act on the cylinder. In Newtonian fluids, one has to integrate the pressure and shear stress on the cylinder surface which yields the net force exerted on the cylinder by the viscous fluid, whose streamwise component forms the drag, while its component normal to the streamwise direction is lift. In micropolar fluids however, the effect of non-symmetric stress tensor must be taken into account as well. Therefore, the drag force in this case is composed of three parts; the part from non-symmetric stress tensor D_τ, the form (pressure) drag D_p due to normal stresses, and the part induced by the shear stresses D_f. D_τ is given by [34]

$$(4.3)_1 \qquad D_\tau = \int_S \tau_{rx}|_{r=1} dS = -\mu_r \int_0^{2\pi} \varOmega_z(1,\theta)\sin\theta d\theta,$$

where $\varOmega_z = (\operatorname{rot}\mathbf{v})_z/2$ and it reduces to $\varOmega_z(1,\theta) = \dfrac{1}{2}\partial v_\theta/\partial r$. Performing the integration one obtains

$$(4.3)_2 \qquad D_\tau = \pi\mu_r\alpha(\varepsilon)[1-\mu_r\beta(\xi)],$$

where

$$(4.4) \qquad \beta(\xi) = \frac{1}{2K_1(1/\xi)}[K_1''(1/\xi) - \xi K_1'(1/\xi) + \xi^2 K_1(1/\xi)].$$

Similarly, for the form drag D_p we have

$$(4.5)_1 \qquad D_p = -\int_0^{2\pi} p(1,\theta)\cos\theta d\theta = \pi(2-\mu_r)\alpha(\varepsilon),$$

where the pressure field $p(r,\theta)$ can be obtained from the linear momentum equation (2.13) as

$$(4.5)_2 \qquad p(r,\theta) = p_0 + \overline{\varOmega}\frac{\sin\theta}{r} - \frac{2-\mu_r}{r}\alpha(\varepsilon)\cos\theta.$$

D_f is also given by

$$(4.6) \qquad D_f = -\left(1-\frac{\mu_r}{2}\right)\int_0^{2\pi} \frac{\partial v_\theta}{\partial r}\bigg|_{r=1} \sin\theta d\theta = \pi(2-\mu_r)\alpha(\varepsilon)[1-\mu_r\beta(\xi)].$$

The total drag force $D_\Sigma = D_\tau + D_p + D_f$ becomes

$$(4.7)_1 \qquad D_\Sigma = \pi\alpha(\varepsilon)[4-\mu_r-2\mu_r\beta(\xi)],$$

which can be rewritten as

$$(4.7)_2 \qquad D_\Sigma = \pi(\eta+\eta_r)U\alpha(\varepsilon)[4-\mu_r-2\mu_r\beta(\xi)].$$

It is obvious that the rotation of a cylinder has no effect on the drag force. It is expected because our solution implies that the flow induced by a rotating cylinder in quiescent fluid is inviscid and hence, produces no drag. Nevertheless, it is not a complete physical view, because the rotation has an effect on drag force in reality. By the way, this effect is much smaller than the effect of shear flow past the cylinder and thus, it is not that far from reality. The flow induced by a rotating cylinder in a quiescent fluid is not inviscid in reality due to end effects near the ends of the cylinder and three-dimensionality of fluid at those regions. In the present solution however, we do not take them into account. The drag force $(4.7)_2$ reduces to the Oseen's drag [34] for the case of a Newtonian fluid ($\mu_r \to 0$) which is given by

$$(4.7)_3 \qquad D_\Sigma = \frac{4\pi\eta U}{S + 1/2}, \qquad S = \ln\left(\frac{4}{\mathrm{Re}}\right) - \gamma.$$

Now, let us calculate the lift force. Drag was calculated by taking into account the streamwise component of the total force. Lift is the remaining component which is normal to the main-stream direction. Non-symmetric stress tensor and shear stress contribution to lift is zero, i.e. $L_\tau = L_f = 0$, and we just have the contribution of the pressure field, i.e. L_p.

$$(4.8) \qquad L_\Sigma = L_p = \int_0^{2\pi} p(1,\theta) \sin\theta d\theta = \pi\overline{\Omega}.$$

Here it is evident that the lift force has no dependence upon viscosity. The reason, as explained above, is that the flow induced by a rotating cylinder in a quiescent fluid is essentially inviscid. This lift force is still different from that of the potential (irrotational) flow theory which is $L_\Sigma = 2\pi\overline{\Omega}$. The potential flow can be considered as the limiting case of very large Reynolds number, i.e. $\mathrm{Re} \to \infty$. On the other hand, the creeping flow is valid for the limiting case of vanishing Reynolds number, i.e. $\mathrm{Re} \to 0$. Therefore, if we define $L_\Sigma/2\pi\overline{\Omega} = f(\mathrm{Re})$, then one deduces that $f(0) = 1/2$ and $f(\infty) = 1$.

5. Concluding remarks

An analytical closed-form solution for slow motion of a rotating cylinder through a micropolar fluid is presented. By the use of the obtained solution, several flow field quantities are studied. The streamlines topology for a flow past a stationary and a rotating cylinder for various rotation speeds of the cylinder are presented. It is deduced that the influence of the cylinder rotation on the flow field is less pronounced with increase in the fluid micropolarity. The microrotation field is also studied and we conclude that for $\mu_r = 0$, the contours are similar to those of conventional Stokes flow. However, for $\mu_r > 0$ the contours topology changes gradually and the formation of a concentrated microrotation zone near the cylinder is observed. The radial and tangential velocity profiles are investigated as well. The results are validated by comparing with the results of ATEFI's analytical solution [10] for the case of $\mu_r = 0$. The diagrams show that the velocity profiles for $\mu_r = 0$ and $\mu_r = 1$ have similar trends while the trends are completely different from those predicted by potential flow theory. In all of the considered cases, the magnitudes of velocity components for $\mu_r = 1$ are greater than those for $\mu_r = 0$ and the differences between them grows with increase in r.

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