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Flow of Herschel–Bulkley fluid through an inclined tube of non-uniform cross-section with multiple stenoses

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The steady flow of Herschel–Bulkley fluid through an inclined tube of non-uniform cross-section with multiple stenoses has been investigated. Assuming the stenoses to be mild, the flow equations have been linearised and the expressions for resistance to the flow and wall shear stress have been derived. The effects of various parameters on these flow variables have been studied. It is found that the flow resistance increases with the heights of the stenoses, yield stress, power law index, but decreases with inclination. Further, the shear stress increases with plug core region radius.

1. Introduction

In the present situation, one of the major health hazards is atherosclerosis, which refers to the narrowing of the arterial lumen i.e. the inner open space or cavity of an artery, due to deposition of fatty substances. This may lead to hypertension, myocardial infarction etc. Hence the formation of stenosis, i.e., the abnormal and unnatural growth disturbs the normal blood flow and there is a considerable evidence that hydrodynamical factors such as wall shear stress, resistance to the flow etc. can play a significant role in the development and progression of this pathological condition. Hence, the detailed knowledge of the flow field in a stenosed tube may help in proper understanding and prevention of arterial diseases.

In view of this, several authors have considered various mathematical models for flows through stenosed/constricted ducts (Young [1], Lee and Fung [2], Shukla *et al.* [3], Chaturani and Samy [4], Radhakrishnamacharya and Srinivasa Rao [5]).

In all these mathematical studies, blood has been characterized as a Newtonian fluid. But Majhi and Nair [6] suggested that blood behaves like a non-Newtonian fluid under certain conditions. It is also realised that the Herschel–Bulkley model is a better model than Casson's model (Blair and Spanner [7]). Further, in small diameter tubes, blood behaves like a Herschel–Bulkley fluid rather than power law and Bingham fluids (Chaturani and Samy [8]). However, all these investigations considered the effect of single stenosis and the tube was taken to be of uniform cross-section. But it is known that many of the blood vessels change their cross-section slowly along their length and may have multiple stenoses at junctions and bends (Schneck *et al.* [9]). Hence, Maruthi Prasad and Radhakrishnamacharya [10] discussed blood flow through an artery having multiple stenoses with non-uniform cross-section, considering blood as a Herschel-Bulkley fluid. It is known that many ducts in physiological systems are not horizontal but have some inclination to the axis. Recently Vajravelu *et al.* [11] studied the peristaltic transport of Herschel–Bulkley fluid through an inclined tube. Hence the study of the effect of inclination of the tube on the flow of non-Newtonian fluid (Herschel–Bulkley model) in the presence of multiple stenoses, may help in better understanding of the role of fluid dynamical factors in the development and progression of arterial diseases.

With this motivation and purpose, a mathematical model for Herschel–Bulkley fluid through an inclined tube with non-uniform cross-section and with two stenoses is considered. Assuming that the stenoses are mild, closed-form solutions have been obtained. Expressions for the resistance to the flow and shear stress at the wall have been derived and the effects of various parameters on these flow variables have been studied.

2. Mathematical formulation

We consider the steady flow of Herschel–Bulkley fluid through a tube of non-uniform cross-section and with two stenoses. Cylindrical polar coordinate system (z, r) is chosen so that the z-axis coincides with the centre line of the channel. It is assumed that the tube is inclined at an angle 'α' to the horizontal direction (see Fig. 1). The stenoses are supposed to be mild and develop in an axially-symmetric manner. The radius of the tube is taken as (Maruthi Prasad and Radhakrishnamachrya [10])

$$h = R(z) \tag{2.1}$$

$$= \begin{cases} R_0 & : 0 \le z \le d_1, \\ R_0 - \dfrac{\delta_1}{2}\left(1 + \operatorname{Cos}\dfrac{2\pi}{L_1}\left(z - d_1 - \dfrac{L_1}{2}\right)\right) & : d_1 \le d_1 + L_1, \\ R_0 & : d_1 + L_1 \le z \le B_1 - \dfrac{L_2}{2}, \\ R_0 - \dfrac{\delta_1}{L_2}\left(1 + \operatorname{Cos}\dfrac{2\pi}{L_2}(z - B_1)\right) & : B_1 - \dfrac{L_2}{2} \le z \le B_1, \\ R^*(z) - \dfrac{\delta_2}{2}\left(1 + \operatorname{Cos}\dfrac{2\pi}{L_2}(z - B_1)\right) & : B_1 \le z \le B_1 + \dfrac{L_2}{2}, \\ R^*(z) & : B_1 + \dfrac{L_2}{2} \le z \le B. \end{cases}$$

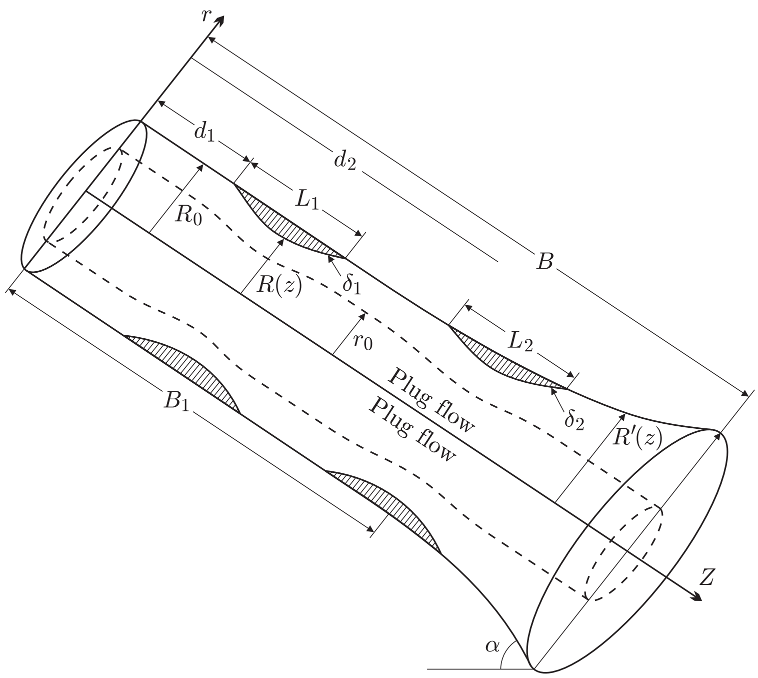

FIG. 1. Geometry of an inclined tube with multiple stenoses.

The following restrictions for mild stenoses [10] are supposed to be satisfied:

$$\delta_i \ll \min\left(R_0, R_{\text{out}}\right),$$

$$\delta_i \ll L_i, \quad \text{where } R_{\text{out}} = R(z) \text{ at } z = B.$$

Here L_i and δ_i $(i = 1, 2)$ are the lengths and maximum heights of two stenoses (the suffixes 1 and 2 refer to the first and second stenosis respectively).

The basic momentum equation governing the flow (VAJRAVELU *et al.* [11]) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) = -\frac{\partial p}{\partial z} + \frac{\operatorname{Sin}\alpha}{F}, \tag{2.2}$$

where τ_{rz} is the shear stress for the Herschel–Bulkley fluid, which is given by

$$\tau_{rz} = \left(-\frac{\partial w}{\partial r}\right)^n + \tau_0, \qquad \tau_{rz} > \tau_0, \tag{2.3}$$

$$\frac{\partial w}{\partial r} = 0, \qquad \tau_{rz} < \tau_0, \tag{2.4}$$

and

$$F = \frac{\mu U^n}{\rho g R_0^{n+1}}.$$

Here w is the axial velocity, p is the pressure, τ_0 is the yield stress, n is the power law index, μ is the fluid viscosity, U is some characteristic velocity, ρ is the density, g is acceleration due to gravity and R_0 is the radius of the tube.

When $\tau_{rz} < \tau_0$ i.e the shear stress is less than the yield stress, there is a core region which flows as a plug (Fig. 1), and Eq. (2.4) corresponds to vanishing velocity gradient in that region. However, the fluid behavior is indicated whenever $\tau_{rz} > \tau_0$.

The boundary conditions are:

$$\tau \text{ is finite at } r = 0, \tag{2.5}$$

$$w = 0 \text{ at } r = h(z). \tag{2.6}$$

3. Solution

Solving Eqs. (2.2) and (2.3) under the boundary conditions (2.5) and (2.6), we obtain the velocity as

$$w = \frac{h^{k+1}(P+f)^{k+1}}{2^{k+1}(k+1)\left(\dfrac{P+f}{2}\right)}\left[\left(1-\frac{2\tau_0}{h(P+f)}\right)^{k+1}-\left(\frac{r}{h}-\frac{2\tau_0}{h(P+f)}\right)^{k+1}\right] \quad \text{for } r_0 \le r \le h, \tag{3.1}$$

where $P = -\partial p/\partial z$, $k = 1/n$ and $f = \mathrm{Sin}\,\alpha/F$.

Using the condition (2.4), we finally get the upper limit of the plug flow region (i.e. the region between $r = 0$ and $r = r_0$ for which $|\tau_{rz}| < \tau_0$) as

$$r_0 = \frac{2\tau_0}{P+f} \tag{3.2}$$

and using the condition $\tau_{rz} = \tau_h$ at $r = h$, we obtain

$$\frac{r_0}{h} = \frac{\tau_0}{\tau_h} = \tau, \qquad 0 < \tau < 1. \tag{3.3}$$

Taking $r = r_0$ in Eq. (3.1), we get the plug core velocity as

$$w_p = \frac{h^{k+1}(P+f)^k}{2^k(k+1)}\left[1-\frac{r_0}{h}\right]^{k+1} \qquad \text{for } 0 \le r \le r_0. \tag{3.4}$$

The volume flow rate is defined by

$$Q = 2\left[\int_0^{r_0} w_p r dr + \int_{r_0}^{h} w r dr\right]. \tag{3.5}$$

Substituting Eq. (3.1) and Eq. (3.4) in Eq. (3.5) and integrating, we finally get

$$(3.6)\quad Q=A\left[\frac{(k+2)(k+3)\left[1-\frac{r_0^2}{2h^2}-2\left(1-\frac{r_0}{h}\right)-\left(1-\frac{r_0}{h}\right)\frac{1}{h^{k+3}}\right]+2\left(1-\frac{r_0}{h}\right)^2}{(k+2)(k+3)}\right],$$

where

$$A=\frac{h^{k+3}(P+f)^k}{2^k(k+1)}\left(1-\frac{r_0}{h}\right)^{k+1}.$$

From Eq. (3.6) we obtain

$$(3.7)\qquad \frac{dp}{dz}=-P$$

$$=B\left[\frac{1}{\left[(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+\frac{1-\tau_0}{h^{k+3}}\right]+2(1-\tau_0)^2\right]^{\frac{1}{k}}}\right]+f,$$

where $B=\dfrac{-2\left[Q(k+1)(k+2)(k+3)\right]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\left(1-\tau_0\right)^{1+\frac{1}{k}}}$. When $k=1$, $\alpha=0$ and $\tau_0\to 0$, Eq. (3.7) reduces to the results of Young [1].

The pressure drop Δp across the stenosis between the cross-sections $z=\pm L/2$ can be obtained by integrating Eq. (3.7) as

$$(3.8)\quad \Delta p=-\int\limits_{\frac{-L}{2}}^{\frac{L}{2}}\left(B\left[\frac{1}{\left\{(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+\frac{1-\tau_0}{h^{k+3}}\right]+2(1-\tau_0)^2\right\}^{\frac{1}{k}}}\right]+f\right)dz.$$

Introducing the following non-dimensional quantities

$$\bar{z}=\frac{z}{L},\qquad \bar{\delta}=\frac{\delta}{R_0},\qquad \bar{R}(z)=\frac{R(z)}{R_0},\qquad \bar{P}=\frac{P}{(\mu UL/R_0^2)},$$

$$\bar{\tau}_0=\frac{\tau_0}{\mu\left(\dfrac{U}{R_0}\right)},\qquad \bar{\tau}_{rz}=\frac{\tau_{rz}}{\mu\left(\dfrac{U}{R_0}\right)},\qquad \bar{Q}=\frac{Q}{\pi R_0^2U},\qquad \bar{F}=\frac{F}{\mu U\lambda}$$

in Eq. (3.8), we finally get (after dropping the bars)

$$\Delta p = -\int_{-1}^{1}\left(B\left[\frac{1}{\left\{(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+\frac{1-\tau_0}{h^{k+3}}\right]+2(1-\tau_0)^2\right\}^{\frac{1}{k}}}\right]+f\right)dz. \tag{3.9}$$

The resistance to the flow, λ, is defined by

$$\lambda = \frac{\Delta p}{Q}. \tag{3.10}$$

Using Eq. (3.9) in Eq. (3.10), we get

$$\lambda = -\frac{1}{Q}\int_{-1}^{1}\left(B\left[\frac{1}{\left\{(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+\frac{1-\tau_0}{h^{k+3}}\right]+2(1-\tau_0)^2\right\}^{\frac{1}{k}}}\right]+f\right)dz. \tag{3.11}$$

The pressure drop in the absence of the stenosis ($h = 1$), denoted by Δp_N, can be obtained from Eq. (3.9) as

$$\Delta p_N = -\int_{-1}^{1}\left(C\left[\frac{1}{\left\{(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+(1-\tau_0)\right]+2(1-\tau_0)^2\right\}^{\frac{1}{k}}}\right]+f\right)dz, \tag{3.12}$$

where $C = \dfrac{-2\left[Q(k+1)(k+2)(k+3)\right]^{\frac{1}{k}}}{(1-\tau_0)^{1+\frac{1}{k}}}$.

The resistance to the flow in absence of the stenosis ($h = 1$) denoted by λ_N, is obtained from Eq. (3.12) as

$$\lambda_N = \frac{\Delta p_N}{Q} \tag{3.13}$$

$$= -\frac{1}{Q}\int_{-1}^{1}\left(C\left[\frac{1}{\left\{(k+2)(k+3)\left[1-\frac{\tau_0^2}{2}-2(1-\tau_0)+(1-\tau_0)\right]+2(1-\tau_0)^2\right\}^{\frac{1}{k}}}\right]+f\right)dz,$$

The normalized resistance to the flow, denoted by $\bar{\lambda}$, is given by

$$\bar{\lambda} = \frac{\lambda}{\lambda_N}. \tag{3.14}$$

Shear stress acting on the surface of the wall can be obtained (VAJRAVELU *et al.* [11]) as

$$\bar{\tau} = \frac{r_0}{h}. \tag{3.15}$$

4. Results

The effects of various parameters on the resistance to the flow ($\bar{\lambda}$) and shear stress ($\bar{\tau}$) acting on the wall, are computed numerically by taking

$$\frac{R^*(z)}{R_0} = \exp\left[\beta B^2 (z - B_1)^2\right]$$

and $d_1 = L_1 = L_2 = 0.2$, $B_1 = 0.8$, $B = 1$, $\beta_1 = 0.01$.

It is observed that the resistance to the flow increases with the heights of both the primary and secondary stenoses (δ_1 and δ_2). However, it may be noted that this increase is significant only when the height of the secondary stenosis exceeds the value 0.04. It is interesting to note that the resistance to the flow increases with yield stress (τ_0) and power law index (n), i.e. the resistance increases with non-Newtonian character of the fluid (Figs. 2–7). But the resistance to the flow decreases with inclination (α) (Figs. 8–10).

The effects of various parameters on shear stress are shown in Figs. 11 and 12. It may be noted that shear stress increases with the heights of the stenoses and the plug core region radius r_0.

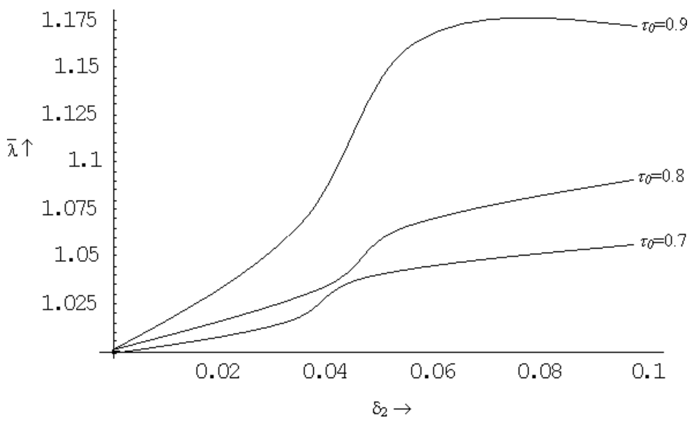

FIG. 2. Effect of δ_2 and τ_0 on $\bar{\lambda}$ ($\delta_1 = 0.0$, $\beta = 0.01$, $k = 8$, $F = 0.1$, $\alpha = 30°$, $B = 1$).

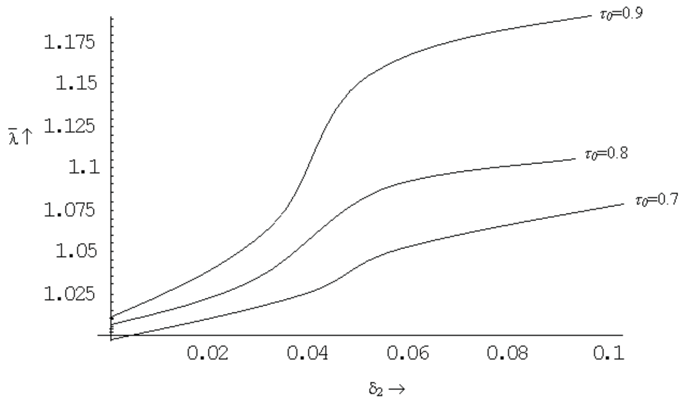

Fig. 3. Effect of δ_2 and τ_0 on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $k = 8$, $F = 0.1$, $\alpha = 30°$, $B = 1$).

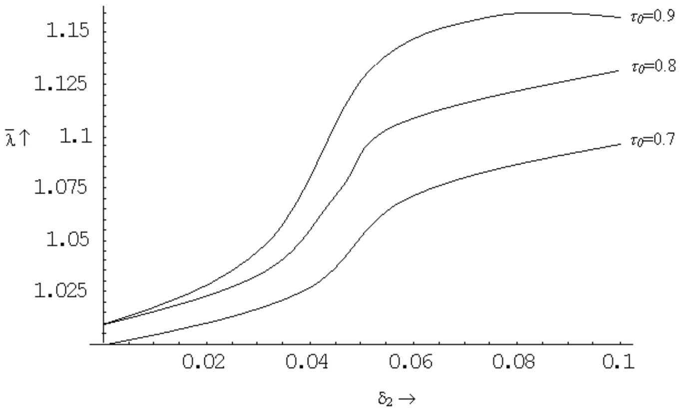

Fig. 4. Effect of δ_2 and τ_0 on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $k = 8$, $F = 0.5$, $\alpha = 30°$, $B = 1$).

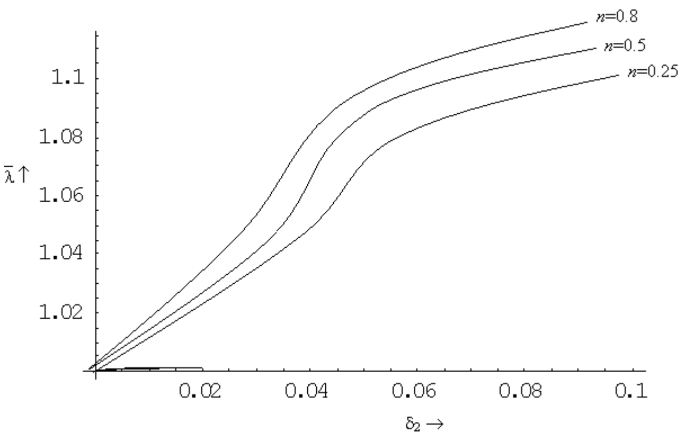

Fig. 5. Effect of δ_2 and n on $\bar{\lambda}$ ($\delta_1 = 0.0$, $\beta = 0.01$, $\tau_0 = 0.8$, $F = 0.1$, $\alpha = 30°$, $B = 1$).

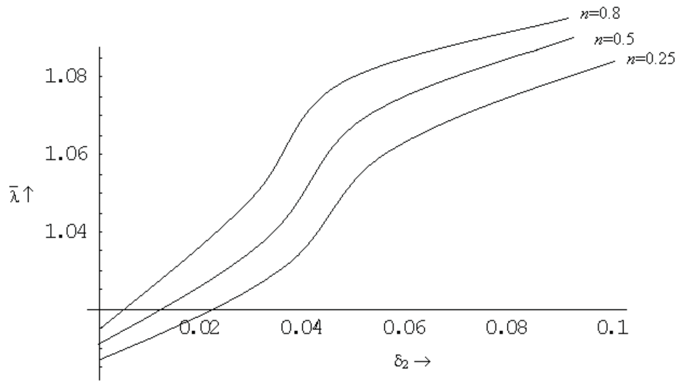

FIG. 6. Effect of δ_2 and n on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $\tau_0 = 0.8$, $F = 0.1$, $\alpha = 30^\circ$, $B = 1$).

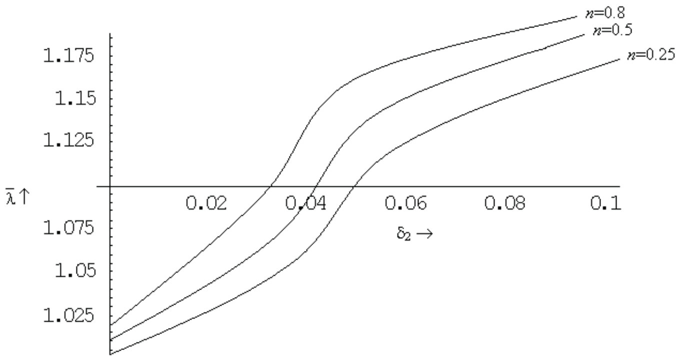

FIG. 7. Effect of δ_2 and n on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $\tau_0 = 0.8$, $F = 0.5$, $\alpha = 30^\circ$, $B = 1$).

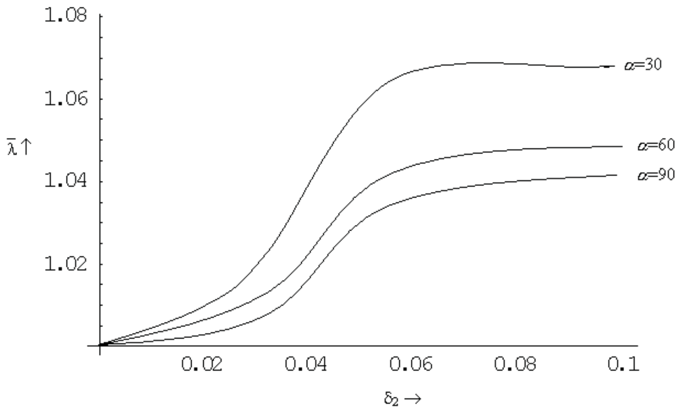

FIG. 8. Effect of δ_2 and α on $\bar{\lambda}$ ($\delta_1 = 0.0$, $\beta = 0.01$, $\tau_0 = 0.1$, $F = 0.1$, $k = 10$, $B = 1$).

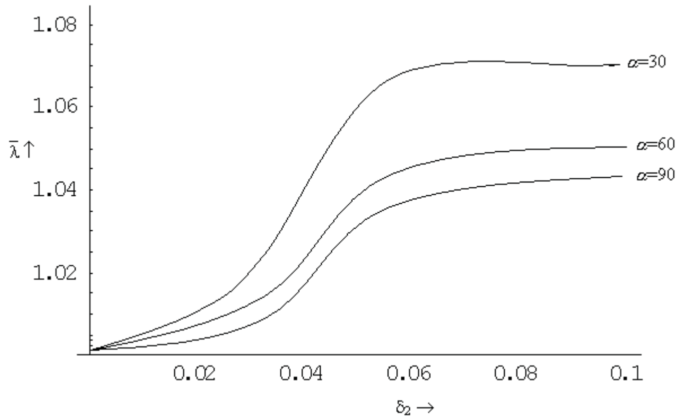

Fig. 9. Effect of δ_2 and α on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $\tau_0 = 0.1$, $F = 0.1$, $k = 10$, $B = 1$).

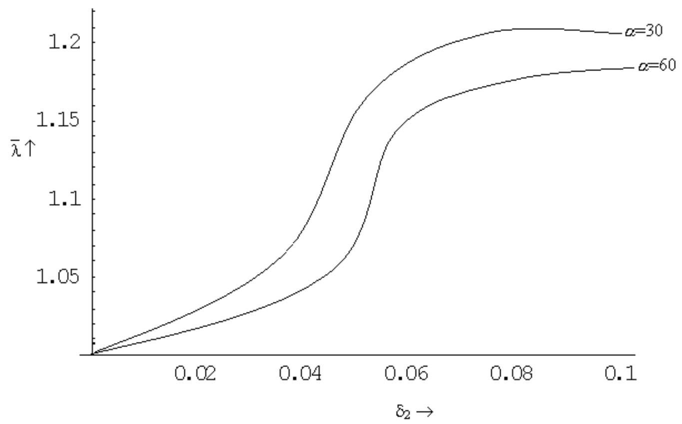

Fig. 10. Effect of δ_2 and α on $\bar{\lambda}$ ($\delta_1 = 0.1$, $\beta = 0.01$, $\tau_0 = 0.1$, $F = 0.5$, $k = 10$, $B = 1$).

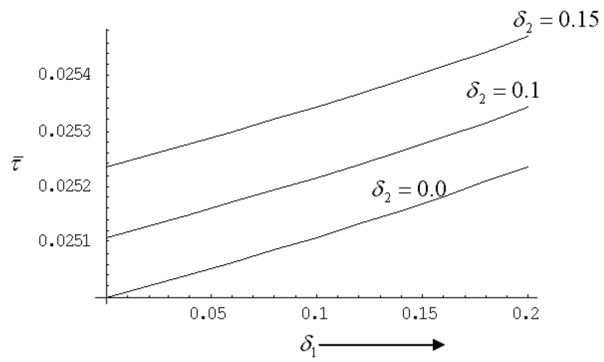

Fig. 11. Effect of δ_1 and δ_2 on $\bar{\tau}$ ($r_0 = 0.01$, $\beta = 0.01$).

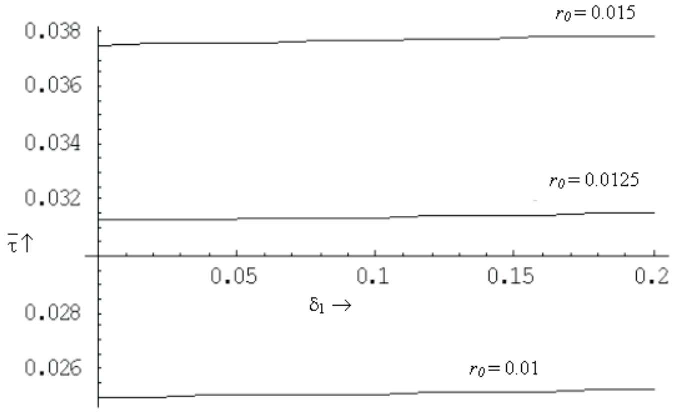

FIG. 12. Effect of δ_1 and r_0 on $\bar{\tau}$ ($\delta_2 = 0.1$, $\beta = 0.01$).

5. Conclusion

A mathematical model for the steady flow of Herschel–Bulkley fluid through an inclined tube of varying cross-section and having two stenoses has been presented. Solutions have been obtained for mild stenosis and it has been shown that the resistance to the flow increases with the heights of the stenoses, yield stress and power law index, but it decreases with inclination.

It is also observed that the shear stress increases with the heights of the stenoses and the plug core-region radius.

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