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# On effects of internal friction in the revised Goodman–Cowin theory with an independent kinematic internal length

CHUNG FANG

*Department of Civil Engineering, National Cheng Kung University*
*No.1, University Road, Tainan City 701, Taiwan*

In the present study, the revised Goodman–Cowin theory with an independent kinematic internal length proposed by Fang et al. for rapid dry granular flows is extended to account for the effects of internal friction. A thermodynamic analysis, based on the Müller–Liu entropy principle, is performed to deduce the ultimate equilibrium expressions of the constitutive equations. Results show that while internal friction contributes significantly to the equilibrium expressions of the constitutive variables, the dependence on internal friction of the Helmholtz free energy becomes a critical point for the present formulation in practical applications. In comparison with other constitutive formulations based on the revised Goodman–Cowin theory, the present formulation is the most general one and shows an ability to take into account the microstructural effects of volume fraction variation, internal friction and evolution of internal length of dry granular flows simultaneously.

**Key words:** Goodman–Cowin theory, Müller–Liu entropy principle, internal friction, internal length.

## 1. Introduction

Rapid dry granular flows are large amounts of solid particles with interstices filled with a fluid or a gas in rapid motions, in which the interstitial fluids/gases play an insignificant role in the transportation processes. Such flows can thus be regarded as single-phase rather than multi-phase flows. They are encountered e.g. in such technical fields as ultrastructural processing of ceramics or new methods of Xerography and powder metal forming. Related to these industrial applications are problems arising in the geophysical or environmental contexts such as avalanches or the formation of dunes. Distinctive features of dry granular mass flows and the arising challenges in understanding their behaviour are summarized in e.g. [1–4].

Although dry granular masses are discrete in nature, they are considered in the context of continuum mechanics continuous media exhibiting elastic, viscous and plastic features, simultaneously associated with microstructural effects. Typical significant microstructural effects include variation of the configurations of the grains, evolution of the internal friction and other non-conservative forces among the grains, the turbulent (fluctuating) motions and the surface roughness of the grains, etc. In 1972, Goodman and Cowin proposed a continuum-based theory for dry granular mass flows, known as the *Goodman–Cowin theory*, in which the microstructural effect of variation of the configurations of the grains was modeled by use of an internal variable $\nu$, called the volume fraction defined as the ratio of the solid volume divided by the total volume of a representative volume element [5–7]. Since then this theory has been widely applied in various fields of science and technologies, and extended to the investigations of granular mixtures with/without interstitial fluids, polar fluids, multi-phase mixtures with chemical reactions and diffusions, porous materials and liquid crystals [8–13].

Recently, the Goodman–Cowin theory has been demonstrated to bear some dimensional inconsistencies. These inconsistencies can be removed by introducing an internal length $\ell$ [14]. A class of constitutive models regarding different considerations of $\ell$ was proposed, in which it is considered as a material constant (Model I), a constitutive variable (Model II), an independent dynamic field quantity (Model III) and an independent kinematic field quantity (Model IV). The thermodynamic and variational analyses were performed to deduce the equilibrium expressions of the constitutive equations [14, 15], and the implemented models were applied to study the Benchmark problems [16]. Despite that the revised Goodman–Cowin theory is more well-motivated physically than the original one, it yet regards dry granular masses as viscoelastic media, and is not capable of dealing with the phenomena closely related to the plastic deformations and internal friction of the material such as rate-independent behaviour, memory effects of loading history and hysteretic behaviour under cyclic shearing (see e.g. [17–19]). To remove this disadvantage, Fang et al. further extended the Models I and III of the revised Goodman–Cowin theory to account for the microstructural effects of the internal friction, and applied the implemented models to study simple shear problems [20–22]. These two models are more able to describe the combined elasto-visco-plastic effects of dry granular mass flows; however, their predictions are not satisfied when the material is subjected to small cyclic loading. Extensions and modifications of the theory become necessary. Since the Model IV provides more general formulations than the other three models, its extension may represent a possible solution to this difficulty[1)].

[1)] Generality means more freedom to identify the equilibrium and non-equilibrium parts of the constitutive equations under the constraints deduced from the thermodynamic analysis. In addition, Model IV is essentially a gradient-flow theory. Although such a gradient-flow consideration is not always physically justified, it might be appropriate for granular matter in view of some recent results (see e.g. [23]).

Thus, we extend in the present study the revised Goodman–Cowin theory, in particular Model IV, to incorporate the effects of internal friction among the grains, to construct a continuum-based elasto-visco-plastic model with microstructural effects for dry granular mass flows. The focus is on the effects of the internal friction of the revised Goodman–Cowin theory, in which the internal length is considered as an independent kinematic field quantity. To this end, the balance equations and the constitutive class assumptions are outlined in Sec. 2, followed by the thermodynamic analysis based on the Müller–Liu entropy principle in Sec. 3. The thermodynamic analysis is completed by identifying the Liu-identities and the restrictions derived from the thermodynamic equilibrium. This paper is summarized in Sec. 4. In this paper, only the theoretical derivations are presented, the implementations and applications of the derived model are deferred to other papers.

## 2. Balance equations and constitutive assumptions

Following the previous work [14], the balance equations of the revised Goodman–Cowin theory with an independent kinematic internal length are given by

$$
\begin{aligned}
0 &= \dot{\gamma}\nu + \gamma\dot{\nu} + \gamma\nu \operatorname{div}\mathbf{v}, && \text{(mass)},\\
\mathbf{0} &= \gamma\nu\dot{\mathbf{v}} - \operatorname{div}\mathbf{t} - \gamma\nu\mathbf{b}, && \text{(linear momentum)},\\
\mathbf{0} &= \mathbf{t} - \mathbf{t}^{\mathrm{T}}, && \text{(angular momentum)},\\
0 &= \gamma\nu(\ell\dot{\nu})^{\cdot} - \operatorname{div}\mathbf{h} - \gamma\nu f, && \text{(equilibrated force)},\\
0 &= \gamma\nu\dot{\ell} - \operatorname{div}\boldsymbol{\Gamma} - \mathit{\Pi}, && \text{(internal length)},\\
0 &= \gamma\nu\dot{e} - \mathbf{t}\cdot\mathbf{D} + \operatorname{div}\mathbf{q}\\
&\quad - \gamma\nu r - \mathbf{h}\cdot\operatorname{grad}(\ell\dot{\nu}) + \gamma\nu f\ell\dot{\nu}, && \text{(internal energy)},\\
0 &= \gamma\nu\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \rho s - \pi, && \text{(entropy)},
\end{aligned}
\tag{2.1}
$$

where $\gamma$ is the true mass density of the grains, $\nu$ is again the volume fraction, $\mathbf{v}$ the velocity, $\mathbf{t}$ the Cauchy stress tensor, $\mathbf{b}$ the specific body force, $\ell$ an independent internal length, $\mathbf{h}$ the equilibrated stress vector, $f$ the equilibrated intrinsic body force, $\boldsymbol{\Gamma}$ and $\mathit{\Pi}$ the flux and production associated with $\ell$, respectively, $e$ the specific internal energy, $\boldsymbol{D}$ the symmetric part of the velocity gradient, known as the stretching tensor, $\mathbf{q}$ the heat flux, $r$ the specific energy supply, $\eta$ the specific entropy, $\boldsymbol{\phi}$ the entropy flux, $s$ the specific entropy supply and $\pi$ the entropy production. The superscript T denotes the transposition; the symbol $\dot{\wp}$ denotes the material time derivative of $\wp$, i.e., $\dot{\wp} = \partial\wp/\partial t + (\operatorname{grad}\wp)\cdot\mathbf{v}$, while $\boldsymbol{A}\cdot\boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}) = \operatorname{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})$ for two arbitrary second-rank tensors, and $\mathbf{a}\cdot\mathbf{b} = \operatorname{tr}(\mathbf{a}\otimes\mathbf{b})$, where $\otimes$ denotes a dyadic product. Equations $(2.1)_{1,2,7}$ are the traditional balances of mass, linear momentum and entropy, respectively, except that the bulk density $\rho$ is decomposed into $\rho = \gamma\nu$. Since the material is not considered as micropolar or Cosserat-type and the effects of particle rotation are excluded, the balance of angular momentum reduces to $(2.1)_3$, i.e., the symmetry of the Cauchy stress tensor. Furthermore, the internal variable $\nu$ is employed to capture the effects of variations of the configurations of the grains and its evolution is described by the balance equation of equilibrated force $(2.1)_4$, in which the product $\ell\dot{\nu}$ is the "pseudo-velocity" of the grains [14]. Since this equation is dynamic in nature, the variation of $\nu$ contributes extra energies to the granular system, which are relevant in the last two terms of the balance of internal energy $(2.1)_6$. The internal length $\ell$, an internal variable, is introduced as an independent field quantity whose evolution is described by the Eq. $(2.1)_5$, which is kinematic in nature and provides no extra energies to the system. It can be regarded as a higher-order closure condition for the formulation. While the Eq. $(2.1)_4$ has been demonstrated to be able to describe the force balances along the lines connecting the centers of the grains, Eq. $(2.1)_5$ is related to the detailed information on the configurations of the grains and can, to some extent take the influences of the characteristic length of the granular flows into account [14–16].

In order to take the effects of internal friction into account, an Euclidean frame-indifferent, stress-like, symmetric tensor-valued spatial internal variable $\mathbf{Z}$ is introduced, which is a phenomenological generalization of the Mohr–Coulomb model for internal friction and other non-conservative forces inside a material point in a granular mass at low energy and high grain volume fraction [24, 25]. $\mathbf{Z}$ is assumed to be an independent internal variable and its evolution is *assigned* by

$$\overset{\circ}{\mathbf{Z}} \equiv \dot{\mathbf{Z}} - [\mathbf{\Omega}, \mathbf{Z}] = \mathbf{\Phi}, \tag{2.2}$$

where $\mathbf{\Omega}$ is any orthogonal rotation of the material point and $\mathbf{\Phi}$ is a tensor-valued constitutive relation for the production of $\mathbf{Z}$, respectively, and the notation $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{AB} - \boldsymbol{BA}$ is employed for two arbitrary second-rank tensors[2)]. The LHS of (2.2) is the "corotational" objective time derivative of $\mathbf{Z}$. It reduces to the Jaumann derivative of $\mathbf{Z}$ when $\mathbf{\Omega}$ is chosen to be $\boldsymbol{W}$, the skew-symmetric

[2)] Similar approaches can also be found in e.g. [20, 25, 26]. In this approach, the granular continuum is investigated by modeling each element of the material body as a deformable "microcontinuum", an idea proposed by MINDLIN [27] and ERINGEN and KADAFAR via their concept of "micromorphic continuum" [28]. From this perspective, $\boldsymbol{t}$ is a constitutive quantity of the microcontinuum, whilst $\mathbf{Z}$ describes the internal friction and other non-conservative forces inside the microcontinuum which cannot be seen outside the microcontinuum. Nevertheless, there should exist a relation between $\mathbf{t}$ and $\mathbf{Z}$. Such a relation is partly an objective of the paper and can be deduced by use of the second law of thermodynamics, as will be shown later on.

part of velocity gradient[3]. It is noted that the evolution of $\mathbf{Z}$ is described by a kinematic equation (2.2), which *per se* yields no dissipation, a "true" variational principle for its energy contributions does not exist, and the internal energy balance $(2.1)_6$ remains thus unchanged. From this perspective, $\mathbf{Z}$ is rather an abstract ideal which does not represent any physical frictional forces on the surface of the grains.

In the present study, the dry granular mass is considered a temperature-dependent elasto-visco-plastic continuum whose constitutive class is postulated in the form

$$\mathcal{Q} = \{\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{g}_4, \boldsymbol{L}, \mathbf{Z}\}, \tag{2.3}$$

for the constitutive variables

$$\mathcal{S} \in \{\mathbf{t}, \mathbf{h}, f, \mathbf{\Gamma}, \mathit{\Pi}, e, \mathbf{q}, \eta, \boldsymbol{\phi}, \mathbf{\Phi}\}, \qquad \mathcal{S} = \hat{\mathcal{S}}(\mathcal{Q}), \tag{2.4}$$

where the hat in $\mathcal{S} = \hat{\mathcal{S}}(\mathcal{Q})$ indicates that $\mathcal{S}$ is a function of $\mathcal{Q}$, $\mathbf{g}_1 = \operatorname{grad}\nu$, $\mathbf{g}_2 = \operatorname{grad}\ell$, $\mathbf{g}_3 = \operatorname{grad}\gamma$, $\mathbf{g}_4 = \operatorname{grad}\theta$ and $\boldsymbol{L} = \operatorname{grad}\mathbf{v}$, the velocity gradient. In (2.3) $\nu_0$ and $\ell_0$ are respectively the values of $\nu$ and $\ell$ in the reference configuration; they are included in the constitutive class due to their influences on the granular masses in rapid motions [30, 31]. The omission of $\mathbf{v}$ in (2.3) is due to the requirement of material frame indifference. Strictly, $\dot{\gamma}$ and $\dot{\theta}$ should be included in (2.3) for consistency of Truesdell's equi-presence principle. Since it is assumed that the variation of $\gamma$ is small, it is possible to neglect $\dot{\gamma}$ as an independent argument in (2.3). In addition, although the exclusion of $\dot{\theta}$ will cause the indifference between the empirical and absolute temperatures, and leads to an infinite propagation speed of a small thermal pulse in the material (see e.g. [32, 33]), these situations are rather rare and unimportant for most dry granular flows such like soil. Thus, from the viewpoint of practical application, $\dot{\theta}$ is omitted in (2.3) for simplicity. While the dependences on $\{\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3\}$ in (2.3) are used to capture the elastic effects, $\mathbf{Z}$ and $\boldsymbol{L}$ are assigned to the plastic and viscous effects, respectively. With the principle of material objectivity, (2.3) can alternatively be represented in the form

$$\mathcal{Q} = \{\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{g}_4, \mathbf{D}, \mathbf{Z}\}. \tag{2.5}$$

Equations (2.4) and (2.5) define the constitutive class employed in this paper, with which $(2.1)_3$, namely the symmetry of the Cauchy stress tensor, is automatically satisfied. In the formulations (2.1), the independent field quantities are $\gamma$, $\nu$, $\ell$, $\mathbf{v}$, $\theta$ (temperature) and $\mathbf{Z}$, totally 13 scalar unknowns, whilst

[3] $\mathbf{\Omega}$ is defined as $\mathbf{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T$ with $\boldsymbol{R}$ being the rotation of the material. It is chosen to be $\boldsymbol{W}$ here to reach the material frame indifference of the time-rate of change of $\mathbf{Z}$, which can also be achieved by other choices of $\mathbf{\Omega}$, see e.g. [29].

$\mathcal{S}$ in (2.4) are treated as constitutive variables. Since the number of the unknown field quantities corresponds to the number of the available equations ($(2.1)_1\times1+(2.1)_2\times3+(2.1)_4\times1+(2.1)_5\times1 +(2.1)_6\times1+(2.2)\times6=13)$[4)], equations $(2.1)_{1,2,4,5,6}$, (2.2), (2.4) and (2.5) form a mathematically well-posed system, and one has the chance to obtain the values of the independent field quantities by integrating these equations simultaneously, provided that the constitutive equations are prescribed. To achieve this, it is assumed that the expression of any constitutive variable $\wp$ can be decomposed into two parts, namely the equilibrium response and the dynamic (non-equilibrium) response, viz.,

$$\wp \equiv \wp|_{\mathrm{E}} + \wp^{\mathrm{D}}, \qquad \wp^{\mathrm{D}}|_{\mathrm{E}} = 0, \tag{2.6}$$

where the subscript E denotes that the indexed quantity is evaluated in thermodynamic equilibrium, while the superscript D denotes the non-equilibrium contribution. We now turn to the possible restrictions of $\wp|_{\mathrm{E}}$ in the context of thermodynamic analysis.

## 3. Thermodynamic analysis

### 3.1. Entropy inequality and its exploitation

The second law of thermodynamics requires that the entropy production $\pi$ should always be non-negative during a physical process, what allows to write $(2.1)_7$ alternatively in the form

$$\pi = \gamma\nu\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \gamma\nu s \geq 0. \tag{3.1}$$

A physically-realizable process should be one in which the entropy inequality (3.1), the balance equations $(2.1)_{1,2,4,5,6}$, the evolution of internal friction (2.2) as well as the constitutive relations (2.4) and (2.5) must hold simultaneously. This can be achieved by regarding these equations as the constraints of the entropy inequality via the method of Lagrange multipliers in the form [34]

$$\begin{aligned}
\pi = \gamma\nu\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \gamma\nu s - \lambda^{\gamma}\big(\dot{\gamma}\nu + \gamma\dot{\nu} + \gamma\nu\operatorname{div}\mathbf{v}\big)& \\
- \boldsymbol{\lambda}^{v}\cdot\big(\gamma\nu\dot{\mathbf{v}} - \operatorname{div}\mathbf{t} - \gamma\nu\mathbf{b}\big) - \lambda^{\nu}\big(\gamma\nu(\ell\dot{\nu})^{\cdot} - \operatorname{div}\mathbf{h} - \gamma\nu f\big)& \\
- \lambda^{\ell}\big(\gamma\nu\dot{\ell} - \operatorname{div}\boldsymbol{\Gamma} - \mathit{\Pi}\big) - \boldsymbol{\lambda}^{Z}\cdot\big(\dot{\mathbf{Z}} - [\boldsymbol{\Omega},\mathbf{Z}] - \boldsymbol{\Phi}\big)& \\
- \lambda^{e}\big(\gamma\nu\dot{e} - \mathbf{t}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} - \gamma\nu r - \mathbf{h}\cdot\operatorname{grad}(\ell\dot{\nu}) + \gamma\nu f\ell\dot{\nu}\big) \geq 0&,
\end{aligned} \tag{3.2}$$

[4)]While $(2.1)_3$ is fulfilled by prescribing the specific constitutive class (2.5), $(2.1)_7$ can be written as an inequality and will be exploited in the thermodynamic analysis, as will be shown later on.

where $\lambda^\gamma$, $\boldsymbol{\lambda}^v$, $\lambda^\nu$, $\lambda^\ell$, $\lambda^e$ and $\boldsymbol{\lambda}^Z$ are the Lagrange multipliers of the balances of mass, linear momentum, equilibrated force, internal length, internal energy and the evolution equation of $\mathbf{Z}$, respectively. Since the material behaviour is required to be independent of the external sources, it follows immediately that

$$-\gamma\nu s + \gamma\nu\boldsymbol{\lambda}^v \cdot \mathbf{b} + \lambda^e\gamma\nu r = 0 \tag{3.3}$$

must hold, which is an identity for the entropy supply and is more general than the classical selection. Once $\boldsymbol{\lambda}^v$ or $\mathbf{b}$ vanishes, it reduces to the traditional expression of the entropy supply (i.e., $s = \lambda^e r$, see e.g. [35]). Since $\dot\theta$ is not included in (2.5), it is plausible to introduce the *Helmholtz free energy* $\Psi$ in the form of $\Psi \equiv e - \theta\eta$, and to conjecture that $\lambda^e = 1/\theta$[5]. Substituting these and (3.3) into (3.2) yields

$$\begin{aligned}
\pi = {} & \frac{\gamma\nu}{\theta}(\dot e - \dot\theta\eta - \dot\Psi) + \operatorname{div}\boldsymbol{\phi} - \lambda^\gamma(\gamma\dot\nu + \dot\gamma\nu + \gamma\nu\operatorname{div}\mathbf{v}) \\
& - \boldsymbol{\lambda}^v \cdot (\gamma\nu\dot{\mathbf{v}} - \operatorname{div}\mathbf{t}) - \lambda^\nu\big(\gamma\nu(\ell\dot\nu)^{\cdot} - \operatorname{div}\mathbf{h} - \gamma\nu f\big) \\
& - \lambda^\ell(\gamma\nu\dot\ell - \operatorname{div}\boldsymbol{\Gamma} - \Pi) - \boldsymbol{\lambda}^Z \cdot \big(\dot{\mathbf{Z}} - [\boldsymbol{\Omega}, \mathbf{Z}] - \boldsymbol{\Phi}\big) \\
& - \frac{1}{\theta}(\gamma\nu\dot e - \mathbf{t}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} - (\mathbf{h}\cdot\operatorname{grad}\ell)\dot\nu - (\mathbf{h}\cdot\operatorname{grad}\dot\nu)\ell + \gamma\nu f\ell\dot\nu) \geq 0.
\end{aligned} \tag{3.4}$$

Incorporating the functional dependences of the constitutive variables (2.4) and (2.5) into (3.4) by use of the chain rule of differentiation, gives rise to the entropy inequality exploited in the form

$$\begin{aligned}
\pi = {} & -\left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\nu} + \lambda^\gamma\gamma - \frac{\mathbf{h}\cdot\mathbf{g}_2}{\theta} + \frac{\gamma\nu f\ell}{\theta}\right\}\dot\nu - \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\dot\nu} + \lambda^\nu\gamma\nu\ell\right\}\ddot\nu \\
& - \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{D}}\cdot\dot{\mathbf{D}} - \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\gamma} + \lambda^\gamma\nu\right\}\dot\gamma - \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\ell} + \lambda^\nu\gamma\nu\dot\nu + \lambda^\ell\gamma\nu\right\}\dot\ell \\
& - \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\theta} + \frac{\gamma\nu\eta}{\theta}\right\}\dot\theta - \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{Z}} + \boldsymbol{\lambda}^Z\right\}\cdot\dot{\mathbf{Z}} - \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_2}\cdot\dot{\mathbf{g}}_2 - \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_3}\cdot\dot{\mathbf{g}}_3 \\
& - \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_4}\cdot\dot{\mathbf{g}}_4 - \gamma\nu\boldsymbol{\lambda}^v\cdot\dot{\mathbf{v}} + \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_1}\otimes\mathbf{g}_1\right\}\cdot\boldsymbol{L} + [\boldsymbol{\lambda}^Z, \mathbf{Z}]\cdot\boldsymbol{\Omega} + \boldsymbol{\lambda}^Z\cdot\boldsymbol{\Phi} \\
& - \lambda^\gamma\gamma\nu\operatorname{div}\mathbf{v} + \lambda^\nu\gamma\nu f + \lambda^\ell\Pi + \frac{\mathbf{t}\cdot\mathbf{D}}{\theta}
\end{aligned} \tag{3.5}$$

[5]In fact, the exclusion of $\dot\theta$ in (2.5) leads only to the result that $\lambda^e = \hat\lambda^e(\theta)$. The specific form that $\lambda^e = 1/\theta$ can only be derived for some simple substances (see e.g. [32, 33]). However, following previous works this conjecture is justified and hence employed here (see e.g. [13, 14, 20, 25, 26]).

(3.5) [cont.]
$$+\left\{\frac{\partial\phi}{\partial\dot\nu}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\dot\nu}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\dot\nu}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\dot\nu}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\dot\nu}+\frac{\ell\mathbf{h}}{\theta}-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_1}\right\}\cdot\operatorname{grad}\dot\nu$$
$$+\left\{\frac{\partial\phi}{\partial\nu_0}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\nu_0}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu_0}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\nu_0}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu_0}\right\}\cdot\operatorname{grad}\nu_0$$
$$+\left\{\frac{\partial\phi}{\partial\ell_0}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\ell_0}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\ell_0}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\ell_0}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\ell_0}\right\}\cdot\operatorname{grad}\ell_0$$
$$+\left\{\frac{\partial\phi}{\partial\nu}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\nu}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\nu}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu}\right\}\cdot\mathbf{g}_1$$
$$+\left\{\frac{\partial\phi}{\partial\ell}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\ell}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\ell}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\ell}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\ell}\right\}\cdot\mathbf{g}_2$$
$$+\left\{\frac{\partial\phi}{\partial\gamma}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\gamma}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\gamma}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\gamma}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\gamma}\right\}\cdot\mathbf{g}_3$$
$$+\left\{\frac{\partial\phi}{\partial\theta}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\theta}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\theta}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\theta}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\theta}\right\}\cdot\mathbf{g}_4$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{g}_1}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{g}_1}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{g}_1}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{g}_1}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{g}_1}\right\}\cdot\operatorname{grad}\mathbf{g}_1$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{g}_2}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{g}_2}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{g}_2}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{g}_2}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{g}_2}\right\}\cdot\operatorname{grad}\mathbf{g}_2$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{g}_3}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{g}_3}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{g}_3}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{g}_3}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{g}_3}\right\}\cdot\operatorname{grad}\mathbf{g}_3$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{g}_4}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{g}_4}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{g}_4}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{g}_4}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{g}_4}\right\}\cdot\operatorname{grad}\mathbf{g}_4$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{D}}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{D}}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{D}}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{D}}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{D}}\right\}\cdot\operatorname{grad}\mathbf{D}$$
$$+\left\{\frac{\partial\phi}{\partial\mathbf{Z}}+\boldsymbol{\lambda}^{v}\frac{\partial\mathbf{t}}{\partial\mathbf{Z}}+\lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{Z}}+\lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{Z}}-\frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{Z}}\right\}\cdot\operatorname{grad}\mathbf{Z}\geq 0.$$

In deriving (3.5), the identities

$$\dot{\mathbf{g}}_1=\operatorname{grad}\dot\nu-\mathbf{g}_1\boldsymbol{L},\qquad \boldsymbol{\lambda}^{Z}\cdot[\boldsymbol{\Omega},\mathbf{Z}]=[\boldsymbol{\lambda}^{Z},\mathbf{Z}]\cdot\boldsymbol{\Omega},\tag{3.6}$$

have been used. Let $\boldsymbol{\mathcal{X}}=\{\dot{\mathbf{v}},\ddot\nu,\dot\ell,\dot\gamma,\dot\theta,\dot{\mathbf{D}},\dot{\mathbf{Z}},\operatorname{grad}\dot\nu,\dot{\mathbf{g}}_2,\dot{\mathbf{g}}_3,\dot{\mathbf{g}}_4,\operatorname{grad}\nu_0,\operatorname{grad}\ell_0,\operatorname{grad}\mathbf{g}_1,\operatorname{grad}\mathbf{g}_2,\operatorname{grad}\mathbf{g}_3,\operatorname{grad}\mathbf{g}_4,\operatorname{grad}\mathbf{D},\operatorname{grad}\mathbf{Z}\}$. It is straightforward to see that the inequality (3.5) possesses the structure

$$\mathbf{a}\cdot\boldsymbol{\mathcal{X}}+b\geq 0,\tag{3.7}$$

where the vector $\mathbf{a}$ and the scalar $b$ are functions of (2.5), but not of $\boldsymbol{\mathcal{X}}$, and the vector $\boldsymbol{\mathcal{X}}$, as defined, depends on time and space derivatives of (2.5). Since (3.5) is linear in $\boldsymbol{\mathcal{X}}$ which can take any values, it would be possible to violate (3.7) unless

$$\mathbf{a} = \mathbf{0}, \qquad \text{and} \qquad b \geq 0, \tag{3.8}$$

where $(3.8)_1$ leads to the so-called *Liu identities* and $(3.8)_2$ gives rise to the *residual entropy inequality.* In particular, the linearity in $\dot{\mathbf{v}}$, $\ddot{\nu}$, $\dot{\gamma}$, $\dot{\ell}$ and $\dot{\mathbf{Z}}$ gives rise to the Lagrangian multipliers in the forms

$$\begin{gathered}\boldsymbol{\lambda}^{v} = \mathbf{0}, \qquad \lambda^{\nu} = -\frac{1}{\ell\theta}\frac{\partial\Psi}{\partial\dot{\nu}} \qquad \lambda^{\gamma} = -\frac{\gamma}{\theta}\frac{\partial\Psi}{\partial\gamma}, \\ \lambda^{\ell} = -\lambda^{\nu}\dot{\nu} - \frac{1}{\theta}\frac{\partial\Psi}{\partial\ell}, \qquad \boldsymbol{\lambda}^{Z} = -\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{Z}},\end{gathered} \tag{3.9}$$

while the linearity in $\dot{\mathbf{D}}$, $\dot{\mathbf{g}}_2$, $\dot{\mathbf{g}}_3$ and $\dot{\mathbf{g}}_4$ results in the constraints of the free energy, viz.,

$$\frac{\partial\Psi}{\partial\mathbf{D}} = \mathbf{0}, \qquad \frac{\partial\Psi}{\partial\mathbf{g}_2} = \mathbf{0}, \qquad \frac{\partial\Psi}{\partial\mathbf{g}_3} = \mathbf{0}, \qquad \frac{\partial\Psi}{\partial\mathbf{g}_4} = \mathbf{0}. \tag{3.10}$$

In addition, (3.5) is also linear in $\operatorname{grad}\nu_0$, $\operatorname{grad}\ell_0$, $\operatorname{grad}\mathbf{D}$, $\operatorname{grad}\mathbf{Z}$, $\operatorname{grad}\dot{\nu}$, $\operatorname{grad}\mathbf{g}_1$, $\operatorname{grad}\mathbf{g}_2$, $\operatorname{grad}\mathbf{g}_3$ and $\operatorname{grad}\mathbf{g}_4$. These conditions result in certain relations among $\boldsymbol{\phi}$, $\mathbf{h}$, $\mathbf{q}$ and $\boldsymbol{\Gamma}$ given by

$$\begin{gathered}\frac{\partial\boldsymbol{\phi}}{\partial\Xi} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\Xi} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\Xi} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\Xi} = \mathbf{0}, \qquad \Xi \in \{\nu_0, \ell_0, \mathbf{D}, \mathbf{Z}\}, \\ \frac{\partial\boldsymbol{\phi}}{\partial\dot{\nu}} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\dot{\nu}} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\dot{\nu}} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\dot{\nu}} + \frac{\ell\mathbf{h}}{\theta} = \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_1}, \\ \operatorname{sym}\left\{\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{\Theta}} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\boldsymbol{\Theta}} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{\Theta}} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\boldsymbol{\Theta}}\right\} = \mathbf{0}, \qquad \boldsymbol{\Theta} \in \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\},\end{gathered} \tag{3.11}$$

which must hold simultaneously, where $\operatorname{sym}\{\boldsymbol{A}\}$ denotes the symmetric part of an arbitrary second-rank tensor $\boldsymbol{A}$. In deriving (3.11), the condition $(3.9)_1$ has been used. Lastly, since the linearity of (3.5) holds equally in $\dot{\theta}$ and $\boldsymbol{\Omega}$, we conclude that

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \qquad [\boldsymbol{\lambda}^{\mathbf{Z}}, \mathbf{Z}] = \mathbf{0}. \tag{3.12}$$

With these, the Liu identities have been identified.

Substituting $(3.9)_1$, (3.10)–(3.12) into (3.5) gives rise to the residual entropy inequality in the form

$$
\begin{aligned}
(3.13)\qquad \pi = &-\left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\nu} + \lambda^{\gamma}\gamma - \frac{\mathbf{h}\cdot\mathbf{g}_2}{\theta} + \frac{\gamma\nu f\ell}{\theta}\right\}\dot{\nu} \\
&+ \left\{\frac{\partial\boldsymbol{\phi}}{\partial\nu} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\nu} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu}\right\}\cdot\mathbf{g}_1 + \left\{\frac{\partial\boldsymbol{\phi}}{\partial\ell} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\ell} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\ell} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\ell}\right\}\cdot\mathbf{g}_2 \\
&+ \left\{\frac{\partial\boldsymbol{\phi}}{\partial\gamma} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\gamma} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\gamma} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\gamma}\right\}\cdot\mathbf{g}_3 + \left\{\frac{\partial\boldsymbol{\phi}}{\partial\theta} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\theta} + \lambda^{\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\theta} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\theta}\right\}\cdot\mathbf{g}_4 \\
&+ \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{g}_1}\otimes\mathbf{g}_1\right\}\cdot\mathbf{D} - \lambda^{\gamma}\gamma\nu\,\mathrm{div}\,\mathbf{v} + \lambda^{\nu}\gamma\nu f + \lambda^{\ell}\Pi + \frac{\mathbf{t}\cdot\mathbf{D}}{\theta} + \boldsymbol{\lambda}^{Z}\cdot\boldsymbol{\Phi} \geq 0.
\end{aligned}
$$

**3.2. Extra entropy flux vector**

In some flow circumstances involving dry granular masses, the free energy depends significantly on spatial variations of the volume fraction, but not on its time-rate of change [4, 13, 14, 20], it is thus plausible to assume that $\Psi$ is not a function of $\dot{\nu}$. With this and the restrictions (3.10), the functional dependence of $\Psi$ reduces to

$$
(3.14)\qquad \Psi = \hat{\Psi}(\ell_0, \nu_0, \nu, \mathbf{g}_1, \ell, \gamma, \theta, \mathbf{Z}),
$$

which can be decomposed into two parts, namely the non-frictional part, $\Psi^e$, and the frictional part, $\Psi^f$, viz.,

$$
\begin{aligned}
(3.15)\qquad \Psi &= \hat{\Psi}^e(\ell_0, \nu_0, \nu, \mathbf{g}_1, \ell, \gamma, \theta) + \hat{\Psi}^f(\mathbf{Z}) \\
&= \hat{\Psi}^e(\ell_0, \nu_0, \nu, \mathbf{g}_1\cdot\mathbf{g}_1, \ell, \gamma, \theta) + \hat{\Psi}^f(I_Z, II_Z, III_Z),
\end{aligned}
$$

where $I_Z$, $II_Z$ and $III_Z$ are the three invariants of $\mathbf{Z}$. In deriving (3.15), $\Psi$ is assumed to be an isotropic scalar function. In addition, we assume that the internal friction effects only will be confined to $\Psi^f$ and the coupling effects between $\mathbf{g}_1$ and $\mathbf{Z}$ are neglected [20, 21, 26, 37].

Define the extra entropy flux vector $\mathbf{k}$, an auxiliary variable accounting for the deviations of the entropy flux from the heat flux, in the form

$$
(3.16)\qquad \boldsymbol{\phi} = \frac{\mathbf{q}}{\theta} - \boldsymbol{\lambda}^{v}\mathbf{t} - \lambda^{\nu}\mathbf{h} + \mathbf{k} = \frac{\mathbf{q}}{\theta} + \mathbf{k}, \qquad \text{with } \boldsymbol{\lambda}^{v} = \mathbf{0},\ \lambda^{\nu} = 0.
$$

Substituting (3.16) into (3.11) yields

$$
\begin{aligned}
&\frac{\partial \mathbf{k}}{\partial \Xi} + \lambda^\ell \frac{\partial \boldsymbol{\Gamma}}{\partial \Xi} = \mathbf{0}, && \Xi \in \{\nu_0, \ell_0, \mathbf{D}, \mathbf{Z}\},\\
&\frac{\partial \mathbf{k}}{\partial \dot\nu} + \lambda^\ell \frac{\partial \boldsymbol{\Gamma}}{\partial \dot\nu} + \frac{\ell \mathbf{h}}{\theta} = \frac{\gamma\nu}{\theta}\frac{\partial \Psi}{\partial \mathbf{g}_1}, \\
&\operatorname{sym}\left\{\frac{\partial \mathbf{k}}{\partial \boldsymbol{\Theta}} + \lambda^\ell \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\Theta}}\right\} = \mathbf{0}, && \boldsymbol{\Theta} \in \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}.
\end{aligned} \tag{3.17}
$$

Since $\Psi \neq \hat\Psi(\cdot, \dot\nu)$, inserting this into $(3.9)_4$ yields an identity of $\lambda^\ell = -(\partial\Psi/\partial\ell)/\theta$. With this and (3.15), the functional dependence of the Lagrangian multiplier $\lambda^\ell$ becomes

$$
\lambda^\ell = -\frac{1}{\theta}\frac{\partial \Psi}{\partial \ell} = -\frac{1}{\theta}\frac{\partial \Psi^e}{\partial \ell} = \hat\lambda^\ell(\ell_0, \nu_0, \nu, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta), \tag{3.18}
$$

since $\lambda^\ell$ is also assumed to be an isotropic scalar function. With (3.18), the last two equations of $(3.17)_1$ and the last three equations of $(3.17)_3$ can be expressed alternatively as

$$
\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\Xi}}(\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma}) = \mathbf{0}, && \boldsymbol{\Xi} \in \{\mathbf{D}, \mathbf{Z}\},\\
&\operatorname{sym}\left\{\frac{\partial}{\partial \boldsymbol{\Theta}}(\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma})\right\} = \mathbf{0}, && \boldsymbol{\Theta} \in \{\mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}.
\end{aligned} \tag{3.19}
$$

Relation $(3.19)_1$ implies that $\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma} \neq \mathit{funct.}(\cdot, \mathbf{D}, \mathbf{Z})$, of which the possible explicit expressions can be obtained by integrating $(3.19)_2$ by pairs. viz.,

$$
\begin{aligned}
\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma} &= \boldsymbol{A}_1 \mathbf{g}_3 + \boldsymbol{B}_1 \mathbf{g}_4 + \boldsymbol{C}_1(\mathbf{g}_3 \otimes \mathbf{g}_4) + \mathbf{d}_1(\ell_0, \nu_0, \nu, \dot\nu, \ell, \gamma, \theta, \mathbf{g}_1, \mathbf{g}_2),\\
&= \boldsymbol{A}_2 \mathbf{g}_2 + \boldsymbol{B}_2 \mathbf{g}_3 + \boldsymbol{C}_2(\mathbf{g}_2 \otimes \mathbf{g}_3) + \mathbf{d}_2(\ell_0, \nu_0, \nu, \dot\nu, \ell, \gamma, \theta, \mathbf{g}_1, \mathbf{g}_4),\\
&= \boldsymbol{A}_3 \mathbf{g}_2 + \boldsymbol{B}_3 \mathbf{g}_4 + \boldsymbol{C}_3(\mathbf{g}_2 \otimes \mathbf{g}_4) + \mathbf{d}_3(\ell_0, \nu_0, \nu, \dot\nu, \ell, \gamma, \theta, \mathbf{g}_1, \mathbf{g}_3),
\end{aligned} \tag{3.20}
$$

where $\boldsymbol{A}_{1-3}, \boldsymbol{B}_{1-3}$ and $\boldsymbol{C}_{1-3}$ are second-rank and third-rank skew-symmetric tensors, respectively, and $\mathbf{d}_{1-3}$ are the undetermined vector functions of which the functional dependences are indicated. In addition, while $\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1$ and $\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2$ are not functions of $\mathbf{g}_3, \mathbf{g}_4$ and $\mathbf{g}_2, \mathbf{g}_3$, respectively, $\boldsymbol{A}_3, \boldsymbol{B}_3, \boldsymbol{C}_3$ are not functions of $\mathbf{g}_2, \mathbf{g}_4$. Moreover, since $\mathbf{q}$ and $\boldsymbol{\Lambda}$ are assumed to be isotropic vectors, the vector $\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma}$ must also be isotropic. It follows immediately that $\boldsymbol{A}_{1-3} = \boldsymbol{B}_{1-3} = \mathbf{0}$ and $\boldsymbol{C}_{1-3} = \mathbf{0}$, because there are no isotropic second and third-rank skew-symmetric tensors. With these and the possible functional dependences of $\mathbf{d}_{1-3}$, (3.20) reduces to

$$
\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma} = \mathbf{d}(\ell_0, \nu_0, \nu, \dot\nu, \mathbf{g}_1, \ell, \gamma, \theta), \tag{3.21}
$$

where $\mathbf{d}$ is an undetermined vector function that should also be isotropic. In view of this, $\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma}$ can be expressed explicitly via the isotropic assumption, viz.,

$$\mathbf{k} + \lambda^\ell \boldsymbol{\Gamma} = \xi \mathbf{g}_1, \qquad \xi = \hat{\xi}(\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta). \tag{3.22}$$

In the next step, from (3.18) and (3.22) it is plausible to assume that $\boldsymbol{\Gamma}$ is collinear with $\mathbf{g}_1$ and hence can be expressed as [14]

$$\boldsymbol{\Gamma} = \kappa \mathbf{g}_1, \qquad \kappa = \hat{\kappa}(\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta). \tag{3.23}$$

Substituting (3.16), (3.22) and (3.23) into the first two equations of $(3.11)_1$ gives rise to

$$\frac{\partial \xi}{\partial \wp} \mathbf{g}_1 = \frac{\partial \lambda^\ell}{\partial \wp} \boldsymbol{\Gamma} = \frac{\partial \lambda^\ell}{\partial \wp} \kappa\, \mathbf{g}_1, \quad \Longrightarrow \quad \frac{\partial \xi}{\partial \wp} = \frac{\partial \lambda^\ell}{\partial \wp} \kappa, \qquad \wp \in \{\nu_0, \ell_0\}, \tag{3.24}$$

which furnish with two restrictions the specific functional forms of the isotropic scalar functions $\xi$, $\kappa$ and $\lambda^\ell$. Next, inserting (3.15), (3.16) and (3.22) into $(3.17)_2$ yields an explicit expression for the equilibrated stress vector $\mathbf{h}$, viz.,

$$\begin{aligned} \mathbf{h} &= \frac{1}{\ell} \left\{ \gamma \nu \frac{\partial \Psi^e}{\partial \mathbf{g}_1} - \theta \frac{\partial \xi}{\partial \dot{\nu}} \mathbf{g}_1 \right\} = \frac{1}{\ell} \left\{ \mathcal{A} - \theta \frac{\partial \xi}{\partial \dot{\nu}} \right\} \mathbf{g}_1, \\ \mathcal{A} &= 2\gamma\nu \frac{\partial \Psi^e}{\partial (\mathbf{g}_1 \cdot \mathbf{g}_1)} = \hat{\mathcal{A}}(\nu_0, \nu, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta). \end{aligned} \tag{3.25}$$

The remaining condition, the first equation of $(3.17)_1$, is now analyzed by substituting (3.18), (3.22) and (3.23) into itself, what results in

$$\operatorname{sym} \left\{ \xi \mathbf{I} + \Big( 2 \frac{\partial \xi}{\partial (\mathbf{g}_1 \cdot \mathbf{g}_1)} - 2 \frac{\partial \lambda^\ell}{\partial (\mathbf{g}_1 \cdot \mathbf{g}_1)} \kappa \Big) \mathbf{g}_1 \otimes \mathbf{g}_1 \right\} = \mathbf{0}, \tag{3.26}$$

which, due to the fact that $\mathbf{g}_1$ varies independently, can only be fulfilled when the conditions

$$\xi = 0, \qquad \frac{\partial \lambda^\ell}{\partial (\mathbf{g}_1 \cdot \mathbf{g}_1)} = 0, \tag{3.27}$$

hold[6)]. In view of (3.18), $(3.24)_2$ and (3.27), the functional dependence of $\lambda^\ell$ is further simplified to

$$\lambda^\ell = \hat{\lambda^\ell}(\nu, \ell, \gamma, \theta). \tag{3.28}$$

[6)] Equation $(3.27)_2$ results from the equality of $(\partial \lambda^\ell / \partial (\mathbf{g}_1 \cdot \mathbf{g}_1))\kappa = 0$. The quantity $\kappa$ is not allowed to vanish, or it leads to a vanishing $\boldsymbol{\Gamma}$, which is a too strong requirement for the present constitutive formulation.

Similarly, applying $(3.27)_1$ to (3.22) and (3.25) yields ultimate expressions of $\mathbf{k}$ and $\mathbf{h}$ in the forms

$$\mathbf{k} = -\lambda^\ell \mathbf{\Gamma}, \qquad \mathbf{h} = \frac{\gamma\nu}{\ell}\frac{\partial \Psi^e}{\partial \mathbf{g}_1}. \tag{3.29}$$

Condition $(3.29)_1$ is significant, which indicates that the entropy flux is not collinear with the heat flux if the flux $\mathbf{\Gamma}$ of $\ell$ and the Lagrange multiplier $\lambda^\ell$ do not vanish. This formulation differs thermodynamically from other constitutive formulations based on the revised Goodman–Cowin theory with internal friction, in which the entropy flux was demonstrated to be identical to the classical selection [20, 21]. Heretofore, the Liu identities have been fully exploited.

Substituting $(3.9)_{3-5}$, (3.15), (3.16), (3.22), $(3.27)_1$ and (3.29) into (3.13) and noting that $\lambda^\nu = 0$ gives rise to an alternative form of the residual entropy inequality, viz.,

$$\begin{aligned}\theta\pi = \Big\{p - \beta + \mathbf{h}\cdot\mathbf{g}_2 - \gamma\nu f\ell\Big\}\dot\nu + \mathbf{\Gamma}\cdot\left\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\right\} - \zeta\Pi \\ + \left\{\Big(\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\theta}\Big)\mathbf{\Gamma} - \frac{\mathbf{q}}{\theta}\right\}\cdot\mathbf{g}_4 + \Big\{\mathbf{t} + \nu p\mathbf{I} + \ell\mathbf{h}\otimes\mathbf{g}_1\Big\}\cdot\mathbf{D} - \gamma\nu\frac{\partial\Psi^f}{\partial\mathbf{Z}}\cdot\mathbf{\Phi} \geq 0,\end{aligned} \tag{3.30}$$

with the abbreviations

$$p \equiv \gamma^2\frac{\partial\Psi^e}{\partial\gamma}, \qquad \beta \equiv \gamma\nu\frac{\partial\Psi^e}{\partial\nu}, \qquad \zeta \equiv \frac{\partial\Psi^e}{\partial\ell}, \tag{3.31}$$

where $p$ and $\beta$ are respectively the thermodynamic and $\nu$-induced configuration pressures, and $\zeta$ is the emerging kinematic constraint induced by the variations of $\ell$ [6, 10, 14, 20, 21]. Equation (3.30) will be further investigated in the context of thermodynamic equilibrium.

### 3.3. Thermodynamic equilibrium

In the current formulation, *thermodynamic equilibrium* is defined to be a time-independent process with homogeneous thermodynamic field quantities and vanishing entropy production [36]

$$\pi|_{\mathrm{E}} = 0. \tag{3.32}$$

In view of this, (2.4)-(2.6) and (3.30), we define the following states:

$$\begin{aligned}\mathbf{Y} &\equiv (\ell_0, \nu_0, \nu, \dot\nu, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{g}_4, \mathbf{D}, \mathbf{Z}),\\ \mathbf{Y}|_{\mathrm{E}} &= (\ell_0, \nu_0, \nu, 0, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{0}, \mathbf{0}, \mathbf{Z}),\\ \mathbf{Y}^{\mathrm{D}} &= (\dot\nu, \mathbf{g}_4, \mathbf{D}), \qquad \mathbf{Y}^{\mathrm{D}}|_{\mathrm{E}} = \mathbf{0},\end{aligned} \tag{3.33}$$

with which (3.30) can be recast as a "quasi-linear" form in $\mathbf{Y}^{\mathrm{D}}$, viz.

$$
\begin{aligned}
&\theta\pi = \boldsymbol{\mathcal{A}} \cdot \mathbf{Y}^{\mathrm{D}} + \mathcal{B} \geq 0, \qquad \boldsymbol{\mathcal{A}}, \mathcal{B} = funct.(\mathbf{Y}),\\
&\boldsymbol{\mathcal{A}} = \left\{ p - \beta + \mathbf{h}\cdot\mathbf{g}_2 - \gamma\nu f\ell, \left(\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\theta}\right)\boldsymbol{\Gamma} - \frac{\mathbf{q}}{\theta}, \mathbf{t} + \nu p\mathbf{I} + \ell\mathbf{h}\otimes\mathbf{g}_1 \right\},\\
&\mathcal{B} = \boldsymbol{\Gamma}\cdot\left\{ \frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3 \right\} - \zeta\varPi - \gamma\nu\frac{\partial\varPsi^f}{\partial\mathbf{Z}}\cdot\boldsymbol{\Phi}.
\end{aligned}
\tag{3.34}
$$

In addition, the entropy production $\pi$ assumes its global minimum located at $\mathbf{Y}|_{\mathrm{E}}$. It follows that under sufficient smoothness properties, $\pi$ has to satisfy the conditions

$$
\pi,_{\dot\nu}|_{\mathrm{E}} = 0, \qquad \pi,_{\mathbf{g}_4}|_{\mathrm{E}} = \mathbf{0}, \qquad \pi,_{\mathbf{D}}|_{\mathrm{E}} = \mathbf{0},
\tag{3.35}
$$

and that the Hessian matrix of $\pi$ with respect to these variables is positive semi-definite at $\mathbf{Y}|_{\mathrm{E}}$, viz.

$$
\mathcal{H}|_{\mathrm{E}} = \left.\begin{pmatrix} \pi,_{\dot\nu\dot\nu} & \pi,_{\dot\nu\mathbf{g}_4} & \pi,_{\dot\nu\mathbf{D}} \\ \pi,_{\mathbf{g}_4\dot\nu} & \pi,_{\mathbf{g}_4\mathbf{g}_4} & \pi,_{\mathbf{g}_4\mathbf{D}} \\ \pi,_{\mathbf{D}\dot\nu} & \pi,_{\mathbf{D}\mathbf{g}_4} & \pi,_{\mathbf{D}\mathbf{D}} \end{pmatrix}\right|_{\mathrm{E}}.
\tag{3.36}
$$

Whereas condition (3.36) constraints the sign of the material parameters in the equilibrium expressions of the constitutive variables, conditions (3.32) and (3.35) yield the restrictions and equilibrium expressions of the dependent constitutive fields, which will be investigated separately.

First, applying (3.32) to (3.34) leads to

$$
\boldsymbol{\Gamma}\big|_E \cdot \left\{ \frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3 \right\} - \zeta\varPi\big|_E - \gamma\nu\frac{\partial\varPsi^f}{\partial\mathbf{Z}}\cdot\boldsymbol{\Phi}\big|_{\mathrm{E}} = 0,
\tag{3.37}
$$

which must be fulfilled by specific forms of $\boldsymbol{\Gamma}|_{\mathrm{E}}$, $\varPsi^e$, $\varPsi^f$, $\varPi|_{\mathrm{E}}$ and $\boldsymbol{\Phi}|_{\mathrm{E}}$. In deriving (3.37) it is noted that the functional dependences of the quantities $\varPsi^e$, $\varPsi^f$, $\lambda^\ell$, $p$, $\beta$, $\zeta$ and $\mathbf{h}$ are the same in both the equilibrium and non-equilibrium situations. (3.37) should serve as a restriction for the functional forms of $\boldsymbol{\Gamma}$, $\varPi$ and $\boldsymbol{\Phi}$[7].

[7] Equation (3.37) can nevertheless be fulfilled by assuming that

$$
\boldsymbol{\Gamma}\big|_{\mathrm{E}} = \mathbf{0}, \qquad \varPi\big|_{\mathrm{E}} = 0, \qquad \boldsymbol{\Phi}|_{\mathrm{E}} = \mathbf{0},
\tag{3.38}
$$

hold. While the last two conditions imply that $\varPi$ and $\boldsymbol{\Phi}$ are production-like quantities, indicating vanishing productions for both the evolutions of the internal length and internal friction, the first one, in view of (3.23), results in a restriction for the quantity $\kappa$, viz.,

$$
\kappa\big|_E = \hat\kappa(\ell_0, \nu_0, \nu, \dot\nu = 0, \mathbf{g}_1\cdot\mathbf{g}_1, \ell, \gamma, \theta) = 0.
\tag{3.39}
$$

Although these simplifications seem to be rational, they should be justified by comparing with experimental outcomes. For generality, we let (3.37) be expressed as a general restriction which should be fulfilled by specific forms of $\boldsymbol{\Gamma}$, $\varPi$ and $\boldsymbol{\Phi}$.

Applying (3.35) to (3.34) yields the equilibrium expressions of the constitutive variables $f$, $\mathbf{t}$ and $\mathbf{q}$ in the forms

$$
(3.40)\qquad
\begin{aligned}
f\big|_{\mathrm{E}} &= \frac{p-\beta+\mathbf{h}\cdot\mathbf{g}_2}{\gamma\nu\ell} + \frac{1}{\gamma\nu\ell}\frac{\partial\boldsymbol{\Gamma}}{\partial\dot{\nu}}\bigg|_{\mathrm{E}} \cdot \left\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\right\} \\
&\qquad - \frac{\zeta}{\gamma\nu\ell}\frac{\partial\mathit{\Pi}}{\partial\dot{\nu}}\bigg|_{\mathrm{E}} - \frac{1}{\ell}\frac{\partial\mathit{\Psi}^f}{\partial\mathbf{Z}}\cdot\frac{\partial\boldsymbol{\Phi}}{\partial\dot{\nu}}\bigg|_{\mathrm{E}}, \\
\mathbf{t}|_{\mathrm{E}} &= -\nu p\mathbf{I} - \ell\mathbf{h}\otimes\mathbf{g}_1 + \zeta\frac{\partial\mathit{\Pi}}{\partial\mathbf{D}}\bigg|_{\mathrm{E}} + \gamma\nu\frac{\partial\mathit{\Psi}^f}{\partial\mathbf{Z}}\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{D}}\bigg|_{\mathrm{E}}, \\
\mathbf{q}|_{\mathrm{E}} &= \theta\Big(\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\theta}\Big)\boldsymbol{\Gamma}\Big|_{\mathrm{E}} - \theta\zeta\frac{\partial\mathit{\Pi}}{\partial\mathbf{g}_4}\bigg|_{\mathrm{E}} - \gamma\nu\theta\frac{\partial\mathit{\Psi}^f}{\partial\mathbf{Z}}\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{g}_4}\bigg|_{\mathrm{E}},
\end{aligned}
$$

in which the conditions $\zeta \neq \hat{\zeta}(\cdot,\dot{\nu},\mathbf{D})$, and $\boldsymbol{\Gamma} \neq \hat{\boldsymbol{\Gamma}}(\cdot,\mathbf{g}_4,\mathbf{D})$ have been used. Equation $(3.40)_1$ is significant, it indicates that both the internal friction and the evolution of the internal length enter the balance equation of equilibrated force $(2.1)_4$. This is in particular important for the internal friction, since it shows that $(2.1)_4$, which was previously employed to describe the force balances along the lines connecting the centers of the grains, is extended rationally to account for the frictional effects of the grains. Equation $(3.40)_2$ implies that the equilibrium stress tensor is also influenced by the internal friction and the evolution of the internal length. It leads to a corollary that a dry granular heap can yet exist under homogeneous distributions of the grains, provided that the internal friction or the production of the internal length exists. $(3.40)_3$ shows a non-vanishing "generalized" heat flux in thermodynamic equilibrium which does not need to correspond to its physical counterpart. It leads to a possible restriction among the functional forms of $\boldsymbol{\Gamma}$, $\mathit{\Pi}$ and $\boldsymbol{\Phi}$ if $\mathbf{q}$ is required to vanish in equilibrium.

Table 1 summarizes certain derived results of the constitutive formulations based on the revised Goodman–Cowin theory, associated with an independent kinematic internal length with/without the effects of internal friction, in which those results without the effects of internal friction are quoted from [14]. All the symbols used in the table have been introduced previously, except that $\mathcal{C}$ denotes the constitutive class. In addition, the restriction which must be fulfilled by specific functional forms of $\boldsymbol{\Gamma}$, $\mathit{\Pi}$ and $\boldsymbol{\Phi}$, namely (3.37), is given together with that without the effects of internal friction derived in [14] in (3.41), for comparison:

$$
(3.41)\qquad
\begin{aligned}
0 &= \boldsymbol{\Gamma}\big|_{\mathrm{E}} \cdot \left\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\right\} - \zeta\mathit{\Pi}\big|_{\mathrm{E}}, \\
0 &= \boldsymbol{\Gamma}\big|_{\mathrm{E}} \cdot \left\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\right\} - \zeta\mathit{\Pi}\big|_{\mathrm{E}} - \gamma\nu\frac{\partial\mathit{\Psi}^f}{\partial\mathbf{Z}}\cdot\boldsymbol{\Phi}\big|_{\mathrm{E}}.
\end{aligned}
$$

**Table 1. Effects of internal friction in the constitutive formulations.**

| | without internal friction [14] | with internal friction |
|---|---|---|
| $\mathcal{C}$ | $\{\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{g}_4, \mathbf{D}\}$ | $\{\ell_0, \nu_0, \nu, \dot{\nu}, \mathbf{g}_1, \ell, \mathbf{g}_2, \gamma, \mathbf{g}_3, \theta, \mathbf{g}_4, \mathbf{D}, \mathbf{Z}\}$ |
| $\mathbf{k}$ | $-\lambda^\ell \mathbf{\Gamma}$ | $-\lambda^\ell \mathbf{\Gamma}$ |
| $\Psi$ | $\hat{\Psi}(\ell_0, \nu_0, \nu, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta)$ | $\hat{\Psi}^e(\ell_0, \nu_0, \nu, \mathbf{g}_1 \cdot \mathbf{g}_1, \ell, \gamma, \theta) + \hat{\Psi}^f(I_Z, II_Z, III_Z)$ |
| $\mathbf{h}$ | $\frac{\gamma\nu}{\ell}\frac{\partial\Psi}{\partial\mathbf{g}_1}$ | $\frac{\gamma\nu}{\ell}\frac{\partial\Psi^e}{\partial\mathbf{g}_1}$ |
| $\mathbf{\Gamma}$ | restriction $(3.41)_1$ | restriction $(3.41)_2$ |
| $\mathbf{t}\vert_{\mathrm{E}}$ | $-\nu p\mathbf{I} - \ell\mathbf{h}\otimes\mathbf{g}_1 + \zeta\frac{\partial\Pi}{\partial\mathbf{D}}\Big\vert_{\mathrm{E}}$ | $-\nu p\mathbf{I} - \ell\mathbf{h}\otimes\mathbf{g}_1 + \zeta\frac{\partial\Pi}{\partial\mathbf{D}}\Big\vert_{\mathrm{E}} + \gamma\nu\frac{\partial\Psi^f}{\partial\mathbf{Z}}\frac{\partial\mathbf{\Phi}}{\partial\mathbf{D}}\Big\vert_{\mathrm{E}}$ |
| $f\vert_{\mathrm{E}}$ | $\frac{p-\beta}{\gamma\nu\ell} + \frac{\mathbf{h}\cdot\mathbf{g}_2}{\gamma\nu\ell} - \frac{\zeta}{\gamma\nu\ell}\frac{\partial\Pi}{\partial\dot{\nu}}\Big\vert_{\mathrm{E}}$ | $\frac{p-\beta}{\gamma\nu\ell} + \frac{\mathbf{h}\cdot\mathbf{g}_2}{\gamma\nu\ell} - \frac{\zeta}{\gamma\nu\ell}\frac{\partial\Pi}{\partial\dot{\nu}}\Big\vert_{\mathrm{E}} - \frac{1}{\ell}\frac{\partial\Psi^f}{\partial\mathbf{Z}}\cdot\frac{\partial\mathbf{\Phi}}{\partial\dot{\nu}}\Big\vert_{\mathrm{E}}$ |
| | $+\frac{1}{\gamma\nu\ell}\frac{\partial\mathbf{\Gamma}}{\partial\dot{\nu}}\Big\vert_{\mathrm{E}}\cdot\Big\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\Big\}$ | $+\frac{1}{\gamma\nu\ell}\frac{\partial\mathbf{\Gamma}}{\partial\dot{\nu}}\Big\vert_{\mathrm{E}}\cdot\Big\{\frac{\partial\zeta}{\partial\nu}\mathbf{g}_1 + \frac{\partial\zeta}{\partial\ell}\mathbf{g}_2 + \frac{\partial\zeta}{\partial\gamma}\mathbf{g}_3\Big\}$ |
| $\mathbf{q}\vert_{\mathrm{E}}$ | $\theta\Big(\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\theta}\Big)\mathbf{\Gamma}\vert_{\mathrm{E}} - \theta\zeta\frac{\partial\Psi}{\partial\mathbf{g}_4}\Big\vert_{\mathrm{E}}$ | $\theta\Big(\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\theta}\Big)\mathbf{\Gamma}\vert_{\mathrm{E}} - \theta\zeta\frac{\partial\Psi}{\partial\mathbf{g}_4}\Big\vert_{\mathrm{E}} - \gamma\nu\theta\frac{\partial\Psi^f}{\partial\mathbf{Z}}\frac{\partial\mathbf{\Phi}}{\partial\mathbf{g}_4}\Big\vert_{\mathrm{E}}$ |
| $\Pi\vert_{\mathrm{E}}$ | restriction $(3.41)_1$ | restriction $(3.41)_2$ |

It is seen that the internal friction enters the equilibrium expressions of $\mathbf{t}|_{\mathrm{E}}$, $f|_{\mathrm{E}}$ and $\mathbf{q}|_{\mathrm{E}}$ obviously, however, not significantly in the free energy. This is due to the decomposition of the free energy $\Psi$ into $\Psi = \Psi^e + \Psi^f$, in which all the effects induced by the internal friction are assumed to be confined to $\Psi^f$. This assumption is indeed not realistic, for it neglects the coupling effects between the internal friction and other physical characteristics, such like elastic effects. However, this decomposition has been demonstrated to be a useful concept in the implementation of the model [20, 21, 26, 37], and is hence employed in the present study. Due to this decomposition, the effects induced by the internal friction are thus decoupled from other contributions and modeled separately in the equilibrium expressions of $\mathbf{t}|_{\mathrm{E}}$, $f|_{\mathrm{E}}$ and $\mathbf{q}|_{\mathrm{E}}$, and the restriction among $\mathbf{\Gamma}$, $\Pi$ and $\mathbf{\Phi}$ shown in $(3.41)_2$. In comparison with other constitutive formulations based on the revised Goodman–Cowin theory associated with an internal length, the present model is the most general one in which the internal friction, the evolutions of the configurations of the grains and the internal length are taken simultaneously into account. This shows a better possibility to describe the complex behaviour of dry granular mass flows.

## 4. Concluding remarks

In the present study, the revised Goodman–Cowin theory with an independent kinematic internal length proposed in [14, 15] for dry granular mass flows, was rationally extended to take the effects of internal friction into account. To this end, we have generalized the Mohr–Coulomb friction model by using an Euclidean frame-indifferent, second-rank symmetric tensor $\mathbf{Z}$ (a spatial internal variable) to account for the internal friction and other non-conservative forces inside a material point of a granular continuum. A thermodynamic analysis, based on the Müller–Liu entropy principle, was extensively performed to deduce the ultimate equilibrium expressions and restrictions of the constitutive equations.

Results show that the Helmholtz free energy $\Psi$ plays a central role in the present constitutive formulation: almost all the equilibrium parts of the constitutive variables are related to its derivatives; they are determined in principle once $\Psi$ is prescribed. However, a drawback is the decomposition of $\Psi$ into a non-frictional part, $\Psi^e$, and a frictional part, $\Psi^f$, with which all the effects of internal friction are confined within $\Psi^f$ and the coupling effects between internal friction and other material characteristics are neglected. This was done due to its simplicity in the implementation of the model when proposing the dynamic contributions [20]. In the context of such a decomposition, the effects of internal friction enter the equilibrium expressions of the constitutive variables obviously but separately: they are determined once $\Psi^f$ and $\mathbf{\Phi}$, the production of the evolution of internal friction, are prescribed. With these, $(3.40)_1$ implies that the contact interactions, including those along and tangential to the connecting lines of the centers of the grains, can be simulated appropriately in the present constitutive formulation, while $(3.40)_2$ leads to a corollary that a dry granular heap can exist under homogeneous distributions of the grains, provided that internal friction or the evolution of internal length exists. This is more realistic in practical circumstances, in particular for the situations immediately after triggering of the flow. The present model provides the more general formulations than other formulations based on the revised Goodman–Cowin theory associated with an internal length with/without internal friction. For implementation of the model, e.g. a Taylor series expansion and a hypoplastic expression can be assigned to $\Psi^e$ and $\mathbf{\Phi}$, respectively [20, 38].

In the present paper, only the theoretical derivations of the equilibrium expressions of the constitutive variables are presented, the postulates of their dynamic (non-equilibrium) responses, the implementation of the model and the applications to Benchmark problems are deferred to other papers.

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## References


1. J. Duran, *Sands, Powders and Grains*, Springer, Wien, New York 2000.
2. K. Hutter, K.R. Rajagopal, *On flows of granular materials*, Continuum Mech. Thermodyn. **6**, 81–139, 1994.
3. S.B. Savage, *Mechanics of Granular Flows*, [in:] *Continuum mechanics in environmental sciences and geophysics*, K. Hutter [Ed.], Springer, Heidelberg, 467–522, 1993.
4. Y. Wang, K. Hutter, *Granular material theories revisited*, [in:] *Geomorphological Fluid Mechanics*, N.J. Balmforth and A. Provenzale [Eds.], Springer, Heidelberg, 79–107, 2001.
5. M.A. Goodman, S.C. Cowin, *Two problems in the gravity flow of granular materials*, J. Fluid Mech., **45**, 321–339, 1971.
6. M.A. Goodman, S.C. Cowin, *A continuum theory for granular materials*, Arch. Rational Mech. Anal., **44**, 249–266, 1972.
7. S.C. Cowin, M.A. Goodman, *A variational principle for granular materials*, ZAMM, **56**, 281–286, 1976.
8. S.C. Cowin, *Polar fluids*, Phys. Fluids, **11**, 9, 1919–1927, 1968.
9. F.M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Rational Mech. Anal., **28**, 265–283, 1968.
10. J.W. Nunziato, S.L. Passman, *Gravitational flows of granular materials with incompressible grains*, J. Rheology, **24**, 395–420, 1980.
11. J.W. Nunziato, E.K. Walsh, *On ideal multiphase mixtures with chemical reactions and diffusions*, Arch. Rational Mech. Anal., **73**, 285–311, 1980.
12. S.L. Passman, J.W. Nunziato, E.K. Walsh, *A theory of multiphase mixtures*, [in:] *Rational Thermodynamics*, C. Truesdell [Ed.], Springer Verlag, Heidelberg, 287–325, 1984.
13. Y. Wang, K. Hutter, *A constitutive theory of fluid-saturated granular materials and its application in gravitational flows*, Rheol. Acta., **38**, 214–223, 1999.
14. C. Fang, Y. Wang, K. Hutter, *A thermo-mechanical continuum theory with internal length for cohesionless granular materials. Part I. A class of constitutive models*, Continuum Mech. Thermodyn., **17**, 8, 545–576, 2006.
15. C. Fang, Y. Wang, K. Hutter, *A variational principle for the revised Goodman-Cowin theory with internal length*, Arch. Appl. Mech., **76**, 119–131, 2006.
16. C. Fang, Y. Wang, K. Hutter, *A thermo-mechanical continuum theory with internal length for cohesionless granular materials. Part II. Non-equilibrium postulates and numerical simulations of simple shear, plane Poiseuille and gravity driven problems*, Continuum Mech. Thermodyn., **17**, 8, 577–607, 2006.

17. J.P. Bardt, *Load dependences for isotropic pressure-sensitive elasto-plastic materials*, J. Appl. Mech., **57**, 498–506, 1990.

18. E. Bauer, W. Wu, *A hypoplastic model for granular soils under cyclic loading*, [in:] *Proc. of the Int. Workshop on Modern Approaches to Plasticity*, D. Kolymbas [Ed.], Elservier, 225–245, 1992.

19. G. Gudehus, *A comprehensive constitutive equation for granular materials*, Soils and Foundations, **36**, 1, 1–12, 1996.

20. C. Fang, Y. Wang, K. Hutter, *Shearing flows of a dry granular material — hypoplastic constitutive theory and numerical simulations*, Int. J. Numer. Anal. Meth. Geomech., **30**, 1409–1437, 2006.

21. C. Fang, *Modeling dry granular mass flows as elasto-visco-hypoplastic continua with microstructural effects. I. Thermodynamically consistent constitutive model*, Acta Mech. [in press].

22. C. Fang, *Modeling dry granular mass flows as elasto-visco-hypoplastic continua with microstructural effects. II. Numerical simulations of benchmark flow problems*, Acta Mech. [in press].

23. K.C. Chen, Y.C. Tai, *Volume-weighted mixture theory for granular materials*, Contin. Mech. Thermodyn., **19**, 7, 457–474, 2008.

24. M. Pitteri, *Continuum equations of balance in classical statistical mechanics*, Arch. Rat. Mech. Anal., **94**, 291–305, 1986.

25. B. Svendsen, K. Hutter, L. Laloui, *Constitutive models for granular materials including quasi-static frictional behaviour: toward a thermodynamic theory of plasticity*, Continuum Mech. Thermodyn., **11**, 263–275, 1999.

26. N. Kirchner, A. Teufel, *Thermodynamically consistent modelling of abrasive granular materials. II: Thermodynamic equilibrium and applications to steady shear flows*, Proc. R. Soc. Lond., A **458**, 3053–3077, 2002.

27. R.D. Mindlin, *Microstructure in linear elasticity*, Arch. Rat. Mech. Anal., **16**, 51–78, 1964.

28. A.E. Eringen, C.B. Kadafar, *Polar field theories*, [in:] *Continuum Physics IV*, Academic Press, 1976.

29. D. Kolymbas, *Introduction to Hypoplasticity*, Balkeman, Rotterdam, 2000.

30. B. Svendsen, K. Hutter, *On the thermodynamics of a mixture of isotropic materials with constraints*, Int. J. Engng. Sci., **33**, 2021–2054, 1995.

31. H.Y. Ko, R.F. Scott, J. Soil Mechanics and Found. Div., Proc. ASCE, **93**, 137–156, 1967.

32. K. Hutter, *The Foundations of thermodynamics, its basic postulates and implications. A review of modern thermodynamics*, Acta Mech., **27**, 1–54, 1976.

33. I. Müller, *Thermodynamics*, Pitmann, 1986.

34. I. Liu, *Method of Lagrange multipliers for exploitation of the entropy principle*, Arch. Rat. Mech. Anal., **46**, 131–148, 1972.

35. C. Truesdell, W. Noll, *The non-linear theories of mechanics*, [in:] *Handbuch der Physik, III/3*, S. Flügge [Ed.], Springer, Berlin Heidelberg New York 1965.

36. K. Hutter, Y. Wang, *Phenomenological thermodynamics and entropy principle*, [in:] *Entropy*, A. Greven, G. Keller and G. Warnecke [Eds.], 1st ed., Princeton University Press, 57–77, 2003.

37. K. Hutter, L. Laloui, L. Vulliet, *Thermodynamically based mixture models of saturated and unsaturated soils*, Mech. Cohes.-Frict. Mater., **4**, 295–338, 1999.

38. W. Wu, *On high-order hypoplastic models for granular materials*, J. Eng. Math., **56**, 23–34, 2006.

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