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Distributed loads in an elastic solid with generalized thermodiffusion

S. CHOUDHARY[1)], S. DESWAL[2)]

[1)] *Department of Mathematics*
Govt. College, Nalwa (Hisar) 125037
Haryana, INDIA
e-mail: sahuksv@rediffmail.com

[2)] *Department of Mathematics*
Guru Jambheshwar University of Science and Technology
Hisar 125001, Haryana INDIA
e-mail: spannu_ gju@yahoo.com

The linear theory of generalized thermoelastic diffusion with one relaxation time is employed to study the interactions in a homogeneous, isotropic elastic solid, when a distributed instantaneous source is acting on the free surface of the body. The eigenvalue approach is adopted for the solution of a two-dimensional problem. The Laplace–Fourier transform technique is used. The expansions of the stresses, displacement components, temperature, concentration and chemical potential are obtained analytically. Numerical results are given and illustrated graphically, employing numerical methods for the inversion for transforms. Comparisons are made with the results predicted by the theory of generalized thermoelasticity and elasticity.

Key words: thermoelastic diffusion, generalized thermoelasticity, eigenvalue approach, distributed load, Laplace and Fourier transforms.

Notations

λ, μ Lamé's constants,
ρ density of the medium,
σ_{ij} components of stress tensor,
e_{ij} components of strain tensor,
u_i components of displacement vector,
C_E specific heat at constant strain,
t time,
T absolute temperature,
T_0 reference temperature chosen so that $\dfrac{|T-T_0|}{T_0} \ll 1$,
$\Theta = T - T_0$,
K thermal conductivity,
e_{kk} dilatation,
δ_{ij} Kronecker delta,

P chemical potential per unit mass,
C non-equilibrium concentration,
C_0 mass concentration at natural state,
$c = C - C_0$,
D thermodiffusion constant,
τ_0 thermal relaxation time,
τ diffusion relaxation time,
a measure of thermodiffusion effect,
b measure of diffusive effects,
$\beta_1 = (3\lambda + 2\mu)\alpha_t$,
$\beta_2 = (3\lambda + 2\mu)\alpha_c$,
α_t coefficient of linear thermal expansion,
α_c coefficient of linear diffusion expansion,
F_0 intensity of the applied mechanical load,
$\mathbf{u}$ displacement vector,
ϕ scalar potential,
ψ vector potential,
$\delta(\cdot)$ Dirac delta function.

1. Introduction

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms. Second, the heat equation is of parabolic type, predicting infinite speeds of propagation for heat waves. Biot [1] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is of mixed hyperbolic-parabolic type. Keeping this shortcoming in view, a generalization to the coupled theory was introduced by Lord and Shulman [2], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. Since the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely the equations of motion and constitutive relations, remain the same as those for the coupled and the uncoupled theories.

This theory was extended by Dhaliwal and Sherief [8] to general anisotropic media in the presence of heat sources. Because of the complicated nature of these equations, few attempts have been made to solve them. Ignaczak [10] studied uniqueness of solutions and Sherief [14] proved uniqueness and stability. Sherief and Hamza [17, 18] solved some two-dimensional problems and studied wave propagation and SHERIEF and ANWAR [13] solved a cylindrically-symmetric problem with a line source of heat. SHERIEF and ANWAR [15] have studied the state space formulation for two-dimensional problem of generalized thermoelasticity with one relaxation time. A detailed study of thermoelastic plane waves was made by [3, 19, 22].

Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibrium is reached. It occurs as a result of second law of thermodynamics which states that the entropy or disorder of any system must always increase with time. Diffusion is important in many life processes. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industry. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced, or the effect of the temperature on this interaction.

Thermodiffusion in the solids is one of a transport process that has great practical importance. NOWACKI [4–7] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. The cross-effects arising from the coupling of fields of temperature, mass diffusion and that of strain in an elastic cylinder, have been discussed by OLESIAK and PYRYEV [28]. SHERIEF *et al.* [23] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. The reflection phenomena of P and SV waves from free surface of an elastic solid with thermodiffusion was considered by SINGH [25]. SHERIEF and SALEH [24] worked on a half-space problem of a thermoelastic half-space with a permeating substance, in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Some problems on distributed loads for an orthotropic micropolar elastic medium have been solved by KUMAR and CHOUDHARY [20, 21]. Recently, AOUADI [26, 27] studied a problem of variable electrical and thermal conductivity in the theory of generalized thermoelastic diffusion, and discussed thermoelastic-diffusion interactions in an infinitely long solid cylinder subjected to a thermal shock on its surface, which is in contact with a permeating substance. The present study is motivated by the importance of thermoelastic diffusion process in the field of oil extraction.

2. Basic equations and problem formulation

Following Sherief *et al.* [23], the governing equations for an isotropic, homogeneous elastic solid with generalized thermodiffusion at uniform temperature T_0 in the undisturbed state, in absence of the body forces and heat loads are:

(i) the equation of motion

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \beta_1 \Theta_{,i} - \beta_2 c_{,i}\,, \tag{2.1}$$

(ii) the generalized energy equation

$$K\Theta_{,ii} = \rho C_E(\dot{\Theta} + \tau_0 \ddot{\Theta}) + \beta_1 T_0(\dot{e_{kk}} + \tau_0 \ddot{e_{kk}}) + a T_0(\dot{c} + \tau_0 \ddot{c}), \tag{2.2}$$

(iii) the generalized diffusion equation

$$D\beta_2 e_{kk,ii} + Da\Theta_{,ii} + \dot{c} + \tau \ddot{c} - Dbc_{,ii} = 0, \tag{2.3}$$

(iv) the constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}(\lambda e_{kk} - \beta_1 \Theta - \beta_2 c), \tag{2.4}$$

$$P = -\beta_2 e_{kk} + bc - a\Theta, \tag{2.5}$$

where τ_0, the thermal relaxation time, ensures that heat conduction equation satisfied by temperature Θ predicts finite speed of heat propagation and τ, the diffusion relaxation time, ensures that the equation satisfied by the concentration c also predicts finite speed of propagation of matter from one medium to the other. The superposed dot denotes the derivative with respect to time.

We use a fixed Cartesian coordinate system (x, y, z) with origin on the surface $z = 0$, which is stress-free and with z-axis directed vertically into the medium. The region $z > 0$ is occupied by the elastic solid with generalized thermodiffusion. A distributed load in (normal or tangential) direction of magnitude F_0 is assumed to be acting on the surface $z = 0$ of the medium.

We restrict our analysis to a plane parallel to xz-plane. The boundary of the medium is assumed to be thermally insulated. The chemical potential is also assumed to be a known function of time.

We shall use the following non-dimensional variables:

$$\begin{aligned}
&x^* = \frac{\omega}{c_1}x, && z^* = \frac{\omega}{c_1}z, && t^* = \omega t,\\
&u_x^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_x, && u_z^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_z, && \sigma_{ij}^* = \frac{\sigma_{ij}}{\beta_1 T_0},\\
&c^* = \frac{c}{C_0}, && P^* = \frac{P}{\beta_2}, && \Theta^* = \frac{\Theta}{T_0},\\
&\tau^* = \omega\tau, && \tau_0^* = \omega\tau_0,
\end{aligned} \tag{2.6}$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad \omega = \frac{\rho C_E c_1^2}{K}. \tag{2.7}$$

Using the quantities given by (2.6) in Eqs. (2.1)–(2.3), we obtain the equations in dimensionless form (dropping the asterisks for convenience) as

$$\frac{\partial^2 u_x}{\partial x^2} + a_1 \frac{\partial^2 u_z}{\partial x \partial z} + a_2 \frac{\partial^2 u_x}{\partial z^2} - \frac{\partial \Theta}{\partial x} - a_3 \frac{\partial c}{\partial x} - \frac{\partial^2 u_x}{\partial t^2} = 0, \tag{2.8}$$

$$a_2 \frac{\partial^2 u_z}{\partial x^2} + a_1 \frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial \Theta}{\partial z} - a_3 \frac{\partial c}{\partial z} - \frac{\partial^2 u_z}{\partial t^2} = 0, \tag{2.9}$$

$$\tau_m \frac{\partial \Theta}{\partial t} + b_1 \tau_m \frac{\partial}{\partial t} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) + b_2 \tau_m \frac{\partial c}{\partial t} - b_3 \nabla^2 \Theta = 0, \tag{2.10}$$

$$\frac{\partial}{\partial x} \left(\nabla^2 u_x \right) + \frac{\partial}{\partial z} \left(\nabla^2 u_z \right) + b_4 \nabla^2 \Theta + b_5 \tau_n \frac{\partial c}{\partial t} - b_6 \nabla^2 c = 0, \tag{2.11}$$

where

$$\begin{aligned}
a_1 &= \frac{\lambda + \mu}{\lambda + 2\mu}, & a_2 &= \frac{\mu}{\lambda + 2\mu}, & a_3 &= \frac{\beta_2 C_0}{\beta_1 T_0}, \\
b_1 &= \frac{\beta_1^2 T_0}{\rho^2 c_1^2 C_E}, & b_2 &= \frac{a C_0}{C_E \rho}, & b_3 &= \frac{K\omega}{\rho c_1^2 C_E}, \\
b_4 &= \frac{a \rho c_1^2}{\beta_1 \beta_2}, & b_5 &= \frac{\rho C_0 c_1^4}{D \beta_1 \beta_2 T_0 \omega}, & b_6 &= \frac{b \rho C_0 c_1^2}{\beta_1 \beta_2 T_0}, \\
\tau_m &= \left(1 + \tau_0 \frac{\partial}{\partial t}\right), & \tau_n &= \left(1 + \tau \frac{\partial}{\partial t}\right), & \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned} \tag{2.12}$$

With the aid of the expressions relating displacement components u_x, u_z to the scalar potential ϕ and vector potential ψ in dimensionless form given by

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \qquad u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \tag{2.13}$$

in the Eqs. (2.8)–(2.11), we obtain

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\phi - \Theta - a_3 c = 0, \tag{2.14}$$

$$\nabla^2\psi - \frac{1}{a_2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{2.15}$$

$$\left[\nabla^2 - \frac{1}{b_3}\tau_m\frac{\partial}{\partial t}\right]\Theta - \frac{b_1}{b_3}\tau_m\frac{\partial}{\partial t}\nabla^2\phi - \frac{b_2}{b_3}\tau_m\frac{\partial c}{\partial t} = 0, \tag{2.16}$$

$$\nabla^2\phi + b_4\nabla^2\Theta + \left[b_5\tau_n\frac{\partial}{\partial t} - b_6\nabla^2\right]c = 0. \tag{2.17}$$

3. Solution of the problem

3.1. Formulation of a vector-matrix differential equation in transform domain

We now apply the Laplace and Fourier transforms defined by

$$\hat{f}(x,z,p) = \int_0^\infty f(x,z,t), e^{-pt}dt, \tag{3.1}$$

$$\tilde{f}(\xi,z,p) = \int_{-\infty}^\infty \hat{f}(x,z,p)e^{\iota\xi x}dx, \tag{3.2}$$

where p and ξ are the Laplace and Fourier transform parameters respectively, so that under the homogeneous initial conditions the Eqs. (2.14)–(2.17) reduce to the form

$$\frac{d^2\tilde{\phi}}{dz^2} = R_{11}\tilde{\phi} + R_{12}\tilde{\Theta} + R_{13}\tilde{c}, \tag{3.3}$$

$$\frac{d^2\tilde{\Theta}}{dz^2} = R_{21}\tilde{\phi} + R_{22}\tilde{\Theta} + R_{23}\tilde{c}, \tag{3.4}$$

$$\frac{d^2\tilde{c}}{dz^2} = R_{31}\tilde{\phi} + R_{32}\tilde{\Theta} + R_{33}\tilde{c}, \tag{3.5}$$

$$\left[\frac{d^2}{dz^2} - \left(\xi^2 + \frac{p^2}{a_2}\right)\right]\tilde{\psi} = 0, \tag{3.6}$$

where

$$
\begin{aligned}
&R_{11} = (p^2+\xi^2), \qquad R_{12} = 1, \qquad R_{13} = a_3,\\
&R_{21} = f_1, \qquad R_{22} = f_2, \qquad R_{23} = f_3,\\
&R_{31} = \frac{g_1}{b_6-a_3}, \qquad R_{32} = \frac{g_2}{b_6-a_3}, \qquad R_{33} = \frac{g_3}{b_6-a_3},\\
&g_1 = p^4 + f_1(1+b_4),\\
&g_2 = p^2 + (f_2-\xi^2)(1+b_4),\\
&g_3 = a_3(p^2-\xi^2) + f_3(1+b_4) + b_5\tau_n^* p + b_6\xi^2,\\
&f_1 = \frac{b_1}{b_3}\tau_m^* p^3, \qquad f_2 = \frac{(1+b_1)}{b_3}\tau_m^* p + \xi^2, \qquad f_3 = \frac{(b_1a_3+b_2)}{b_3}\tau_m^* p,\\
&\tau_m^* = 1+\tau_0 p, \qquad \tau_n^* = 1+\tau p.
\end{aligned} \tag{3.7}
$$

The system of Eqs. (3.3)–(3.5) can be written in the form of a vector-matrix differential equation as follows:

$$
\frac{d}{dz}V(\xi,z,p) = A(\xi,p)V(\xi,z,p), \tag{3.8}
$$

where

$$
\begin{aligned}
&V = \begin{bmatrix} U \\ D^*U \end{bmatrix}, \quad A = \begin{bmatrix} O & I \\ A_1 & O \end{bmatrix}, \quad U = \begin{bmatrix} \tilde{\phi} \\ \tilde{\Theta} \\ \tilde{c} \end{bmatrix}, \quad O = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix},\\
&I = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} R_{11}&R_{12}&R_{13}\\R_{21}&R_{22}&R_{23}\\R_{31}&R_{32}&R_{33} \end{bmatrix},
\end{aligned} \tag{3.9}
$$

where D^* denotes the differentiation with respect to z i.e. d/dz.

3.2. Solution of the vector-matrix differential equation

We now proceed to solve Eq. (3.8) by the eigenvalue approach. To solve Eq. (3.8), we take

$$
V(\xi,z,p) = X(\xi,p)e^{qz}, \tag{3.10}
$$

so that

$$
A(\xi,p)V(\xi,z,p) = qV(\xi,z,p), \tag{3.11}
$$

which leads to an eigenvalue problem. The characteristic equation corresponding to the matrix A is given by

$$\det[A - qI] = 0, \tag{3.12}$$

which on expansion provides us with

$$q^6 - \lambda_1 q^4 + \lambda_2 q^2 - \lambda_3 = 0, \tag{3.13}$$

where

$$\begin{aligned}
\lambda_1 &= R_{11} + R_{22} + R_{33},\\
\lambda_2 &= R_{11}R_{22} + R_{22}R_{33} + R_{33}R_{11} - R_{12}R_{21} - R_{23}R_{32} - R_{31}R_{13},\\
\lambda_3 &= R_{11}(R_{22}R_{33} - R_{23}R_{32}) + R_{12}(R_{23}R_{31} - R_{21}R_{33})\\
&\qquad + R_{13}(R_{21}R_{32} - R_{22}R_{31}).
\end{aligned} \tag{3.14}$$

The roots of Eq. (3.13), which are the eigenvalues of the matrix A, are $\pm q_i$, $i = 1, 2, 3$. We assume that real parts of q_i are positive. The eigenvector $X(\xi, p)$ corresponding to the eigenvalues q_i can be determined by solving the homogeneous equation

$$[A - qI]X(\xi, p) = 0. \tag{3.15}$$

The set of eigenvectors $X_i(\xi, p)$, $(i = 1, 2, 3, 5, 6, 7)$ may be obtained as

$$X_i(\xi, p) = \begin{bmatrix} X_{i1}(\xi, p)\\ X_{i2}(\xi, p)\end{bmatrix}, \tag{3.16}$$

where

$$\begin{aligned}
&X_{i1}(\xi, p) = \begin{bmatrix} s_i\\ r_i\\ 1\end{bmatrix}, \qquad X_{i2}(\xi, p) = \begin{bmatrix} s_i q_i\\ r_i q_i\\ q_i\end{bmatrix},\\
&\qquad q = q_i; \qquad i = 1, 2, 3\\
&X_{j1}(\xi, p) = \begin{bmatrix} s_i\\ r_i\\ 1\end{bmatrix}, \qquad X_{j2}(\xi, p) = \begin{bmatrix} -s_i q_i\\ -r_i q_i\\ -q_i\end{bmatrix},\\
&\qquad j = i + 4, \qquad q = -q_i; \qquad i = 1, 2, 3,\\
&\qquad s_i = \frac{s_{i1} - s_{i2} - R_{23}s_{i3}}{R_{21}s_{i3}},\\
&\qquad r_i = \frac{R_{31}R_{13} - (R_{33} - q_i^2)(R_{11} - q_i^2)}{R_{32}(R_{11} - q_i^2) - R_{12}R_{31}},\\
&\qquad s_{i1} = (R_{11} - q_i^2)(R_{22} - q_i^2)(R_{33} - q_i^2),\\
&\qquad s_{i2} = R_{31}R_{13}(R_{22} - q_i^2),\\
&\qquad s_{i3} = (R_{32}(R_{11} - q_i^2) - R_{12}R_{31}); \qquad i = 1, 2, 3.
\end{aligned} \tag{3.17}$$

The solution of Eq. (3.8) is given by

$$V(\xi, z, p) = \sum_{i=1}^{3} [B_i X_i(\xi, p) e^{q_i z} + B_{i+4} X_{i+4}(\xi, p) e^{-q_i z}] \tag{3.18}$$

and solution of Eq. (3.6) is

$$\tilde{\psi} = B_4 e^{q_4 z} + B_8 e^{-q_4 z}, \tag{3.19}$$

where B_i $(i = 1, 2, 3, 4, 5, 6, 7, 8)$ are arbitrary constants and

$$q_4 = \sqrt{\xi^2 + \frac{p^2}{a_2}}. \tag{3.20}$$

The Eqs. (3.18) and (3.19) represent a solution of the general problem in the case of generalized thermodiffusion elasticity by employing the eigenvalue approach and therefore, they can be applied to a broad class of problems in the domain of Laplace and Fourier transforms.

4. Application: interactions due to distributed load

In this section, the general solution for displacement, stresses, temperature field, concentration and chemical potential presented in Eqs. (3.18) and (3.19), will be used to yield the response of a half-space subjected to a load distributed over a strip of width $2l$, at the free surface of the medium. The constants B_i will be determined by imposing the proper boundary conditions. These constants, when substituted in Eqs. (3.18) and (3.19), enable us to obtain the required physical quantities in the Fourier and Laplace transformed (ξ, z, p) domain. The final solution in the original domain (x, z, t) is obtained by a numerical inversion of both transforms.

Case 1. *Load in normal direction.* In the half-space, the load $F(x)$ is acting in normal direction. The surface $z = 0$ is assumed to be thermally insulated so that there is no variation of temperature and concentration on it. Therefore, for this loading case the boundary conditions are

$$\sigma_{zz} = -F(x)\delta(t), \qquad \sigma_{zx} = 0, \qquad \frac{\partial \Theta}{\partial z} = 0, \qquad \frac{\partial c}{\partial z} = 0, \text{ at } z = 0, \tag{4.1}$$

where $F(x) = F_0(H(x + l) - H(x - l))$.

Case 2. *Load in tangential direction.* In the half-space, the load $F(x)$ is acting in the tangential direction. The boundary conditions in this case are

$$\sigma_{zz}=0,\qquad \sigma_{zx}=-F(x)\delta(t),\qquad \frac{\partial\Theta}{\partial z}=0,\qquad \frac{\partial c}{\partial z}=0,\ \text{at } z=0. \tag{4.2}$$

It can be seen that eight unknowns are to be determined in Eqs. (3.18) and (3.19) and only four boundary conditions appear in each case. For the half-space the radiation conditions imply outgoing waves with decreasing amplitudes in the positive z-direction. Therefore, the radiation conditions require that $B_1=B_2=B_3=B_4=0$.

We obtain the expressions for the displacement components, stresses, temperature field, concentration and potential as

$$\tilde{\sigma_{zz}}=e_1B_5e^{-q_1z}+e_2B_6e^{-q_2z}+e_3B_7e^{-q_3z}+e_4B_8e^{-q_4z}, \tag{4.3}$$

$$\tilde{\sigma_{zx}}=k_1B_5e^{-q_1z}+k_2B_6e^{-q_2z}+k_3B_7e^{-q_3z}-k_4B_8e^{-q_4z}, \tag{4.4}$$

$$\tilde{u_x}=-\iota\xi[s_1B_5e^{-q_1z}+s_2B_6e^{-q_2z}+s_3B_7e^{-q_3z}]+q_4B_8e^{-q_4z}, \tag{4.5}$$

$$\tilde{u_z}=-[s_1q_1B_5e^{-q_1z}+s_2q_2B_6e^{-q_2z}+s_3q_3B_7e^{-q_3z}]-\iota\xi B_8e^{-q_4z}, \tag{4.6}$$

$$\tilde{\Theta}=r_1B_5e^{-q_1z}+r_2B_6e^{-q_2z}+r_3B_7e^{-q_3z}, \tag{4.7}$$

$$\tilde{c}=B_5e^{-q_1z}+B_6e^{-q_2z}+B_7e^{-q_3z}, \tag{4.8}$$

$$\tilde{P}=M_1B_5e^{-q_1z}+M_2B_6e^{-q_2z}+M_3B_7e^{-q_3z}, \tag{4.9}$$

where

$$\begin{aligned}
B_{i+4}&=\frac{2\sin(\xi l)\Delta_i}{\xi\Delta};\qquad i=1,2,3,4,\\
M_i&=-e^*(q_i^2-\xi^2)s_i+\frac{bC_0}{\beta_2}-\frac{aT_0r_i}{\beta_2};\qquad i=1,2,3,\\
\Delta&=(r_3-r_2)q_2q_3(e_1k_4+e_4k_1)+(r_1-r_3)q_1q_3(e_2k_4+e_4k_2)\\
&\qquad+(r_2-r_1)q_1q_2(e_3k_4+e_4k_3),\\
e_i&=q_i^2s_i-a^*s_i-r_i-b^*;\qquad i=1,2,3,\qquad e_4=\iota\xi q_4\left(1-\frac{\lambda}{\rho c_1^2}\right),\\
k_i&=\frac{2\iota\xi s_iq_i\mu}{\rho c_1^2};\qquad i=1,2,3,\qquad k_4=\frac{\mu}{\rho c_1^2}(\xi^2+q_4^2),\\
e^*&=\frac{\beta_1T_0}{\rho c_1^2},\qquad a^*=\frac{\lambda\xi^2}{\rho c_1^2},\qquad b^*=\frac{\beta_2C_0}{\beta_1T_0}.
\end{aligned} \tag{4.10}$$

Case 1. *In normal direction.* The values of Δ_i; $i=1,2,3,4$, when the distributed load is acting in normal direction, are

$$
(4.11)\quad
\begin{aligned}
\Delta_1 &= F_0 k_4 (r_2 - r_3) q_2 q_3,\\
\Delta_2 &= F_0 k_4 (r_3 - r_1) q_3 q_1,\\
\Delta_3 &= F_0 k_4 (r_1 - r_2) q_1 q_2,\\
\Delta_4 &= F_0 \left[k_1 (r_2 - r_3) q_2 q_3 + k_2 (r_3 - r_1) q_1 q_3 + k_3 (r_1 - r_2) q_1 q_2\right].
\end{aligned}
$$

Case 2. *In tangential direction.* The solution for this case as in Eqs. (4.3)–(4.9), only with the replacement of Δ_i; $i = 1, 2, 3, 4$ as:

$$
(4.12)\quad
\begin{aligned}
\Delta_1 &= F_0 e_4 (r_2 - r_3) q_2 q_3,\\
\Delta_2 &= F_0 e_4 (r_3 - r_1) q_3 q_1,\\
\Delta_3 &= F_0 e_4 (r_1 - r_2) q_1 q_2,\\
\Delta_4 &= -F_0 [e_1 (r_2 - r_3) q_2 q_3 + e_2 (r_3 - r_1) q_1 q_3 + e_3 (r_1 - r_2) q_1 q_2].
\end{aligned}
$$

Particular Case I. By taking $c = D = a = b = \beta_2 = 0$, we obtain the expressions for displacement components, stresses and temperature field in the generalized thermoelastic medium as:

$$(4.13)\qquad \tilde{\sigma_{zz}} = e_1^* B_4^* e^{-q_1^* z} + e_2^* B_5^* e^{-q_2^* z} + e_3^* B_6^* e^{-q_3^* z},$$

$$(4.14)\qquad \tilde{\sigma_{zx}} = k_1^* B_4^* e^{-q_1^* z} + k_2^* B_5^* e^{-q_2^* z} - k_3^* B_6^* e^{-q_3^* z},$$

$$(4.15)\qquad \tilde{u_x} = -\iota\xi [s_1^* B_4^* e^{-q_1^* z} + s_2^* B_5^* e^{-q_2^* z}] + q_3^* B_6^* e^{-q_3^* z},$$

$$(4.16)\qquad \tilde{u_z} = -[s_1^* q_1^* B_4^* e^{-q_1^* z} + s_2^* q_2^* B_5^* e^{-q_2^* z}] - \iota\xi B_6^* e^{-q_3^* z},$$

$$(4.17)\qquad \tilde{\Theta} = B_4^* e^{-q_1^* z} + B_5^* e^{-q_2^* z},$$

where

$$(4.18)\qquad q_i^{*2} = \frac{\lambda_1^* + (-1)^{i+1}\sqrt{\lambda_1^{*2} - 4\lambda_2^*}}{2}; \quad i = 1, 2,$$

are the roots of the equation

$$(4.19)\qquad q^4 - \lambda_1^* q^2 + \lambda_2^* = 0,$$

where

$$
\begin{aligned}
\lambda_1^* &= R_{11} + R_{22}, \qquad \lambda_2^* = R_{11}R_{22} - R_{21}R_{12},\\
q_3^* &= q_4^2, \qquad B_{i+3}^* = 2\sin(\xi l)\Delta_i^*/\xi\Delta^*; \qquad i = 1,2,3,\\
\Delta^* &= q_1^*(e_2^*k_3^* + e_3^*k_2^*) - q_2^*(e_3^*k_1^* + e_1^*k_3^*),\\
e_i^* &= q_i^{*2}s_i^* - a^*s_i^* - 1; \qquad i = 1,2, \qquad e_3^* = \iota\xi q_3^*\left(1 - \frac{\lambda}{\rho c_1^2}\right),\\
k_i^* &= \frac{2\mu}{\rho c_1^2}\left(\iota\xi q_i^* s_i^*\right); \qquad i = 1,2, \qquad k_3^* = \frac{\mu}{\rho c_1^2}\left(q_3^{*2} + \xi^2\right),\\
s_i^* &= -\frac{R_{22} - q_i^{*2}}{R_{21}}; \qquad i = 1,2.
\end{aligned}
\tag{4.20}
$$

Case 1. *In normal direction.* The values of $\Delta_i^*; i = 1,2,3$, when the distributed load is acting in normal direction, are

$$
\Delta_1^* = F_0 q_2^* k_3^*, \qquad \Delta_2^* = -F_0 q_1^* k_3^*, \qquad \Delta_3^* = F_0\left[q_2^* k_1^* - q_1^* k_2^*\right].
\tag{4.21}
$$

Case 2. *In tangential direction.* The solution for this case as in Eqs. (4.13)–(4.17), only with the replacement of Δ_i^*; $i = 1,2,3$, as:

$$
\Delta_1^* = F_0 q_2^* e_3^*, \qquad \Delta_2^* = -F_0 q_1^* e_3^*, \qquad \Delta_3^* = F_0\left[q_1^* e_2^* - q_2^* e_1^*\right].
\tag{4.22}
$$

Particular Case II. If we neglect the thermodiffusion effect from the medium considered, the corresponding expressions for displacement components and stresses are given by:

$$
\tilde{\sigma_{zz}} = e_1' B_3' e^{-q_1' z} + e_2' B_4' e^{-q_2' z},
\tag{4.23}
$$

$$
\tilde{\sigma_{zx}} = k_1' B_3' e^{-q_1' z} - k_2' B_4' e^{-q_2' z},
\tag{4.24}
$$

$$
\tilde{u_x} = -(\iota\xi) B_3' e^{-q_1' z} + q_2' B_4' e^{-q_2' z},
\tag{4.25}
$$

$$
\tilde{u_z} = -q_1' B_3' e^{-q_1' z} + \iota\xi B_4' e^{-q_2' z},
\tag{4.26}
$$

where

$$
\begin{aligned}
q_1' &= \sqrt{p^2 + \xi^2}, \qquad q_2' = \sqrt{\frac{p^2}{a_2} + \xi^2},\\
B_{i+2}' &= 2\sin(\xi l)\Delta_i'/\xi\Delta'; \qquad i = 1,2, \qquad \Delta' = -(e_1'k_2' + e_2'k_1'),\\
e_1' &= q_1'^2 - a^*, \qquad e_2' = \iota\xi q_2'\left(1 - \frac{\lambda}{\rho c_1^2}\right),\\
k_1' &= \frac{2\mu}{\rho c_1^2}(\iota\xi q_1'), \qquad k_2' = \frac{\mu}{\rho c_1^2}(\xi^2 + q_2'^2).
\end{aligned}
\tag{4.27}
$$

Case 1. *In normal direction.* The values of $\varDelta_i'$; $i = 1, 2$, when the load is distributed in normal direction, are

$$\varDelta_1' = F_0 k_2', \qquad \varDelta_2' = F_0 k_1'. \tag{4.28}$$

Case 2. *In tangential direction.* The solution for this case are as in Eqs. (4.23)–(4.26), only with the replacement of $\varDelta_i'$; $i = 1, 2$ as:

$$\varDelta_1' = F_0 e_2', \qquad \varDelta_2' = -F_0 e_1'. \tag{4.29}$$

5. Inversion of transforms

The transformed displacements, stresses, temperature field, concentration and chemical potential (4.3)–(4.9), (4.13)–(4.17) and (4.23)–(4.26) are functions of z, the parameters of Laplace and Fourier transforms p and ξ, respectively, and hence are of the form $\tilde{f}(\xi, z, p)$. To get the function $f(x, z, t)$ in the physical domain, first we invert the Fourier transform using

$$\begin{aligned} \hat{f}(x, z, p) &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-\iota\xi x} \tilde{f}(\xi, z, p) d\xi, \\ &= \frac{1}{\pi} \int\limits_{0}^{\infty} \{\cos(\xi x) \tilde{f}_e - \iota \sin(\xi x) \tilde{f}_o\} d\xi, \end{aligned} \tag{5.1}$$

where $\tilde{f}_e$ and $\tilde{f}_o$ are even and odd parts of the function $\tilde{f}(\xi, z, p)$ respectively. Thus, expressions (5.1) gives us the Laplace transform $\hat{f}(x, z, p)$ of function $f(x, z, t)$.

Now, for the fixed values of ξ, x and y, the $\overline{f}(x, y, p)$ in the expression (5.1) can be considered as the Laplace transform $\overline{g}(p)$ of some function $g(t)$. Following Honig and Hirdes [11], the Laplace transformed function $\overline{g}(p)$ can be inverted as given below.

The function $g(t)$ can be obtained by using

$$g(t) = \frac{1}{2\pi\iota} \int\limits_{v-\iota\infty}^{v+\iota\infty} e^{pt} \overline{g}(p) dp, \tag{5.2}$$

where v is an arbitrary real number greater than all the real parts of the singularities of $\overline{g}(p)$. Taking $p = v + \iota y$, we get

$$g(t) = \frac{e^{vt}}{2\pi} \int\limits_{-\infty}^{\infty} e^{\iota t y} \overline{g}(v + \iota y) dy. \tag{5.3}$$

Now, taking $e^{-vt}g(t)$ as $h(t)$ and expanding it as Fourier series in $[0, 2L]$, we obtain approximately the formula

$$g(t) = g_{\infty}(t) + E_D, \tag{5.4}$$

where

$$\begin{aligned} g_{\infty}(t) &= \frac{v_o}{2} + \sum_{k=1}^{\infty} v_k, \qquad 0 \le t \le 2L, \\ v_k &= \frac{e^{vt}}{L} \Re\Big[e^{\frac{\iota k\pi t}{L}} \overline{g}\Big(v + \frac{\iota k\pi}{L} \Big)\Big], \end{aligned} \tag{5.5}$$

E_D is the discretization error and can be made arbitrarily small by choosing v large enough. The value of v and L are chosen according to the criteria outlined by Honig and Hirdes [11].

Since the infinite series in Eqs. (5.5) can be summed up only to a finite number of N terms, so the approximate value of $g(t)$ becomes

$$g_N(t) = \frac{v_o}{2} + \sum_{k=1}^{N} v_k, \qquad 0 \le t \le 2L. \tag{5.6}$$

Now, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error in evaluating $g(t)$ using the above formula. Two methods are used to reduce the total error. The discretization error is reduced by using the 'Korrektur'-method, Honig and Hirdes [11] and then 'ϵ-algorithm' is used to reduce the truncation error and hence to accelerate the convergence.

The 'Korrektur'-method formula, to evaluate the function $g(t)$, is

$$g(t) = g_{\infty}(t) - e^{-2vL} g_{\infty}(2L + t) + E_{D'}, \tag{5.7}$$

where

$$|E_{D'}| \ll |E_D|. \tag{5.8}$$

Thus, the approximate value of $g(t)$ becomes

$$g_{N_k}(t) = g_N(t) - e^{-2vL} g_{N'}(2L + t), \tag{5.9}$$

where N' is an integer such that $N' < N$.

We shall now describe the ϵ-algorithm which is used to accelerate the convergence of the series in Eq. (5.6). Let N be a natural number and $S_m = \sum_{k=1}^{m} v_k$

be the sequence of partial sums of Eq. (5.6). We define the ϵ-sequence by

$$\begin{aligned}\epsilon_{0,m} &= 0, \qquad \epsilon_{1,m} = S_m,\\ \epsilon_{n+1,m} &= \epsilon_{n-1,m+1} + \frac{1}{\epsilon_{n,m+1} - \epsilon_{n,m}}; \qquad n, m = 1, 2, 3, \ldots\end{aligned}$$

It can be shown, HONIG and HIRDES [11], that the sequence $\epsilon_{1,1}, \epsilon_{3,1}, \ldots, \epsilon_{N,1}$ converges to $g(t) + E_D - v_o/2$ faster than the sequence of partial S_m, $m = 1, 2, 3, \ldots$. The actual procedure to invert the Laplace transform reduces to the study of Eq. (5.9) together with the ϵ-algorithm.

The last step is to evaluate the integral in Eq. (5.1). The method for evaluating this integral by PRESS *et al.* [12] involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinement of the extended trapezoidal rule, followed by extrapolation of the results to the limit when the step size tends to zero.

6. Numerical results and discussion

With an aim to illustrate the problem, we will present some numerical results. The material chosen for the purpose of numerical computation is copper, the physical data for which are given by THOMAS [9] in SI units:

$$\begin{aligned}T_0 &= 293 \text{ K}, \qquad \rho = 8954 \text{ kg/m}^3, \qquad \tau_0 = 0.02 \text{ s}, \qquad \tau = 0.2 \text{ s},\\ C_E &= 383.1 \text{ J/(kg K)}, \qquad \alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, \qquad K = 386 \text{ W/(m K)},\\ \lambda &= 7.76 \times 10^{10} \text{ kg/(m s}^2), \qquad \mu = 3.86 \times 10^{10} \text{ kg/(m s}^2),\\ \alpha_c &= 1.98 \times 10^{-4} \text{ m}^3\text{/kg}, \qquad D = 0.85 \times 10^{-8} \text{ kg s/m}^3,\\ a &= 1.2 \times 10^4 \text{ m}^2/(\text{s}^2 \text{ K}), \qquad b = 0.9 \times 10^6 \text{ m}^5/(\text{kg s}^2).\end{aligned}$$

The computations are performed at $z = 1.0$ in the range $0 \leq x \leq 10$ for the value of non-dimensional length $l(= l/h) = 1.0$, where h is a parameter of dimension of length and initial concentration $C_0 = 1$. The numerical values of dimensionless normal displacement $u_z(= u_z/F_0)$, normal stress $\sigma_{zz}(= \sigma_{zz}/F_0)$, temperature $\Theta(= \Theta/F_0)$ and deviation of concentration $c(= c/F_0)$ for three different cases; a solid with thermoelastic diffusion (THED), a thermoelastic solid (THE) and an elastic solid due to normal and tangential distributed loads of width $2l$, are computed for the exact solutions obtained in physical domain. The variations of field variables with respect to distance x, are presented graphically in Figs. 1–8. The computations are carried out for two values of non-dimensional time, namely $t = 0.075, 0.100$.

Case I. *Normal load applied.* The comparison of dimensionless normal displacement u_z, normal stress σ_{zz}, temperature Θ and mass concentration c for the three different theories, when normal load is applied, are shown in Figs. 1–4. The variations of normal displacement u_z with distance x are similar in nature for THED and elastic theories for both the times, whereas for THE theory the behaviour is different from them, as shown in Fig. 1. The original values for THED theory have been divided by 10 to depict the comparison of all the curves simultaneously in the same figure. Very near to the origin, the values of normal displacement are smaller for THED and elastic theories as compared to the values for THE theory. For THED and elastic theories the variations show the sharp increase in initial range. For all the three theories, it is observed that very near to the point of application of load, the values of displacement for time 0.075 are smaller than those for time 0.100.

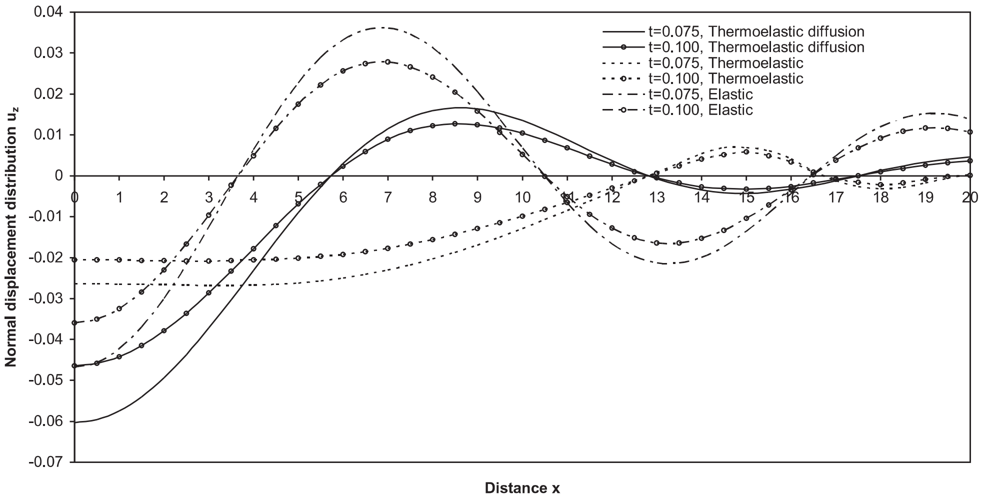

Fig. 1. Distribution of normal displacement u_z (due to normal load) versus distance.

Figure 2 depicts the variations of normal stress σ_{zz} for all the three theories, after dividing the values for THE and elastic theories by 10. The flow of variation for THED theory is just opposite to that for THE and elastic theories in almost the whole range. For THED theory as time increases, the value of normal stress decreases in the range near to origin, whereas this type of behaviour is just reverse in THE and elastic theories. The difference between the three curves at any fixed point as well as at fixed time for the three theories is clearly visible from this figure. The distribution of temperature for both the theories i.e. THED and THE, is observed from Fig. 3, for both the times. It should be observed that the variations for THE theory have been depicted after dividing the original values by 100. It is clear that diffusion in thermoelastic medium plays an important role in temperature distribution in the medium as Θ varies very smoothly in thermoelastic medium without diffusion, whereas its variation is more oscillatory in thermoelastic medium with diffusion. Variation of concentration about initial concentration is represented by Fig. 4 for THED theory. The difference in the values of c at a particular point for two different times can be easily observed from the graphs. It is also clearly depicted in the figure that the values of concentration c are maximum at the origin for both the times, and after a small oscillatory behaviour, they seem to be vanishing in the range far from origin.

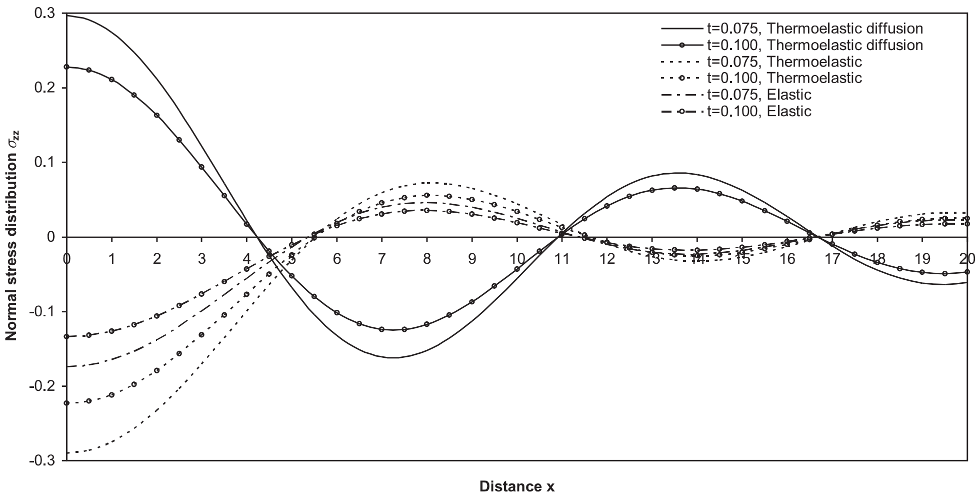

FIG. 2. Distribution of normal stress σ_{zz} (due to normal load) versus distance.

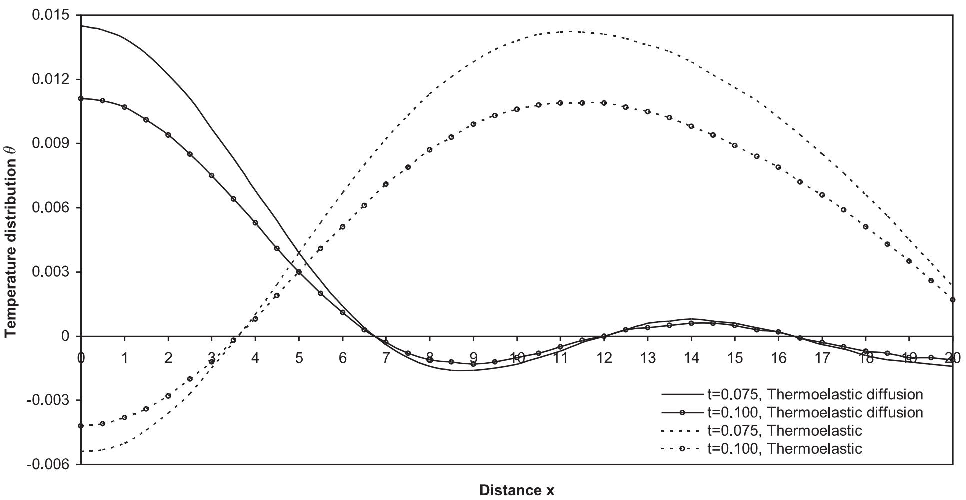

FIG. 3. Distribution of temperature θ (due to normal load) versus distance.

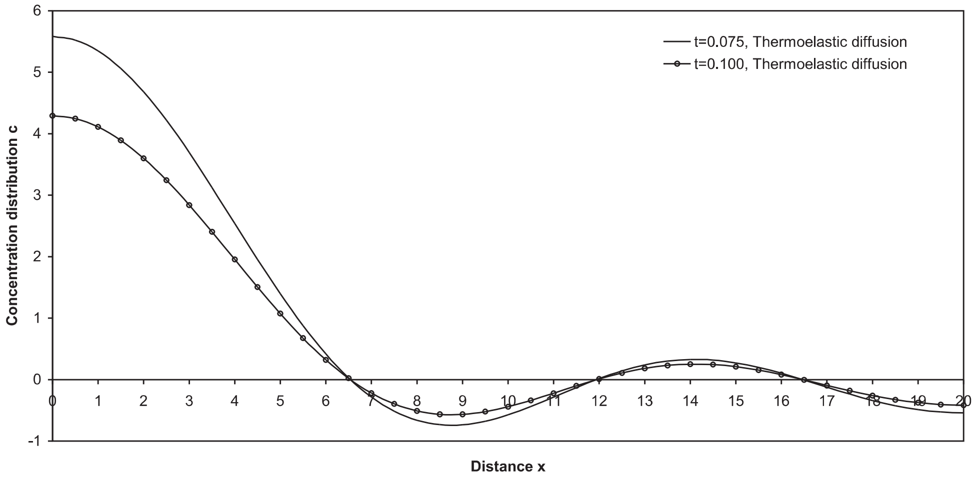

Fig. 4. Distribution of concentration c (due to normal load) versus distance.

Case II. *Tangential load applied.* The comparison of dimensionless normal displacement u_z, normal stress σ_{zz}, temperature Θ and concentration deviation c for the three different theories and at both the times, when tangential load is applied, are presented in Figs. 5–8. The variations in normal displacement u_z are shown in Fig. 5, where the original values for THE and elastic theories have been multiplied by 100. Initially, the trend of change in displacement for THED and THE theories is opposite in nature to the elastic theory at both the times,

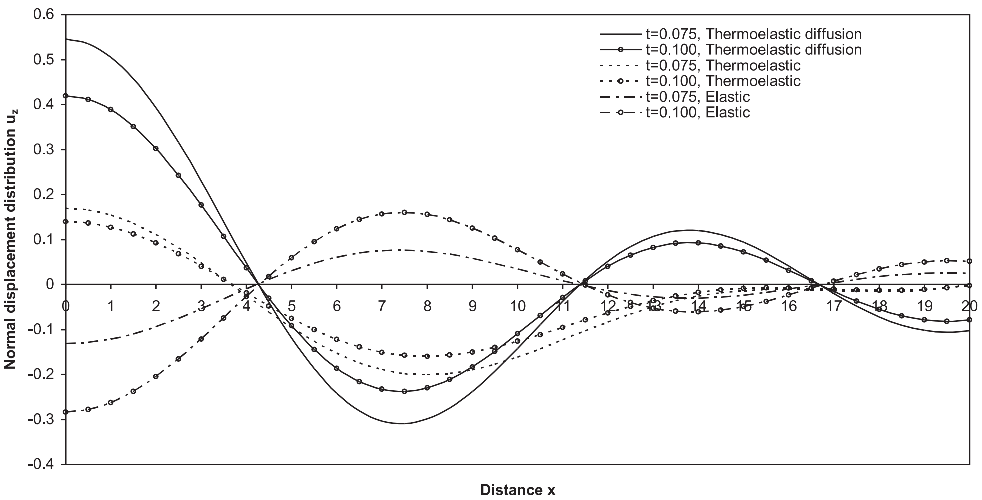

Fig. 5. Distribution of normal displacemnet u_z (due to tangential load) versus distance.

whereas the behaviour of variations in time for all the three theories is same. The numerical values for all the theories lie in the range −0.4 to 0.6 and have oscillating behaviour.

The variations of σ_{zz} have been depicted in Fig. 6 after multiplying the original values by 100 for THE and elastic theories. The behaviour of variations is oscillating in the whole range whereas the values are decreasing with the increasing range of distance x. The variations of temperature for THED and THE theories are shown in Fig. 7 at both the times. It is observed that initially,

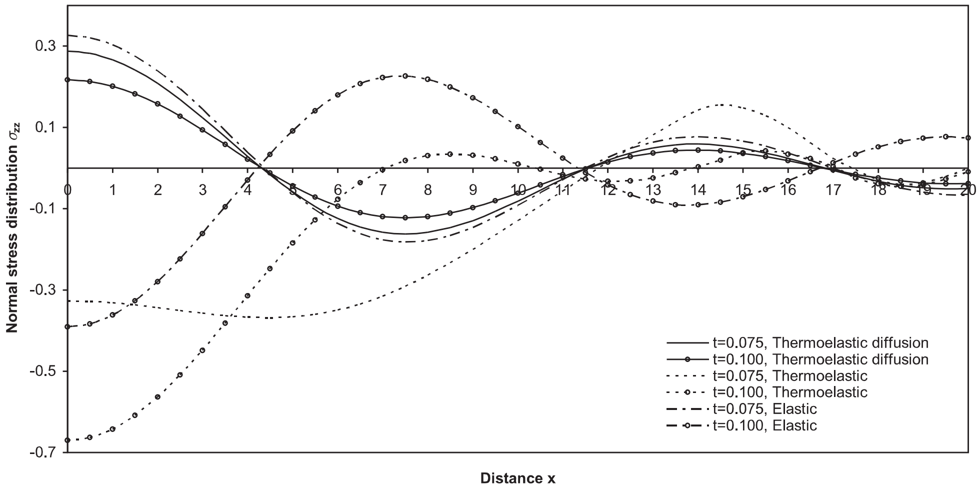

FIG. 6. Distribution of normal stress σ_{zz} (due to tangential load) versus distance.

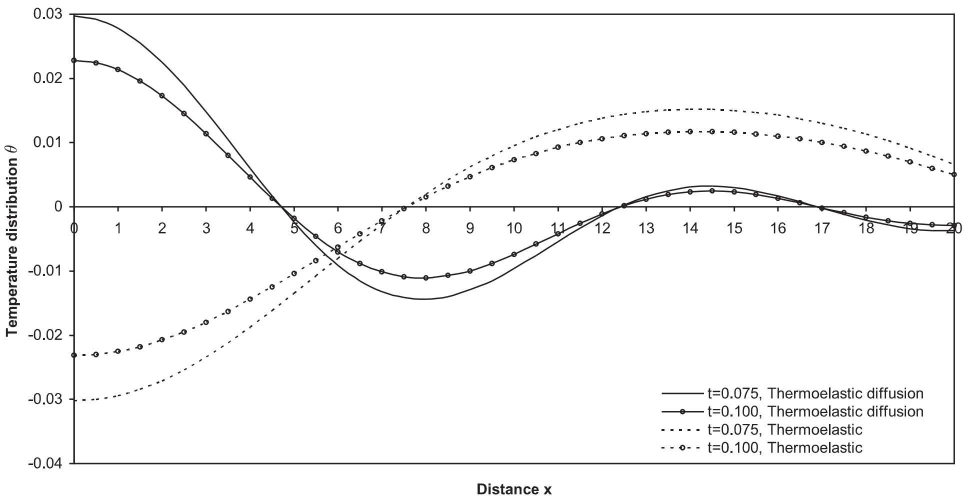

FIG. 7. Distribution of temperature θ (due to tangential load) versus distance.

the trends of variations for both the theories are just opposite to each other for both the times. Deviation of concentration from the mean value for THED theory has been depicted in Fig. 8. The values of concentration for a particular range show considerable difference for the two times.

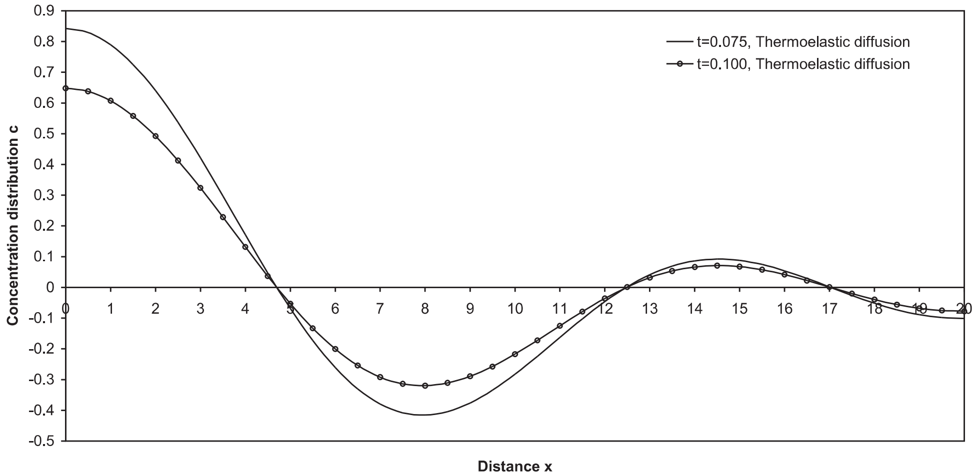

Fig. 8. Distribution of concentration c (due to tangential load) versus distance.

7. Conclusion

Analysis of normal displacement, normal stress component, temperature and mass concentration developed in a body due to a distributed source (normal and tangential) is an interesting problem of mechanics, having its applications in determining the stability of a medium. It can be observed that the varying load time has a significant effect on the normal displacement, normal stress, temperature and concentration. The results for all the functions for THED theory are distinctly different from those obtained for THE and Elastic theory. This is due to the presence of diffusion in thermoelastic solid. The eigenvalue approach is used, which has the advantage of finding the solution of equations in the coupled form directly in matrix notations, whereas the potential function approach requires decoupling of equations. The method used in the present article is applicable to a wide range of problems in thermodynamics.

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