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SIXTY YEARS OF THE ARCHIVES OF MECHANICS

Space of $SO(3)$-orbits of elasticity tensors

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We construct an eighteen-dimensional orbifold that is in a one-to-one correspondence with the space of $SO(3)$-orbits of elasticity tensors. This allows us to obtain a local parametrization of $SO(3)$-orbits of elasticity tensors by six $SO(6)$-invariant and twelve $SO(3)$-invariant parameters. This process unravels the structure of the space of the orbits of the elasticity tensors.

Key words: elasticity tensor, space of orbits, parametrization, orbifold.

1. Introduction

The problem of classifying linear elastic materials — Hookean solids — goes at least as far back as the work of Lord Kelvin [14, 15], and has been investigated by many researchers; notably, by Love [16], Voigt [25], Fedorov [10], Backus [1], Rychlewski [20], Walpole [26], Cowin and Mehrabadi [8], Helbig [13], Forte and Vianello [11] and Chadwick *et al.* [5].

The behavior of a material governed by Hooke's law is encoded in its mass density and elasticity tensor.

In discussing the elasticity tensors, it is often important to make no distinction between different orientations of the same tensor. Thus, we want to identify an elasticity tensor with its orbit under the action of the three-dimensional orthogonal group. The eighteen-dimensional space of these orbits has a complicated structure; it has been studied by Boehler *et al.* [2], who showed that this space can be embedded into a thirty-seven-dimensional Euclidean space. Boehler *et al.* [2] states that "what is needed is a parametrization of distinct orbits" and for this purpose they construct a set of polynomial invariants that "can be used to designate a certain set of materials". Herein, we construct an eighteen-dimensional orbifold, which is in a one-to-one correspondence with the space of orbits of elasticity tensors. Orbifolds are described by Thurston [24, Chapter 13]. This orbifold is modeled by $\mathbb{R}^{18}$, modulo the action of certain symmetry groups that we discuss below. Therefore, the parametrizations will be given up to the action of these symmetry groups. We propose a local parametrization of this orbifold by six $SO(6)$-invariant parameters and twelve $SO(3)$-invariant parameters; we note that the $SO(6)$ invariants are also $SO(3)$ invariants.

The complicated structure of the space of orbits of elasticity tensors does not allow for global charts, and hence a global parametrization.

An elasticity tensor can be viewed either as a fourth-rank tensor in $\mathbb{R}^3$ with intrinsic symmetries, or as a second-rank symmetric tensor in $\mathbb{R}^6$. Examination of rotations of the elasticity tensor in the context of the first viewpoint requires the study of fourth-order rotations of orthogonal group $O(3)$ in the space of elasticity tensors. This approach has been investigated by Coleman and Noll [7], Podio-Guidugli and Varga [19], Huo and Del Piero [28], Forte and Vianello [11], Chadwick *et al.* [5].

The second viewpoint has been formulated and investigated independently by Walpole [26], Rychlewski [20, 21], and Cowin and Mehrabadi [17, 9]. Lord Kelvin [15, p. 110] described, albeit without the use of tensors, aspects of matrix representation of elasticity tensors with respect to an orthonormal basis of the six-dimensional space. Such a representation was considered also by Fedorov [10], who gave an explicit relation between Kelvin's and Voigt's [25] notations. Examination of rotations in the context of the second viewpoint requires the study of second-order rotations of a subgroup of the orthogonal group $O(6)$ in the space of elasticity tensors, which has been investigated by Rychlewski [20, 21], Cowin and Mehrabadi [17, 9], Helbig [13], Yang *et al.* [27], and Bóna *et al.* [4].

In order to exploit both representations of the elasticity tensor, we build a group morphism, ψ, between the special orthogonal groups $SO(3)$ and $SO(6)$ that commutes with the linear action of these two groups. Matrix representation of this map was considered first by Mehrabadi and Cowin [17], and used by Helbig [13] and Chapman [6]. This morphism allows us to identify the twenty-one-dimensional space of elasticity tensors, Ela, with a quotient space of $\mathbb{R}^6 \times SO(6)$, via an equivalence relation. Thus, each elasticity tensor is uniquely represented by a class of pairs $(\lambda, \mathcal{A}) \in \mathbb{R}^6 \times SO(6)$. Using this morphism, we can identify the twelve-dimensional quotient space $SO(6)/\psi(SO(3))$ with the twelve-dimensional Stiefel manifold $V_3(\mathbb{R}^6)$, which allows us to identify the space of orbits $Ela/SO(3)$ with a quotient space of $\mathbb{R}^6 \times V_3(\mathbb{R}^6)$. Thus, each orbit of elasticity tensors is uniquely represented by a class of pairs $(\lambda, V) \in \mathbb{R}^6 \times V_3(\mathbb{R}^6)$. Consequently, we propose a parametrization of the fifteen-dimensional group $SO(6)$ that induces a parametrization of the quotient group $SO(6)/\psi(SO(3))$, which will be useful for the parametrization of the space of orbits.

The construction of parametrization of the space of $SO(3)$ orbits of elasticity tensors shows the structure of this space. This structure is important in understanding many properties of elasticity tensors and thus we believe that our work can serve to further the research in the field.

2. Notation

As stated above, elasticity tensors can be viewed as fourth-rank tensors in $\mathbb{R}^3$ with intrinsic symmetries or as a second-rank symmetric tensor in $\mathbb{R}^6$. For either viewpoint, we present the action of the corresponding special orthogonal groups; namely, $SO\,(3)$ or $SO\,(6)$. We discuss the corresponding orbits and symmetry groups of elasticity tensors with respect to $SO\,(3)$ or $SO\,(6)$.

Consider the Euclidean three-dimensional space, $\mathbb{R}^3$, and the three-dimensional special orthogonal group, $SO\,(3)$, which is the group of rotations in $\mathbb{R}^3$. Also consider $L_{2,s}\left(\mathbb{R}^3\right)$, the six-dimensional space of symmetric bilinear maps on $\mathbb{R}^3$, which is a Euclidean space with the scalar product given by $\omega \cdot \tau = \operatorname{Tr}\left(\omega^t \tau\right)$. For an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ we define the orthonormal basis of $L_{2,s}\left(\mathbb{R}^3\right)$ given by

$$\varepsilon_{\alpha(i,j)}\,(e) = 2^{-1/(2-\delta_{ij})}\left(e_i \otimes e_j + e_j \otimes e_i\right). \tag{2.1}$$

Expression (2.1) is a concise notation for the Cartesian base vectors used by MEHRABADI and COWIN [17]. Here $\alpha : \{(i,j)\,|1 \le i < j \le 3\} \longrightarrow \{1, 2, \ldots, 6\}$ is the bijection given by

$$\alpha\,(i,j) = i\delta_{ij} + (1 - \delta_{ij})\,(9 - i - j) \tag{2.2}$$

that corresponds to VOIGT notation [25], and δ_{ij} is the Kronecker delta. In this paper, the Greek indices, α, β, are from 1 to 6 and the Latin indices, i, j, k, l, are from 1 to 3. An elasticity tensor is a linear map

$$c : L_{2,s}\left(\mathbb{R}^3\right) \longrightarrow L_{2,s}\left(\mathbb{R}^3\right),$$

which is symmetric with respect to the scalar product, $c\,(\omega) \cdot \tau = \omega \cdot c\,(\tau)$, and is positive-definite, $c\,(\omega) \cdot \omega > 0$ for $\omega \neq 0$. We denote the components of an elasticity tensor with respect to the orthonormal basis given by expression (2.1) by

$$c_{ijkl} = c\left(\varepsilon_{\alpha(i,j)}\right) \cdot \varepsilon_{\alpha(k,l)}, \tag{2.3}$$

and by Ela the twenty-one-dimensional space of these tensors. The use of the components of elasticity tensors (2.3) was suggested first by Lord KELVIN without using the tensorial notation, which was not known at the time, [15, p. 110]. Each elasticity tensor can be written as

$$c = \sum_{\alpha=1}^{6} \lambda_\alpha \omega_\alpha \otimes \omega_\alpha, \tag{2.4}$$

due to its symmetry and positive definiteness. This eigendecomposition was considered by Rychlewski [20], Mehrabadi and Cowin [17] and Helbig [13] to study properties of the elasticity tensors. Recently, Bóna *et al.* [4] proved that such eigendecomposition provides necessary and sufficient conditions for identifying each of the eight symmetry classes of elasticity tensors. Herein, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_6$ are the eigenvalues of the elasticity tensor, which represent the Kelvin moduli, as referred to by Rychlewski [20], with the corresponding eigentensors $\{\omega_\alpha\}$; this means that $c(\omega_\alpha) = \lambda_\alpha \omega_\alpha, \alpha \in \{1, 2, \ldots, 6\}$. The ordering of the eigenvalues allows us to distinguish between the corresponding eigentensors. We denote the space of ordered positive eigenvalues by

$$\Lambda^6 = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_6) \in \mathbb{R}^6 \mid 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_6 \right\}.$$

The six eigentensors ω_α constitute an orthonormal basis of $L_{2,s}\left(\mathbb{R}^3\right)$.

The action of the special orthogonal group, $SO(3)$, on the space of the second-rank symmetric tensors, $L_{2,s}\left(\mathbb{R}^3\right)$, is given by $(A, \omega) \in SO(3) \times L_{2,s}\left(\mathbb{R}^3\right) \mapsto A \cdot \omega \in L_{2,s}\left(\mathbb{R}^3\right)$, where

$$(A \cdot \omega)(u, v) = \omega(Au, Av), \forall u, v \in \mathbb{R}^3. \tag{2.5}$$

Rotation A acts on the six second-rank symmetric tensors of the orthonormal basis given by expression (2.1) as follows:

$$A \cdot \varepsilon_\alpha(e) = \varepsilon_\alpha(Ae) = 2^{-1/(2-\delta_{ij})}(Ae_i \otimes Ae_j + Ae_j \otimes Ae_i). \tag{2.6}$$

The action of the special orthogonal group $SO(3)$ on the space of the second-rank symmetric tensors can be extended to the space of elasticity tensors, Ela, as follows:

$$(A * c)(\omega) = A \cdot c\left(A^t \cdot \omega\right). \tag{2.7}$$

Rotation A acts on an elasticity tensor given by expression (2.4) as follows:

$$A * c = \sum_{\alpha=1}^{6} \lambda_\alpha A \cdot \omega_\alpha \otimes A \cdot \omega_\alpha. \tag{2.8}$$

For an elasticity tensor, c, consider its symmetry group

$$G_c = \{A \in SO(3) \mid A * c = c\}.$$

A symmetry group is a closed subgroup of the connected three-dimensional Lie group, $SO(3)$. Therefore, the quotient space $SO(3)/G_c$ is a differentiable manifold and its dimension is given by $\dim(SO(3)/G_c) = 3 - \dim G_c$. The orbit of an elasticity tensor is given by

$$O_c = \left\{c' \in Ela \mid \exists A \in SO(3),\ c' = A * c\right\}.$$

The orbit is at most a three-dimensional subspace of the twenty-one-dimensional space of elasticity tensors, Ela. More precisely, there is a diffeomorphism between O_c and $SO(3)/G_c$, which implies that $\dim O_c = 3 - \dim G_c$. If we apply this result to each of the eight symmetry classes of the elasticity tensor, we obtain the dimensions of the corresponding orbits as follows.

If c has isotropic symmetry, then its symmetry group is $G_c = SO(3)$ and therefore its orbit is zero-dimensional; in fact, in such a case its orbit contains only one element, $O_c = \{c\}$.

If c has transversely isotropic symmetry, then its symmetry group is $G_c = SO(2)$, which is one-dimensional and therefore its orbit is a two-dimensional subspace of Ela; hence, it can be parametrized by the two angles that determine the axis of rotation.

If the elasticity tensor belongs to one of the other six symmetry classes, its symmetry group is discrete and therefore its orbit is three-dimensional; such an orbit can be parametrized by the three Euler angles that determine one of the natural coordinate systems.

Two elasticity tensors, c_1 and c_2, that belong to the same orbit, which means that there exists $A \in SO(3)$ such that $c_1 = A * c_2$, have conjugate symmetry groups; in other words, $G_{c_1} = A G_{c_2} A^t$. There are elasticity tensors with conjugate symmetry groups that do not belong to the same orbit. It has been shown by FORTE and VIANELLO [11], CHADWICK *et al.* [5], and BÓNA *et al.* [3], that there are exactly eight nonconjugate subgroups of $SO(3)$, which are the symmetry classes of the elasticity tensor.

3. Spaces of elasticity tensors

There is no canonical way of defining a parametrization of the space of elasticity tensors, Ela, that distinguishes between the orbits. In this section, we construct a twenty-one-dimensional space that is in a one-to-one correspondence with the space of elasticity tensors and allows us to find a parametrization that simplifies the recognition of the orbits. Due to the material symmetries, such a parametrization is given only modulo the action of certain symmetry groups. Moreover, the complicated structure of the space of orbits does not allow for global parametrizations. Therefore the parametrization we propose is only local and allows for identification of orbits of elasticity tensors that belong to the same domain of parametrization. We use the fact that elasticity tensors can be viewed as second-rank symmetric tensors in $\mathbb{R}^6$, as it has been described in the previous section.

For the rest of the paper, we consider the canonical orthonormal basis of $\mathbb{R}^6$: $\{N_\alpha\}$, the canonical orthonormal basis of $\mathbb{R}^3$: $\{e_i\}$, and the orthonormal basis of $L_{2,s}\left(\mathbb{R}^3\right)$: $\{\varepsilon_\alpha\}$, which is given by expression (2.1). We define linear map $g: \ L_{2,s}\left(\mathbb{R}^3\right) \longrightarrow \mathbb{R}^6$ by $g(\varepsilon_\alpha) = N_\alpha, \ \alpha \in \{1, 2, \ldots, 6\}$. This map is a linear isomorphism and preserves the scalar products of $L_{2,s}\left(\mathbb{R}^3\right)$ and $\mathbb{R}^6$, and therefore this map is an isometry.

In our approach, we want to relate coordinate transformations in $\mathbb{R}^3$ to coordinate transformations in $\mathbb{R}^6$. Thus, we define a map between $SO(3)$ and $SO(6)$ that preserves the group properties and the linear actions of these two groups. In other words, we define a group morphism $\psi: \ SO(3) \longrightarrow SO(6)$, such that the following diagram is commutative.

$$\begin{array}{ccc} SO(3) \times L_{2,s}\left(\mathbb{R}^3\right) & \longrightarrow & L_{2,s}\left(\mathbb{R}^3\right) \\ {\scriptstyle \psi \times g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ SO(6) \times \mathbb{R}^6 & \longrightarrow & \mathbb{R}^6. \end{array} \tag{3.1}$$

The commutativity of diagram (3.1) means that – for all $A \in SO(3)$ and $\omega \in L_{2,s}\left(\mathbb{R}^3\right)$ – we have

$$g(A \cdot \omega) = \psi(A)\, g(\omega). \tag{3.2}$$

Expression (3.2) determines uniquely the map ψ. If we let $g(\omega) \in \{N_1, \ldots, N_6\}$, we obtain $\psi(A)\, N_\alpha = g\left(A \cdot g^{-1}(N_\alpha)\right)$, and, therefore, the matrix representation of $\psi(A)$ with respect to the fixed orthonormal bases can be written using bijection α given by expression (2.2) as follows:

$$\psi(A)_{\alpha(i,j)\alpha(k,l)} = 2^{-(\delta_{ij}+\delta_{kl})/2}\left(A_{ik}A_{jl} + A_{il}A_{jk}\right). \tag{3.3}$$

This matrix was considered first by Mehrabadi and Cowin [17] and used by Helbig [13], Chapman [6], Bóna *et al.* [4], to study properties of the elasticity tensors under the action of $\psi(SO(3))$. Herein, A_{ij} is the matrix representation of rotation A in $SO(3)$ with respect to the orthonormal basis, $\{e_1, \ e_2, \ e_3\}$, of $\mathbb{R}^3$. The next theorem presents algebraic properties of map ψ.

THEOREM 1. *Map ψ, given by expression* (3.2)*, or — in terms of coordinates — by expression* (3.3)*, is a group isomorphism between the special orthogonal group, $SO(3)$, and its image, $\psi(SO(3)) \subset SO(6)$.*

P r o o f. Isomorphism g is an isometry and ψ preserves the action of the two special orthogonal groups, namely $SO(3)$ and $SO(6)$. Thus, we obtain $\psi(A) \in SO(6)$ for all $A \in SO(3)$. Using expression (3.2), we show that ψ is a group morphism between $SO(3)$ and $SO(6)$. Namely, $\psi(A \cdot B)(g(\omega)) = g((AB) \cdot \omega) = g(A \cdot B \cdot \omega) = \psi(A)\psi(B)(g(\omega))$, for all $A, B \in SO(3)$.

Since $\psi : SO(3) \to SO(6)$ is a group morphism, using expression (3.3), we see that if $A \in SO(3)$ so that $\psi(A) = I_6$, then $A = I_3$. This means that the kernel of ψ is the identity; in other words, $\ker\psi = \{I_3\}$. Consequently, ψ is a monomorphism of groups and, hence, it is an isomorphism between $SO(3)$ and its image $\psi(SO(3))$. □

Each elasticity tensor is determined uniquely by its eigenvalues and the corresponding eigenspaces. These eigenspaces are defined by an orthonormal frame in $\mathbb{R}^6$ identified with an element of $SO(6)$, which might be not unique. The relation between pairs $(\lambda, \mathcal{A}) \in \Lambda^6 \times SO(6)$ and elasticity tensors can be expressed by map $f : \Lambda^6 \times SO(6) \longrightarrow Ela$,

$$f(\lambda, \mathcal{A}) = \sum_{\alpha=1}^{6} \lambda_\alpha g^{-1}(\mathcal{A}N_\alpha) \otimes g^{-1}(\mathcal{A}N_\alpha). \tag{3.4}$$

Map f is well-defined; it is onto but it is not one-to-one. For the elasticity tensor $f(\lambda, \mathcal{A})$, λ_α, $\alpha \in \{1, \ldots, 6\}$ are its eigenvalues with the corresponding eigentensors $g^{-1}(\mathcal{A}N_\alpha)$. The elasticity tensor $c = f(\lambda, \mathcal{A})$ can be viewed as a second-rank symmetric tensor in $\mathbb{R}^6$ denoted by $C = F(\lambda, \mathcal{A})$, where $F : \Lambda^6 \times SO(6) \longrightarrow L_{2,s}(\mathbb{R}^6)$,

$$F(\lambda, \mathcal{A}) = \sum_{\alpha=1}^{6} \lambda_\alpha \mathcal{A}N_\alpha \otimes \mathcal{A}N_\alpha.$$

The symmetry group of $C = F(\lambda, \mathcal{A})$ is a closed subgroup of the Lie group $SO(6)$, namely,

$$G_{F(\lambda,\mathcal{A})} = \{\mathcal{B} \in SO(6) \,|\, F(\lambda, \mathcal{A}) = F(\lambda, \mathcal{B}\mathcal{A})\}. \tag{3.5}$$

Using map f, we want to build a new map whose domain will be in a one-to-one correspondence with the space of elasticity tensors, Ela. To identify the pairs in $\Lambda^6 \times SO(6)$ that correspond to the same elasticity tensor, we define the following equivalence relation $(\lambda, \mathcal{A}) \sim (\lambda', \mathcal{A}')$ if $f(\lambda, \mathcal{A}) = f(\lambda', \mathcal{A}')$. Each equivalence class $[(\lambda, \mathcal{A})] = f^{-1}(f(\lambda, \mathcal{A}))$ corresponds uniquely to an elasticity tensor. For each elasticity tensor $f(\lambda, \mathcal{A})$, the corresponding set in $\Lambda^6 \times SO(6)$ is given by

$$f^{-1}(f(\lambda, \mathcal{A})) = F^{-1}(F(\lambda, \mathcal{A})) = \{\lambda\} \times G_{F(\lambda,\mathcal{A})}\mathcal{A}, \tag{3.6}$$

due to expression (3.5). We conclude this discussion by the following proposition and theorem.

Proposition 1. *1. The quotient space $\Lambda^6 \times SO(6) / \sim$ is in a one-to-one correspondence with the twenty-one-dimensional space of elasticity tensors.*

2. Two elasticity tensors, $f(\lambda, \mathcal{A})$ and $f(\lambda', \mathcal{A}')$, coincide with one another if and only if $\lambda = \lambda'$ and $\mathcal{A}' \cdot \mathcal{A}^{-1} \in G_{F(\lambda,\mathcal{A})}$.

To understand better the set $\{\lambda\} \times G_{F(\lambda,\mathcal{A})}\mathcal{A}$, which determines uniquely the elasticity tensor $f(\lambda, \mathcal{A})$, we write $G_{F(\lambda,\mathcal{A})}$, which is a subgroup of $SO(6)$, as

$$G_{F(\lambda,\mathcal{A})} = SO(m_1) \times \cdots \times SO(m_r), \tag{3.7}$$

where r is the number of distinct Kelvin moduli, λ_α, and $m_\alpha \geq 1$ are the corresponding multiplicities, with $m_1 + \cdots + m_r = 6$, with $SO(m_\alpha)$ acting on the eigenspace corresponding to λ_α.

Remark 1. Expression (3.6) tells us that each elasticity tensor $c = f(\lambda, \mathcal{A})$ is determined by the pair $(\lambda, \mathcal{A}) \in \Lambda^6 \times SO(6)$ modulo the symmetry group $G_{F(\lambda,\mathcal{A})}$, which is given by expression (3.7).

To exemplify this, consider a pair $(\lambda, \mathcal{A})$, where all Kelvin moduli λ_α are distinct. In this case, the symmetry group of $F(\lambda, \mathcal{A}) \in L_{2,s}\left(\mathbb{R}^6\right)$ is given by

$$G_{F(\lambda,\mathcal{A})} = \{\operatorname{diag}(\eta_1, \eta_2, \ldots, \eta_6), \eta_\alpha \in \{\pm 1\}, \eta_1 \cdots \eta_6 = 1\}. \tag{3.8}$$

According to Proposition 1, the elasticity tensor $f(\lambda, \mathcal{A})$ is uniquely determined by its distinct eigenvalues λ_α and eigenvectors $\pm g^{-1}(\mathcal{A}N_\alpha)$. This case corresponds to one of the following symmetry classes: orthotropic, monoclinic or anisotropic. Thus, an elasticity tensor $c = f(\lambda, \mathcal{A})$ that is either orthotropic, monoclinic or anisotropic, is uniquely determined by the pair $f(\lambda, \mathcal{A})$ modulo rotations by π around $\mathcal{A}N_\alpha$.

We will treat this case of distinct eigenvalues in more details below, after we discuss the parametrization of elasticity tensors modulo the corresponding rotations.

4. Space of orbits of elasticity tensors

In this section, we use the space of elasticity tensors derived in the previous section, namely,

$$Ela \simeq \Lambda^6 \times SO(6) / \sim,$$

to describe the space of orbits of elasticity tensors, $Ela/SO(3)$. We showed in the previous section that an elasticity tensor $c = f(\lambda, \mathcal{A})$ could be described by the pair $(\lambda, \mathcal{A})$ modulo $G_{F(\lambda,\mathcal{A})}$. Therefore, its orbit could be described by the pair $(\lambda, \mathcal{A})$ modulo $G_{F(\lambda,\mathcal{A})} \cdot \psi(SO(3))$. We show that the actions of the two groups commute. Using these properties and the fact that the quotient space $SO(6)/\psi(SO(3))$ is the Stiefel manifold $V_3(\mathbb{R}^6)$, we can describe the orbit of an elasticity tensor $c = f(\lambda, \mathcal{A})$ by the corresponding element of $V_3(\mathbb{R}^6)$ modulo $G_{F(\lambda,\mathcal{A})}$.

The next theorem gives an explicit form for the orbit and the symmetry group of an elasticity tensor.

THEOREM 2. *For a pair* $(\lambda, \mathcal{A}) \in \Lambda^6 \times SO(6)$*, the orbit of the elasticity tensor* $f(\lambda, \mathcal{A})$ *is given by*

$$O_{f(\lambda,\mathcal{A})} = \{f(\lambda, \psi(B)\mathcal{A}) \mid B \in SO(3)\}. \tag{4.1}$$

For a pair $(\lambda, \mathcal{A}) \in \Lambda^6 \times SO(6)$*, the symmetry group of the corresponding elasticity tensor* $f(\lambda, \mathcal{A})$ *is given by*

$$G_{f(\lambda,\mathcal{A})} = \{B \in SO(3) \mid f(\lambda, \mathcal{A}) = f(\lambda, \psi(B)\mathcal{A})\}. \tag{4.2}$$

P r o o f. Using expressions (3.4) and (2.8) we obtain

$$B * f(\lambda, \mathcal{A}) = \sum_{\alpha=1}^{6} \lambda_\alpha B \cdot g^{-1}(\mathcal{A}N_\alpha) \otimes B \cdot g^{-1}(\mathcal{A}N_\alpha). \tag{4.3}$$

If we apply g^{-1} to both sides of expression (3.2), we obtain $B\cdot\omega = g^{-1}(\psi(B)g(\omega))$. For $\omega = g^{-1}(\mathcal{A}N_\alpha)$, this implies that

$$B \cdot g^{-1}(\mathcal{A}N_\alpha) = g^{-1}\left(\psi(B)\, g\left(g^{-1}(\mathcal{A}N_\alpha)\right)\right) = g^{-1}(\psi(B)\mathcal{A}N_\alpha). \tag{4.4}$$

Using expression (4.4), one can write expression (4.3) as follows:

$$\begin{aligned} B * f(\lambda, \mathcal{A}) &= \sum_{\alpha=1}^{6} \lambda_\alpha g^{-1}(\psi(B)\mathcal{A}N_\alpha) \otimes g^{-1}(\psi(B)\mathcal{A}N_\alpha) \\ &= f(\lambda, \psi(B)\mathcal{A}). \end{aligned} \tag{4.5}$$

Expression (4.5) implies that the orbit of the elasticity tensor $f(\lambda, \mathcal{A})$ is given by Eq. (4.1), while the symmetry group is given by expression (4.2). □

In the previous section, we have seen that each elasticity tensor can be expressed uniquely by an equivalence class $[(\lambda, \mathcal{A})] = \{\lambda\} \times G_{F(\lambda,\mathcal{A})}\mathcal{A}$. Using the first part of Theorem 2, the orbit of a given elasticity tensor expressed by the equivalence class $[(\lambda, \mathcal{A})]$ is

$$(4.6) \qquad \begin{aligned} O_{[(\lambda,\mathcal{A})]} &= \{[(\lambda, \psi(B)\mathcal{A})] \,|B \in SO(3)\} \\ &= \{\lambda\} \times \psi(SO(3))\mathcal{A} \subset \Lambda^6 \times SO(6)\,. \end{aligned}$$

Since the eigenspaces of a second-rank tensor are preserved by its symmetry group, we write

$$\begin{aligned} O_{[(\lambda,\mathcal{A})]} &= \left\{(\lambda, \psi(B) D\mathcal{A}) \mid B \in SO(3)\,,\, D \in G_{F(\lambda,\mathcal{A})}\right\} \\ &= \{\lambda\} \times \psi(SO(3))\, G_{F(\lambda,\mathcal{A})}\mathcal{A}. \end{aligned}$$

The one-to-one correspondence between $\Lambda^6 \times SO(6)/\sim$ and Ela induces a one-to-one correspondence between the corresponding orbits $O_{[(\lambda,\mathcal{A})]}$ and $O_{f(\lambda,\mathcal{A})}$ under the action of $SO(3)$. Since for a fixed $(\lambda, \mathcal{A})$,

$$\psi(SO(3))\, G_{F(\lambda,\mathcal{A})}\mathcal{A} = G_{F(\lambda,\mathcal{A})}\psi(SO(3))\,\mathcal{A},$$

we can write

$$Ela/SO(3) \simeq \left(\Lambda^6 \times SO(6)/\sim\right)/\psi(SO(3)) \simeq \left(\Lambda^6 \times SO(6)/\psi(SO(3))\right)/\sim .$$

Both groups $SO(3)$ and $SO(6)$ are compact connected Lie groups of dimensions three and fifteen, respectively. Map ψ, given by expression (3.3), is an embedding. It follows that $\psi(SO(3))$ is a closed Lie subgroup of $SO(6)$ and therefore the quotient space $SO(6)/\psi(SO(3))$ is a twelve-dimensional manifold. This manifold is diffeomorphic with the twelve-dimensional Stiefel compact manifold,

$$(4.7) \qquad V_3\left(\mathbb{R}^6\right) = \left\{V \in M_{6\times 3}(\mathbb{R}) \mid V^t \cdot V = I_3\right\},$$

of isometries from $\mathbb{R}^3$ to $\mathbb{R}^6$. Therefore, we can view the space of orbits of elasticity tensors $Ela/SO(3)$ as

$$Ela/SO(3) \simeq \Lambda^6 \times V_3\left(\mathbb{R}^6\right)/\sim .$$

Note that all the manifolds presented above do not have global charts, therefore parametrizations of such manifolds will be only local. Our aim is to find parametrizations of the fifteen-dimensional space $SO(6)$ that induces parametrizations of the twelve-dimensional space $V_3\left(\mathbb{R}^6\right) \simeq SO(6)/\psi(SO(3))$.

A local parametrization of the eighteen-dimensional orbifold $(\Lambda^6 \times V_3(\mathbb{R}^6)/\sim$ will determine a local parametrization of the eighteen-dimensional space of orbits of elasticity tensors $Ela/SO(3)$.

5. Parametrization of elasticity tensors

In this section, we propose a parametrization of the eighteen-dimensional space $\Lambda^6 \times V_3(\mathbb{R}^6)$ that will result in a parametrization of the space of orbits of elasticity tensors. The proposed parameters do not determine uniquely an elasticity tensor $c = f(\lambda, \mathcal{A})$, they are given modulo $G_{F(\lambda,\mathcal{A})}$. We propose a parametrization of the twenty-one-dimensional space of elasticity tensors by eighteen parameters and the three Euler angles in such a way that the Euler angles determine the orientation of a natural coordinate system. Six of these parameters will represent the rigidity moduli λ_α of the elasticity tensor, another twelve parameters γ_a will determine the six orthonormal eigentensors ω_α with respect to the three orthonormal eigenvectors of ω_1. The twelve parameters γ_a and the three Euler angles parametrize $SO(6)$ in such a way that $SO(6)/\psi(SO(3))$ is parametrized by γ. This parametrization is a local chart of the manifold $SO(6)/\psi(SO(3))$.

We consider the canonical orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and the corresponding orthonormal basis $\{\varepsilon_\alpha\}$ of $L_{2,s}(\mathbb{R}^3)$ given by expression (2.1). We define first the six orthonormal second-rank symmetric tensors,

$$\omega_\alpha : (\gamma_a, A) \in \mathbb{R}^{12} \times SO(3) \mapsto \omega_\alpha(\gamma_a, A) \in L_{2,s}(\mathbb{R}^3),$$

as follows:

$$\omega_1(\gamma, A) = \gamma_1 A \cdot \varepsilon_1 + \gamma_2 A \cdot \varepsilon_2 + x_1(\gamma) A \cdot \varepsilon_3. \tag{5.1}$$

We note that $Ae = \{Ae_1, Ae_2, Ae_3\}$ are the eigenvectors of the second-rank symmetric tensor ω_1, while $x_1(\gamma)$ is chosen in such a way that ω_1 is unitary, which means that $\gamma_1^2 + \gamma_2^2 + x_1^2 = 1$. Second-rank symmetric tensor ω_1 depends on two parameters γ_1, γ_2 and the three Euler angles represented by the orthogonal matrix A.

The second tensor, ω_2, is defined by

$$\begin{aligned}\omega_2(\gamma, A) = \gamma_3 A \cdot \varepsilon_1 + \gamma_4 A \cdot \varepsilon_2 + x_2(\gamma) A \cdot \varepsilon_3 \\ + x_3(\gamma) A \cdot \varepsilon_4 + \gamma_7 A \cdot \varepsilon_5 + \gamma_8 A \cdot \varepsilon_6.\end{aligned} \tag{5.2}$$

If $x_1(\gamma) \neq 0$, then the function $x_2(\gamma)$ is uniquely determined from the orthogonality condition $\omega_1 \perp \omega_2$, which implies $\gamma_1\gamma_3 + \gamma_2\gamma_4 + x_1(\gamma) x_2(\gamma) = 0$. Function $x_3(\gamma)$ is determined up to the sign by the condition that ω_2 is unitary, which means that $\gamma_3^2 + \gamma_4^2 + \gamma_7^2 + \gamma_8^2 + x_2^2 + x_3^2 = 1$. We observe that ω_2 depends on four parameters $\gamma_3, \gamma_4, \gamma_7, \gamma_8$.

The remaining four tensors are defined as follows:

$$\begin{aligned}\omega_3(\gamma, A) = \gamma_5 A \cdot \varepsilon_1 + x_4(\gamma) A \cdot \varepsilon_2 + x_5(\gamma) A \cdot \varepsilon_3 \\ + x_6(\gamma) A \cdot \varepsilon_4 + \gamma_9 A \cdot \varepsilon_5 + \gamma_{10} A \cdot \varepsilon_6,\end{aligned} \tag{5.3}$$

$$\omega_4(\gamma, A) = x_7(\gamma) A \cdot \varepsilon_1 + x_8(\gamma) A \cdot \varepsilon_2 + x_9(\gamma) A \cdot \varepsilon_3 + x_{10}(\gamma) A \cdot \varepsilon_4 + \gamma_{11} A \cdot \varepsilon_5 + \gamma_{12} A \cdot \varepsilon_6, \tag{5.4}$$

$$\omega_5(\gamma, A) = x_{11}(\gamma) A \cdot \varepsilon_1 + x_{12}(\gamma) A \cdot \varepsilon_2 + x_{13}(\gamma) A \cdot \varepsilon_3 + x_{14}(\gamma) A \cdot \varepsilon_4 + x_{15}(\gamma) A \cdot \varepsilon_5 + \gamma_6 A \cdot \varepsilon_6, \tag{5.5}$$

$$\omega_6(\gamma, A) = x_{16}(\gamma) A \cdot \varepsilon_1 + x_{17}(\gamma) A \cdot \varepsilon_2 + x_{18}(\gamma) A \cdot \varepsilon_3 + x_{19}(\gamma) A \cdot \varepsilon_4 + x_{20}(\gamma) A \cdot \varepsilon_5 + x_{21}(\gamma) A \cdot \varepsilon_6. \tag{5.6}$$

Functions $x_4(\gamma), \ldots, x_{21}(\gamma)$ are determined up to the sign by the condition that $\{\omega_\alpha\}$ forms an orthonormal basis of $L_{2,s}(\mathbb{R}^3)$.

For a rotation $\mathcal{A} \in SO(6)$, we consider the orthonormal basis $\{g^{-1}(\mathcal{A}N_\alpha)\}$ of $L_{2,s}(\mathbb{R}^6)$. One can associate with this orthonormal basis twelve parameters γ and the three Euler angles determined by expressions (5.1)–(5.6) by setting $\omega_\alpha = g^{-1}(\mathcal{A}N_\alpha)$. We note that the parameters γ are given uniquely and the three Euler angles are given up to an element in the symmetry group of $\omega_1 = g^{-1}(\mathcal{A}N_1)$. Since these spaces do not have global charts, one cannot obtain a global parametrization of $SO(6)$. One can see that the process of associating parameters γ described by expressions (5.1)–(5.6) does not cover the case when $x_1(\gamma) = 0$ (which corresponds to the case when one of the eigenvalues of $\omega_1 = g^{-1}(\mathcal{A}N_1)$ is zero) and for similar cases when $x_4(\gamma) = 0$, etc. To build another chart of a complete atlas for $SO(6)$, we have to change γ_2 with x_1 in (5.1), γ_5 with x_4 in (5.2), etc.

This way we have parametrized $SO(6)$ by twelve $SO(3)$-invariant parameters γ and the three Euler angles contained in $A \in SO(3)$. Consider $U \subset SO(6)$, the domain of the parametrization described by expressions (5.1)–(5.6), and let $\varphi : U \subset SO(6) \longrightarrow \varphi(U) \subset \mathbb{R}^{12} \times SO(3)$ be the map that realizes this parametrization. Due to the nonuniqueness of the three Euler angels, the domain of such a map can be defined in several ways. The matrix representation of $\mathcal{A} \in SO(6)$ in terms of γ and A is given by

$$\mathcal{A}(\gamma, A) N_\alpha = g(\omega_\alpha(\gamma, A)), \quad \alpha \in \{1, 2, \ldots, 6\}. \tag{5.7}$$

With respect to this local chart of $SO(6)$, the map f given by expression (3.4) can be expressed in our parameters as follows:

$$\hat{f} = f \circ \left(Id \times \varphi^{-1}\right) : \Lambda^6 \times \varphi(U) \subset \Lambda^6 \times \mathbb{R}^{12} \times SO(3) \longrightarrow Ela,$$

where

$$\hat{f}(\lambda, \gamma, A) = \sum_{\alpha=1}^{6} \lambda_\alpha \omega_\alpha(\gamma, A) \otimes \omega_\alpha(\gamma, A). \tag{5.8}$$

We note that function $\hat{f}$ is well-defined, it is onto but not one-to-one. A different approach was proposed by RYCHLEWSKI [20] and COWIN *et al.* [9], where the authors start from the space of elasticity tenors and associate with each elasticity tensor eighteen parameters λ_α and γ_a and the three Euler angles. This process, which corresponds to considering $\hat{f}^{-1}$, is not well-defined since to an elasticity tensor with eigenvalues of higher multiplicity one cannot associate an orthonormal basis of eigentensors ω_α in a unique way. Also, the Euler angles corresponding to the basis in which ω_1 is diagonal, are given up to a rotation form the symmetry group of ω_1.

For an elasticity tensor c consider the six eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_6$ with corresponding eigentensors ω_α. The eigenvalues are invariant under the action of the group $SO(6)$ and therefore, they are invariant under the action of the group $SO(3)$. Consider $\{Ae_1, Ae_2, Ae_3\}$: three eigenvectors of eigentensor ω_1. We associate twelve $SO(3)$ invariant parameters γ_a that are uniquely defined by expressions (5.1)–(5.6) with the fixed frame $\{\omega_\alpha\}$. We can associate different orthonormal frames $\{\omega_\alpha\}$ with the same elasticity tensor c, two such orthonormal frames being connected by a symmetry of the corresponding tensor C.

The relation among different sets of parameters γ that are determined by the same elasticity tensor, can be stated as follows:

PROPOSITION 2. *Two triples (λ, γ, A) and (λ', γ', A') represent the same elasticity tensor, from the domain of the same local parametrization, if and only if $\lambda = \lambda'$ and*

$$\mathcal{A}(\gamma, A)\,\mathcal{A}^{-1}\left(\gamma', A'\right) \in G_C. \tag{5.9}$$

This is a reformulation of Proposition 1.

We will use the remainder of this section to describe the class of parameters we associate with an elasticity tensor in more detail. As discussed above, each elasticity tensor is described by its six ordered eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_6$ and the corresponding normalized eigentensors $\omega_1, \ldots, \omega_6$. This description is not unique: the eigentensors are given up to the multiplication by elements from G_C and/or by -1. The set of the six orthonormal eigentensors can be parametrized by the Euler angles that define a basis in which ω_1 is in a diagonal form, its two eigenvalues γ_1, γ_2, and the ten parameters $\gamma_3, \ldots, \gamma_{12}$ that describe the orientation of the remaining ωs with respect to ω_1. The Euler angles are not unique, they are given up to rotations from the symmetry group $G_{\omega_1} \subset SO(3)$ of ω_1.

In the case when the eigenvalues λ_α are distinct, the normalized eigentensors are given up to the multiplication by -1. The space of elasticity tensors with distinct eigenvalues can be parametrized by:

$$
\begin{aligned}
&\{\lambda_1,\dots,\lambda_6\}\, SO\,(3)\,/G_{\omega_1}\\
&\quad\times\left\{(\gamma_1,\gamma_2,x_1,0,0,0)\,|\gamma_1^2+\gamma_2^2\le 1, (\gamma_1,\gamma_2,0,0,0,0)\sim-(\gamma_1,\gamma_2,0,0,0,0)\right\}\\
&\quad\times\left\{\,(\gamma_3,\gamma_4,x_2,x_3,\gamma_7,\gamma_8)\,|\gamma_3^2+\gamma_4^2+\gamma_7^2+\gamma_8^2+x_2^2\le 1,\right.\\
&\qquad\qquad\qquad\left.(\gamma_3,\gamma_4,x_2,0,\gamma_7,\gamma_8)\sim-(\gamma_3,\gamma_4,x_2,0,\gamma_7,\gamma_8)\,\right\}\\
&\quad\times\left\{\,(\gamma_5,x_4,x_5,x_6,\gamma_9,\gamma_{10})\,|\gamma_5^2+x_4^2+x_5^2+\gamma_9^2+\gamma_{10}^2\le 1,\right.\\
&\qquad\qquad\qquad\left.(\gamma_5,x_4,x_5,0,\gamma_9,\gamma_{10})\sim-(\gamma_5,x_4,x_5,0,\gamma_9,\gamma_{10})\,\right\}\\
&\quad\times\left\{\,(x_7,x_8,x_9,x_{10},\gamma_{11},\gamma_{12})\,|x_7^2+x_8^2+x_9^2+\gamma_{11}^2+\gamma_{12}^2\le 1,\right.\\
&\qquad\qquad\qquad\left.(x_7,x_8,x_9,0,\gamma_{11},\gamma_{12})\sim-(x_7,x_8,x_9,0,\gamma_{11},\gamma_{12})\,\right\}\\
&\quad\times\left\{\,(x_{11},x_{12},x_{13},x_{14},x_{15},\gamma_6)\,|x_{11}^2+x_{12}^2+x_{13}^2+x_{14}^2+\gamma_6^2\le 1,\right.\\
&\qquad\qquad\qquad\left.(x_{11},x_{12},x_{13},x_{14},0,\gamma_6)\sim-(x_{11},x_{12},x_{13},x_{14},0,\gamma_6)\,\right\},
\end{aligned}
$$

where $x_1,\dots,x_{14}$ are determined by the orthonormality of ωs and the last four braces represent parametrization of $\omega_2,\omega_3,\omega_4$ and ω_5. From this expression we can see that parameters λ and γ parametrize orbits of $SO\,(3)\,/G_{\omega_1}$. To parametrize the orbits of $SO\,(3)$, we need to identify parameters which describe ω that differ by action of G_{ω_1}. To do so, we consider the case when the eigenvalues of ω_1 are all distinct. In such a case,

$$G_{\omega_1}=\{A=\mathrm{diag}(\pm1,\pm1,\pm1)|\ \det A=1\}\,,$$

and we have to identify

$$
\begin{aligned}
&(\gamma_3,\gamma_4,x_2,x_3,\gamma_7,\gamma_8)\sim(-\gamma_3,-\gamma_4,x_2,-x_3,-\gamma_7,-\gamma_8)\\
&\qquad\sim(\gamma_3,-\gamma_4,-x_2,-x_3,\gamma_7,\gamma_8)\sim(-\gamma_3,\gamma_4,x_2,-x_3,-\gamma_7,-\gamma_8)
\end{aligned}
$$

for ω_2 and similarly for the rest of ωs.

This section completes the description of a local parametrization of the space of orbits of elasticity tensors.

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