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On the exponential decay for viscoelastic mixtures

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This paper concerns the study of mixtures composed of a thermoelastic solid and a viscous fluid. For these mixtures, the dissipation effects are connected with the viscosity rate of one constituent and with the relative velocity vector. Using the time-weighted surface power method, associated with the linear process, we obtain some spatial decay estimates, characterized by time–independent and time–dependent decay rates, respectively. The first type of estimate is appropriate for large values of time, while the other is useful for short values of the same variable.

Key words: mixtures, viscoelastic materials, spatial behaviour, positive definite energy.

1. Introduction

The theory concerning thermomechanical mixtures can be found in works by Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3, 4], Green and Naghdi [5, 6], Müller [7], Bowen and Wiese [8], Bowen [9], Atkin and Craine [10, 11], Bedford and Drumheller [12] and also in books by Samohyl [13] and Rajagopal and Tao [14]. Moreover, Steel presents, in [15], a linearized version of the above-mentioned theory for an isotropic mixture of two solids, studying the propagation of plane waves. While an Eulerian description is natural for mixtures containing fluids or gases as constituents, a Lagrangian approach is suitable for mixtures in which one component is a solid. The latter description is presented by Bedford and Stern [16, 17]. Following them, Ieşan derives, in [18], a theory for binary mixtures of thermoelastic solids and establishes, in [19], a counterpart of the Boussinesq–Somigliana–Galerkin solution, obtaining the fundamental solutions in the equilibrium theory of homogeneous and isotropic mixtures, under the hypotheses of positive definiteness of the internal energy density. Moreover, Ieşan proves the existence and uniqueness theorems. Uniqueness results in the dynamical theory were also established in [20, 21].

It is possible to study the classical Kelvin–Voigt viscoelastic model considering a mixture made of an elastic solid and a viscous fluid. In [22], Quintanilla studies the linear problem of thermomechanical deformations for the previous model. Taking into account the results of linear operators semigroup theory, the existence of solution is proved. Ieşan [23] studies, through a Lagrangian description, a viscoelastic mixture resulting from a combination of a porous elastic solid with a viscous fluid. He establishes basic equations for both the nonlinear and linear theories.

Furthermore, we have to remark that Saint–Venant's principle has a central role in many theoretical and applied questions of elasticity. An important review of the research on spatial decay for solutions of time-dependent problems is given by Horgan and Knowles [24], Horgan [25, 26] and Chirita [27]. Although it is possible to cite several papers regarding the subject in concern, such as [28–31] we have to remark that, in [32], Ciarletta and Chirita obtain, through an innovative method based on a time-weighted surface power function, a domain of influence in linear elastodynamics and viscoelastodynamics. They also get spatial decay estimates with time-independent decay rate inside the domain of influence. In the context of linear thermoelasticity, they also obtain spatial estimates characterized by time-independent as well as time-dependent decay and growth rates. Following [32], Ciarletta and Passarella [33] establish, for a binary mixture, a spatial decay estimate of Saint–Venant type, with time-independent decay rate for the inside of the domain of influence, in order to obtain a precise determination of it.

In this article, using the time-weighted surface power function method, with regard to viscoelastic mixtures, we describe the spatial behaviour of solutions of given data in a certain time interval, also considering the heat conduction effects. In Sec. 2, we show the linear theory connected with the problem in concern using a Lagrangian description while, in Sec. 3, we derive the first-order differential inequality satisfied by an appropriate time-weighted surface power function; then, we obtain an exponential spatial decay estimate of Saint–Venant type for bounded bodies, whereas we arrive at an alternative estimate of Phragmén–Lindelöf type for unbounded bodies. In both cases, these estimates are characterized by a time–independent decay rate. In Sec. 4, introducing another appropriate function but applying a method similar to the one used in Sec. 3, we establish spatial decay estimates with time-dependent decay rates for bounded and unbounded bodies. The results obtained in Sec. 3 are suitable for large values of time, while estimates of Sec. 4 are appropriate for short values of time.

With respect to classical methods used to estimate decay of solutions, the time–weighted surface power function method refers to mathematical techniques useful to obtain explicit expressions for the decay rate in terms of single constitutive parameters. This gives a possibility to regulate the velocity of decay in function of a single component. In particular, we can affirm that the paper in concern is interesting, considering its possible applications to the dissipation of acoustic and seismic energies.

2. Statement of the problem

Let B be a (bounded or unbounded) regular region of $\mathbb{R}^3$, occupied by a mixture of two interacting materials s_1 and s_2, in a given reference configuration. As described by Ieşan [23], we will assume that s_1 is a viscous fluid, while s_2 represents an elastic solid.

The motion of the body is referred to a fixed orthonormal frame in $\mathbb{R}^3$. We will denote the components of tensors of order $p \geq 1$ by Latin subscripts, ranging over $\{1,2,3\}$. Summation over repeated subscripts will be implied. Superposed dot or subscript k preceded by a comma will mean partial derivative with respect to the time variable or to the corresponding coordinate x_k, respectively. Greek indices will range from 1 to 2 and, in this case, summation convention will not be used. In this connection, we will disregard regularity questions, simply understanding a degree of smoothness sufficient to ensure the analysis to be valid.

Following Bedford and Stern [16, 17], we assume that, in the reference configuration, the typical particles of s_1 and s_2 occupy the same position and the mass of each s_α ($\alpha = 1,2$) is conserved. Furthermore, let us suppose that chemical reactions between constituent materials are not possible and that the partial stress tensors and the partial entropy vanish in the natural state.

Let $\mathbf{u}$ and $\mathbf{w}$ be the displacement vector fields of typical particles of s_1 and s_2 at time t, and ϑ be the absolute temperature measured from the one in the reference configuration, T_0.

The linear theory for homogeneous and isotropic mixtures is described by the following system [22, 23]

$$t^{(1)}_{lk,l} - p_k + \varrho_1 f^{(1)}_k = \varrho_1 \ddot{u}_k, \qquad t^{(2)}_{lk,l} + p_k + \varrho_2 f^{(2)}_k = \varrho_2 \ddot{w}_k$$

equations of motion,

$$\varrho_0 T_0 \dot{\eta} = q_{l,l} + \varrho_0 R \qquad \text{equation of energy,}$$

(2.1)
$$\begin{aligned}
t^{(1)}_{lk} &= (\lambda+\nu)e_{rr}\delta_{lk} + 2(\mu+\zeta)e_{lk} + (\alpha+\nu)g_{rr}\delta_{lk} + (2\beta+\zeta)g_{lk} \\
&\quad + (2\gamma+\zeta)g_{kl} - (\varkappa_1+\varkappa_2)\vartheta\delta_{lk} + \lambda^*\dot{e}_{rr}\delta_{lk} + 2\mu^*\dot{e}_{lk},\\
t^{(2)}_{lk} &= \nu e_{rr}\delta_{lk} + 2\zeta e_{kl} + \alpha g_{rr}\delta_{kl} + 2\beta g_{kl} + 2\gamma g_{lk} - \varkappa_2\vartheta\delta_{lk},\\
p_k &= \xi d_k + \xi^*\dot{d}_k + b^*\vartheta_{,k}, \qquad \varrho_0\eta = \varkappa_1 e_{ll} + \varkappa_2 g_{ll} + a\vartheta,\\
q_k &= \kappa\vartheta_{,k} + f^*\dot{d}_k
\end{aligned}$$

constitutive equations,

where δ_{lk} is the Kroneker's delta and

$$e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \qquad g_{kl} = u_{l,k} + w_{k,l}, \qquad d_k = u_k - w_k. \tag{2.2}$$

In these equations, $t_{lk}^{(\alpha)}$ are the components of the partial stress tensors associated with s_α, p_k are the components of the vector field characterizing the mechanical interaction between the constituents, $f_k^{(\alpha)}$ are the components of the body forces per unit mass acting on s_α, ϱ_α is the density of the material s_α in the reference state and $\varrho_0 = \varrho_1 + \varrho_2$. Furthermore, η is the entropy per unit mass of the mixture, q_k are the components of the heat flux vector, R represents the external heat supply per unit mass and unit time, while λ, ν, μ, α, β, ζ, γ, $\varkappa_1$, $\varkappa_2$, ξ, a, κ, λ^*, μ^*, ξ^*, b^*, f^* are constitutive coefficients. Unlike the theories about solid-fluid mixtures studied in [9–14], in the present work the diffusive force p_k depends on d_k, $\dot{d}_k$ and $\vartheta_{,k}$.

We suppose the constituents at the same temperature and that every thermodynamic process that takes place in the considered mixture satisfies the Clausius–Duhem inequality, so that [22, 23]

$$\begin{aligned} &3\lambda^* + 2\mu^* \geq 0, \qquad \mu^* \geq 0, \qquad \kappa \geq 0, \qquad \xi^* \geq 0, \\ &4\kappa\xi^* \geq T_0\left(b^* + \frac{1}{T_0}f^*\right)^2. \end{aligned} \tag{2.3}$$

Defining ℓ and c_1 as (positive) constants with dimensions of length and velocity, respectively, it is possible to introduce the following dimensionless quantities

$$x'_k = \frac{x_k}{\ell}, \qquad t' = \frac{c_1 t}{\ell}, \qquad u'_k = \frac{u_k}{\ell}, \qquad w'_k = \frac{w_k}{\ell}, \qquad \vartheta' = \frac{\vartheta}{T_0}.$$

By extension, subscripts preceded by a comma will now mean partial differentiation with respect to the variable x'_k, while a superposed dot will denote differentiation with respect to the variable t'. For convenience, in what follows, we will assume all variables to be dimensionless, avoiding the use of symbol $'$.

Denoting by

$$(\lambda_0, \nu_0, \mu_0, \zeta_0, \alpha_0, \beta_0, \gamma_0) = \frac{1}{\varrho_1 c_1^2}(\lambda, \nu, \mu, \zeta, \alpha, \beta, \gamma),$$

$$(\varkappa_1^0, \varkappa_2^0) = \frac{T_0}{\varrho_1 c_1^2}(\varkappa_1, \varkappa_2), \qquad (\lambda_0^*, \mu_0^*) = \frac{1}{\ell \varrho_1 c_1}(\lambda^*, \mu^*),$$

$$\xi_0 = \frac{\ell^2\xi}{\varrho_1 c_1^2}, \qquad \xi_0^* = \frac{\ell\xi^*}{\varrho_1 c_1}, \qquad b_0^* = \frac{T_0 b^*}{\varrho_1 c_1^2}, \qquad a_0 = T_0 a,$$

$$K_0 = \frac{\kappa}{\ell a_0 c_1}, \qquad \chi = \frac{T_0 a_0}{\varrho_1 c_1^2}, \qquad f_0^* = \frac{f^*}{T_0 a_0}, \qquad \rho = \frac{\varrho_2}{\varrho_1},$$

$$F_k^{(1)} = \frac{\ell f_k^{(1)}}{c_1^2}, \qquad F_k^{(2)} = \frac{\rho\ell f_k^{(2)}}{c_1^2}, \qquad S = \frac{\ell\varrho_0 R}{T_0 a_0 c_1},$$

and

$$\pi_{lk}^{(1)} = (\lambda_0+\nu_0)e_{rr}\delta_{lk} + 2(\mu_0+\zeta_0)e_{lk} + (\alpha_0+\nu_0)g_{rr}\delta_{lk} + (2\beta_0+\zeta_0)g_{lk} + (2\gamma_0+\zeta_0)g_{kl},$$

$$\pi_{lk}^{(2)} = \nu_0 e_{rr}\delta_{lk} + 2\zeta_0 e_{kl} + \alpha_0 g_{rr}\delta_{kl} + 2\beta_0 g_{kl} + 2\gamma_0 g_{lk}, \tag{2.4}$$

$$\pi_{lk}^* = \lambda_0^*\dot{e}_{rr}\delta_{lk} + 2\mu_0^*\dot{e}_{lk},$$

the system (2.1) reduces to

$$\tilde{\pi}_{lk,l}^{(1)} - \tilde{\Pi}_k + F_k^{(1)} = \ddot{u}_k, \qquad \tilde{\pi}_{lk,l}^{(2)} + \tilde{\Pi}_k + F_k^{(2)} = \rho\ddot{w}_k, \qquad \tilde{Q}_{l,l} - \dot{h} + S = 0, \tag{2.5}$$

where

$$\tilde{\pi}_{lk}^{(1)} = \pi_{lk}^{(1)} + \pi_{lk}^* - (\varkappa_1^0 + \varkappa_2^0)\vartheta\delta_{lk}, \qquad \tilde{\pi}_{lk}^{(2)} = \pi_{lk}^{(2)} - \varkappa_2^0\vartheta\delta_{lk},$$

$$\tilde{\Pi}_k = \xi_0 d_k + \xi_0^*\dot{d}_k + b_0^*\vartheta_{,k}, \qquad \chi h = (\varkappa_1^0 + \varkappa_2^0)u_{l,l} + \varkappa_2^0 w_{l,l} + \chi\vartheta, \tag{2.6}$$

$$\tilde{Q}_k = K_0\vartheta_{,k} + f_0^*\dot{d}_k.$$

Throughout this work, we will assume that

$$\varrho_1 > 0, \qquad \varrho_2 > 0, \qquad \chi > 0; \tag{2.7}$$

furthermore, we will suppose that the following quadratic forms

$$\mathcal{E}_0 = \frac{1}{2}\xi_0 d_k d_k,$$

$$\mathcal{E}_1 = \frac{1}{2}\lambda_0 e_{ll}e_{kk} + \mu_0 e_{lk}e_{lk} + \frac{1}{2}\alpha_0 g_{ll}g_{kk} + \beta_0 g_{lk}g_{kl} + \gamma_0 g_{lk}g_{lk} + \nu_0 e_{ll}g_{kk} + 2\zeta_0 e_{lk}g_{lk} \tag{2.8}$$

are positive definite. The only eigenvalue associated with $\mathcal{E}_0$ is $\varsigma_0 = \xi_0$, while eigenvalues associated with $\mathcal{E}_1$ are

$$\varsigma_1 = \gamma_0 - \beta_0, \qquad \varsigma_{2,3} = \frac{1}{2}\left\{\beta_0 + \gamma_0 + \mu_0 \pm \sqrt{(\beta_0 + \gamma_0 - \mu_0)^2 + 4\zeta_0^2}\right\},$$

$$\varsigma_{4,5} = \frac{1}{2}\left\{\beta_0 + \gamma_0 + 2\mu_0 \pm \sqrt{(\beta_0 + \gamma_0 - 2\mu_0)^2 + 8\zeta_0^2}\right\},$$

$$\varsigma_{6,7} = \frac{1}{4}\Big\{3\lambda_0 + 2\mu_0 + 3\alpha_0 + 2\beta_0 + 2\gamma_0 \pm \sqrt{[3\lambda_0 + 2\mu_0 - (3\alpha_0 + 2\beta_0 + 2\gamma_0)]^2 + 4(3\nu_0 + 2\zeta_0)^2}\Big\}. \tag{2.9}$$

We remark that the isothermal internal energy density is $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$. It is possible to establish that, for an isotropic mixture, the conditions necessary and sufficient to ensure $\mathcal{E}$ to be positive in the variables d_k, e_{lk} and g_{kl} are

$$\begin{gathered} \xi_0 > 0, \qquad |\beta_0| < \gamma_0, \qquad \mu_0 > 0, \qquad 3\lambda_0 + 2\mu_0 > 0, \\ 3\alpha_0 + 2\beta_0 + 2\gamma_0 > 0, \qquad (\beta_0 + \gamma_0)\mu_0 > {\zeta_0}^2, \\ (3\alpha_0 + 2\beta_0 + 2\gamma_0)(3\lambda_0 + 2\mu_0) > (3\nu_0 + 2\zeta_0)^2. \end{gathered} \tag{2.10}$$

The inequalities (2.3) can be rewritten as follows

$$\begin{gathered} 3\lambda_0^* + 2\mu_0^* \geq 0, \qquad \mu_0^* \geq 0, \qquad K_0 \geq 0, \qquad \xi_0^* \geq 0, \\ 4\chi K_0 \xi_0^* \geq (b^* + \chi f_0^*)^2. \end{gathered} \tag{2.11}$$

Now, Eqs. (2.4)–(2.6) lead to a system formulated in terms of displacements and temperature

$$\begin{aligned} & A_1 u_{k,ll} + A_2 u_{l,lk} + B_1 w_{k,ll} + B_2 w_{l,lk} - \xi_0(u_k - w_k) \\ & \qquad - M_1^0 \vartheta_{,k} - \xi_0^*(\dot{u}_k - \dot{w}_k) + \mu_0^* \dot{u}_{k,ll} + (\lambda_0^* + \mu_0^*)\dot{u}_{l,lk} + F_k^{(1)} = \ddot{u}_k, \\ & B_1 u_{k,ll} + B_2 u_{l,lk} + C_1 w_{k,ll} + C_2 w_{l,lk} + \xi_0(u_k - w_k) \\ & \qquad - M_2 \vartheta_{,k} + \xi_0^*(\dot{u}_k - \dot{w}_k) + F_k^{(2)} = \rho \ddot{w}_k, \\ & \chi K_0 \vartheta_{,ll} - \big[(\varkappa_1^0 + \varkappa_2^0)\dot{u}_{l,l} + \varkappa_2^0 \dot{w}_{l,l}\big] - \chi\dot{\vartheta} + \chi S = 0, \end{aligned} \tag{2.12}$$

where

$$A_1 = \mu_0 + 2\beta_0 + 2\zeta_0, \qquad A_2 = \lambda_0 + \mu_0 + \alpha_0 + 2\nu_0 + 2\gamma_0 + 2\zeta_0,$$
$$B_1 = 2\gamma_0 + \zeta_0, \qquad B_2 = \alpha_0 + \nu_0 + 2\beta_0 + \zeta_0,$$
$$C_1 = 2\beta_0, \qquad C_2 = 2\gamma_0 + \alpha_0,$$
$$M_1^0 = \varkappa_1^0 + \varkappa_2^0 + b_0^*, \qquad M_2^0 = \varkappa_2^0 - b_0^*.$$

Considering the isothermal theory, we obtain equations describing the behaviour of a linear Kelvin–Voigt material, modelled as a mixture, i.e.

$$A_1 u_{k,ll} + A_2 u_{l,lk} + B_1 w_{k,ll} + B_2 w_{l,lk} - \xi_0 d_k - \xi_0^* \dot{d}_k + \mu_0^* \dot{u}_{k,ll} + (\lambda_0^* + \mu_0^*)\dot{u}_{l,lk} + F_k^{(1)} = \ddot{u}_k,$$
$$B_1 u_{k,ll} + B_2 u_{l,lk} + C_1 w_{k,ll} + C_2 w_{l,lk} + \xi_0 d_k + \xi_0^* \dot{d}_k + F_k^{(2)} = \rho \ddot{w}_k.$$

We consider the problem $\mathcal{P}$ defined by Eqs. (2.12) with the following initial conditions

$$(2.13) \qquad u_k = u_k^0, \quad \dot{u}_k = \dot{u}_k^0, \quad w_k = w_k^0, \quad \dot{w}_k = \dot{w}_k^0, \quad \vartheta = \vartheta^0, \quad \text{on} \quad \bar{B} \times \{0\}$$

and the following boundary conditions (Dirichlet problem)

$$(2.14) \qquad u_k = \tilde{u}_k, \quad w_k = \tilde{w}_k, \quad \vartheta = \tilde{\vartheta}, \quad \text{on} \quad \partial B \times [0, +\infty),$$

where $\bar{B}$ and ∂B are, respectively, the closure and the smooth boundary of B. The right-hand terms of Eqs. (2.13), (2.14) stand for (sufficiently smooth) assigned fields; these are, along with $F_k^{(\alpha)}$ and S, external data of the problem in concern. Fixed $\tau_0 > 0$, we introduce the support $\hat{D}_{\tau_0}$ of external data on the time interval $[0, \tau_0]$. For convenience, we assume that $\hat{D}_{\tau_0}$ is a bounded regular region of $\bar{B}$ and we introduce the following sets

$$D_r = \left\{ \mathbf{x} \in \bar{B} : \ \overline{S(\mathbf{x}, r)} \cap \hat{D}_{\tau_0} \neq \emptyset \right\}, \qquad B_r = B \backslash D_r, \qquad B(r_1, r_2) = B_{r_2} \backslash B_{r_1},$$

where $r \geq 0$, $r_2 \leq r_1$ and $\overline{S(\mathbf{x}, r)}$ is the closed ball with radius r and center at $\mathbf{x}$. Then, we denote by S_r the subsurface of ∂B_r contained inside B, and whose outward unit normal vector is directed towards the exterior of D_r.

3. A time-weighted surface power function and related estimates

In this section, we describe the spatial behaviour of solutions, in the time interval $[0,\tau_0]$, for bounded or unbounded bodies under hypotheses (2.7), (2.10), (2.11).

Let us start from introducing the following forms

$$\mathcal{H}_0 = \lambda_0^* \dot e_{ll}\dot e_{kk} + 2\mu_0^* \dot e_{lk}\dot e_{lk}, \quad \mathcal{H}_1 = \xi_0^* \dot d_k \dot d_k + \chi K_0 \vartheta_{,k}\vartheta_{,k} + (b_0^* + \chi f_0^*)\,\vartheta_{,k}\dot d_k. \tag{3.1}$$

Let σ_m^*, ς_m^* and σ_M^*, ς_M^* be the smallest and greatest eigenvalues associated with $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, that is

$$\sigma_m^* = \min\{2\mu_0^*, 3\lambda_0^* + 2\mu_0^*\}, \quad \varsigma_m^* = \frac{1}{2}\left[\chi K_0 + \xi_0^* - \sqrt{(\chi K_0 - \xi_0^*)^2 + (b_0^* + \chi f_0^*)^2}\right],$$

$$\sigma_M^* = \max\{2\mu_0^*, 3\lambda_0^* + 2\mu_0^*\}, \quad \varsigma_M^* = \frac{1}{2}\left[\chi K_0 + \xi_0^* + \sqrt{(\chi K_0 - \xi_0^*)^2 + (b_0^* + \chi f_0^*)^2}\right].$$

Eqs. (2.7), (2.11) imply that all eigenvalues of $\mathcal{H}_0$ and $\mathcal{H}_1$ are positive. Consequently, $\mathcal{H}_0$ and $\mathcal{H}_1$ are positive definite and it results that

$$\begin{gathered} \sigma_m^* \dot e_{lk}\dot e_{lk} \le \mathcal{H}_0 \le \sigma_M^* \dot e_{lk}\dot e_{lk}, \\ \varsigma_m^* (\dot d_k \dot d_k + \vartheta_{,k}\vartheta_{,k}) \le \mathcal{H}_1 \le \varsigma_M^* (\dot d_k \dot d_k + \vartheta_{,k}\vartheta_{,k}). \end{gathered} \tag{3.2}$$

Corresponding to a (regular) solution $\mathbf{U} = \{\mathbf{u}, \mathbf{w}, \vartheta\}$ of the initial-boundary value problem $\mathcal{P}$, we define the following time-weighted surface power function

$$\Gamma(r,t) = -\int_0^t \int_{S_r} e^{-\omega s} \left[\tilde\pi_{lk}^{(1)}(s)\dot u_k(s) + \tilde\pi_{lk}^{(2)}(s)\dot w_k(s) + \chi\vartheta(s)\tilde Q_l(s)\right] n_l\, da\, ds, \tag{3.3}$$

where ω is a prescribed strictly positive parameter. At a fixed $t \in [0,\tau_0]$, the function $\Gamma(r,t)$ is non-increasing with respect to r. In fact, using the divergence theorem, we obtain

$$\Gamma(r_1,t) - \Gamma(r_2,t) = -\int_0^t \int_{B(r_1,r_2)} e^{-\omega s}\Big[\tilde\pi_{lk}^{(1)}(s)\dot u_k(s) + \tilde\pi_{lk}^{(2)}(s)\dot w_k(s) \\ + \chi\vartheta(s)\tilde Q_l(s)\Big]_{,l} dV\, ds \tag{3.4}$$

for $r_2 < r_1$. In view of relations (2.5) and defining the kinetic energy as

$$\mathcal{T} = \frac{1}{2}(\dot{u}_k\dot{u}_k + \rho\dot{w}_k\dot{w}_k), \tag{3.5}$$

we have

$$\begin{aligned}\tilde{\pi}^{(1)}_{lk}(s)\dot{u}_{k,l}(s) + \tilde{\pi}^{(2)}_{lk}(s)\dot{w}_{k,l}(s) + \tilde{\Pi}_k(s)\dot{d}_k(s) + \chi\dot{h}(s)\vartheta(s) \\ = \left[\tilde{\pi}^{(1)}_{lk}(s)\dot{u}_k(s) + \tilde{\pi}^{(2)}_{lk}(s)\dot{w}_k(s) + \chi\vartheta(s)\tilde{Q}_l(s)\right]_{,l} - \chi K_0\vartheta_{,k}(s)\vartheta_{,k}(s) \\ - \chi f_0^*\dot{d}_k(s)\vartheta_{,k}(s) + \dot{u}_k(s)F^{(1)}_k(s) + \rho\dot{w}_k(s)F^{(2)}_k(s) + \chi\vartheta(s)S(s) - \frac{\partial}{\partial s}\mathcal{T}(s).\end{aligned} \tag{3.6}$$

On the other hand, from Eqs. (2.4), (2.6), (2.8), (3.1), we get

$$\begin{aligned}\tilde{\pi}^{(1)}_{lk}(s)\dot{u}_{k,l}(s) + \tilde{\pi}^{(2)}_{lk}(s)\dot{w}_{k,l}(s) + \tilde{\Pi}_k(s)\dot{d}_k(s) + \chi\dot{h}(s)\vartheta(s) \\ = \frac{\partial}{\partial s}\left[\mathcal{E}_0(s) + \mathcal{E}_1(s) + \frac{\chi}{2}\vartheta^2(s)\right] + \mathcal{H}_0(s) + \mathcal{H}_1(s) \\ - \chi K_0\vartheta_{,k}(s)\vartheta_{,k}(s) - \chi f_0^*\dot{d}_k(s)\vartheta_{,k}(s).\end{aligned} \tag{3.7}$$

If we take into account Eqs. (3.4)–(3.7) and consider that external data vanish on B_r and S_r, we deduce

$$\begin{aligned}\Gamma(r_1,t) - \Gamma(r_2,t) \\ = -\int_0^t \int_{B(r_1,r_2)} e^{-\omega s}\left\{\frac{\partial}{\partial s}\left[\mathcal{T}(s) + \mathcal{E}_0(s) + \mathcal{E}_1(s) + \frac{\chi}{2}\vartheta^2(s)\right]\right. \\ \left. + \mathcal{H}_0(s) + \mathcal{H}_1(s)\right\} dV ds;\end{aligned} \tag{3.8}$$

therefore, by means of an integration by parts, we conclude that, for $r_2 < r_1$, it results

$$\begin{aligned}\Gamma(r_1,t) - \Gamma(r_2,t) = -\int_{B(r_1,r_2)} e^{-\omega t}\left[\mathcal{T}(t) + \mathcal{E}_0(t) + \mathcal{E}_1(t) + \frac{\chi}{2}\vartheta^2(t)\right] dV \\ - \int_0^t \int_{B(r_1,r_2)} e^{-\omega s}\left\{\omega\left[\mathcal{T}(s) + \mathcal{E}_0(s) + \mathcal{E}_1(s) + \frac{\chi}{2}\vartheta^2(s)\right] + \mathcal{H}_0(s)\right. \\ \left. + \mathcal{H}_1(s)\right\} dV ds.\end{aligned} \tag{3.9}$$

Through hypotheses (2.7), (2.10), (2.11) and Eq. (3.9), we prove that $\Gamma(r,t)$ is a non-increasing function of r, for all fixed $t \in [0,\tau_0]$.

For a bounded body B, we can remark that

$$\Gamma(L,t)=0, \quad \forall t \in [0,\tau_0], \tag{3.10}$$

where $L = \max_{\mathbf{x}\in \bar{B}} \min_{\mathbf{y}\in \widehat{D}_T^*} \sqrt{|\mathbf{x}-\mathbf{y}|} < \infty$. Putting $r_1 = L$ and $r_2 = r$ in Eq. (3.9), Eq. (3.10) shows that $\Gamma(r,t)$ is a measure

$$\Gamma(r,t) \geq 0 \quad \text{for all} \quad r \in [0,L],\ t \in [0,\tau_0]; \tag{3.11}$$

in particular, it is

$$\Gamma(r,t) = \int_{B_r} e^{-\omega t}\left[\mathcal{T}(t)+\mathcal{E}_0(t)+\mathcal{E}_1(t)+\frac{\chi}{2}\vartheta^2(t)\right] dV$$
$$+\int_0^t\int_{B_r} e^{-\omega s}\left\{\omega\left[\mathcal{T}(s)+\mathcal{E}_0(s)+\mathcal{E}_1(s)+\frac{\chi}{2}\vartheta^2(s)\right]+\mathcal{H}_0(s)+\mathcal{H}_1(s)\right\} dV ds.$$

For an unbounded body B, $\Gamma(r,t)$ is still non-increasing with respect to r and, at a fixed $t \in [0,\tau_0]$, it is

$$\begin{aligned} &\text{either} \quad \Gamma(r,t)\geq 0 \quad \forall r \geq 0, \qquad \text{or} \\ &\exists\ \bar{r}_t \geq 0 \quad \text{such that} \quad \Gamma(r,t) < 0 \quad \forall r \geq \bar{r}_t. \end{aligned} \tag{3.12}$$

In the next part of the work, we will need the following lemma.

LEMMA 1. *Let* $\mathbf{U}$ *be a solution of the initial-boundary-value problem* $\mathcal{P}$ *and let* $\widehat{D}_T$ *be the bounded support of external data, in the time interval* $[0,\tau_0]$. *Let the hypotheses (2.7), (2.10), (2.11) be true. Function* $\Gamma(r,t)$ *satisfies the following first–order differential inequality*

$$\frac{\omega}{\upsilon_1}|\Gamma(r,t)| + \frac{\partial}{\partial r}\Gamma(r,t) \leq 0, \qquad \textit{for any} \qquad 0 \leq r,\ t \in [0,\tau_0], \tag{3.13}$$

where

$$\upsilon_1 = \sqrt{\frac{6\varsigma_M + \sigma_M^*\omega\,(1+\epsilon_0)}{2\min\{1,\rho\}}} \tag{3.14}$$

and ϵ_0 *is the positive root of the algebraic equation*

$$\epsilon_0^2 + \left(1+\frac{6\varsigma_M}{\sigma_M^*\omega} - \frac{6[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2]}{\chi\sigma_M^*\omega} - \frac{2\chi\max\{f_0^{*2},K_0^2\}\min\{1,\rho\}}{\varsigma_m^*\sigma_M^*}\right)\epsilon_0$$
$$-\left(1+\frac{6\varsigma_M}{\sigma_M^*\omega}\right)\frac{6[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2]}{\chi\sigma_M^*\omega} = 0. \tag{3.15}$$

P r o o f. Eq. (3.9) directly leads to

$$\frac{\partial \Gamma}{\partial r}(r,t) = -\int\limits_{S_r} e^{-\omega t}\left[\mathcal{T}(t)+\mathcal{E}_0(t)+\mathcal{E}_1(t)+\frac{\chi}{2}\vartheta^2(t)\right] da \tag{3.16}$$

$$-\int\limits_0^t\int\limits_{S_r} e^{-\omega s}\left\{\omega\left[\mathcal{T}(s)+\mathcal{E}_0(s)+\mathcal{E}_1(s)+\frac{\chi}{2}\vartheta^2(s)\right]+\mathcal{H}_0(s)+\mathcal{H}_1(s)\right\} da ds.$$

In order to estimate $\Gamma(r,t)$ in terms of $\dfrac{\partial \Gamma}{\partial r}$ (r,t), we rewrite π^*_{lk} and $\mathcal{H}_0$ as

$$\pi^*_{lk} = A_{rskl}\dot{e}_{rs}, \qquad \mathcal{H}_0 = A_{rskl}\dot{e}_{rs}\dot{e}_{lk}, \tag{3.17}$$

with

$$A_{rskl} = \lambda^*_0\delta_{rs}\delta_{lk} + 2\mu^*_0\delta_{rl}\delta_{sk}.$$

Through Schwarz's inequality and Eqs. $(3.2)_1$, (3.17), we obtain

$$\begin{aligned}\pi^*_{lk}\pi^*_{lk} &= A_{rskl}\dot{e}_{rs}\pi^*_{lk} \le (A_{rskl}\dot{e}_{rs}\dot{e}_{kl})^{1/2}(A_{rskl}\pi^*_{rs}\pi^*_{lk})^{1/2}\\ &\le (A_{rskl}\dot{e}_{rs}\dot{e}_{kl})^{1/2}(\sigma^*_M\pi^*_{lk}\pi^*_{lk})^{1/2}\end{aligned}$$

and we deduce

$$\pi^*_{lk}\pi^*_{lk} \le \sigma^*_M\mathcal{H}_0. \tag{3.18}$$

Now, for each couple of second-order tensors $\{\omega^{(1)}_{lk},\omega^{(2)}_{lk}\}$, we introduce the vector space $\mathcal{V}$ of all vector fields such that

$$\mathbf{E} \equiv \{E_{11},E_{22},E_{33},E_{12},E_{13},E_{23},G_{11},G_{22},G_{33},G_{12},G_{13},G_{23},G_{21},G_{31},G_{32}\},$$

with

$$E_{lk} = E_{kl} = \frac{1}{2}\left(\omega^{(1)}_{lk}+\omega^{(1)}_{kl}\right), \qquad G_{lk} = \omega^{(1)}_{kl}+\omega^{(2)}_{lk}.$$

It results

$$E_{lk}E_{lk} = E_{lk}\omega^{(1)}_{lk} \le \frac{1}{2}\left[\omega^{(1)}_{lk}\omega^{(1)}_{lk}+\frac{1}{2}\left(\omega^{(1)}_{lk}\omega^{(1)}_{lk}+\omega^{(1)}_{kl}\omega^{(1)}_{kl}\right)\right] = \omega^{(1)}_{lk}\omega^{(1)}_{lk}. \tag{3.19}$$

Moreover, for each couple of second-order tensors $\{L_{lk},F_{lk}\}$, using the well-known property

$$(L_{lk}+F_{lk})(L_{lk}+F_{lk}) \le (1+\epsilon)L_{lk}L_{lk} + \left(1+\frac{1}{\epsilon}\right)F_{lk}F_{lk}, \tag{3.20}$$

we get

$$G_{lk}G_{lk} = (\omega^{(1)}_{kl} + \omega^{(2)}_{lk})(\omega^{(1)}_{kl} + \omega^{(2)}_{lk}) \le 2(\omega^{(1)}_{lk}\omega^{(1)}_{lk} + \omega^{(2)}_{lk}\omega^{(2)}_{lk}). \tag{3.21}$$

Let us define the functional

$$\begin{aligned}\mathcal{F}[\mathbf{E},\bar{\mathbf{E}}] \equiv \frac{1}{2}\lambda_0 E_{kk}\bar{E}_{ll} + \mu_0 E_{lk}\bar{E}_{lk} + \frac{1}{2}\alpha_0 G_{kk}\bar{G}_{ll} + \beta_0 G_{kl}\bar{G}_{lk} \\ + \gamma_0 G_{lk}\bar{G}_{lk} + \frac{1}{2}\nu_0\left(E_{kk}\bar{G}_{ll} + G_{kk}\bar{E}_{ll}\right) + \zeta_0(E_{lk}\bar{G}_{lk} + G_{lk}\bar{E}_{lk}\right),\end{aligned}$$

with $\bar{\mathbf{E}} \equiv \{\bar{E}_{11}, ..., \bar{E}_{23}, \bar{G}_{11}, ..., \bar{G}_{32}\} \in \mathcal{V}$. Clearly, it follows that

$$\mathcal{F}[\mathbf{E},\bar{\mathbf{E}}] = \mathbf{E}\cdot\mathcal{B}\bar{\mathbf{E}}, \qquad \mathcal{F}[\mathbf{E},\mathbf{E}] \le \varsigma_M(E_{lk}E_{lk} + G_{lk}G_{lk}), \tag{3.22}$$

where $\mathcal{B}$ is the symmetric matrix associated with the positive definite quadratic form $\mathcal{E}_1$ and ς_M is the greatest between eigenvalues of $\mathcal{B}$ defined in (2.9). Equations (3.19), (3.21), (3.22) imply that

$$\mathcal{F}[\mathbf{E},\mathbf{E}] \le 3\varsigma_M(\omega^{(1)}_{lk}\omega^{(1)}_{lk} + \omega^{(2)}_{lk}\omega^{(2)}_{lk}). \tag{3.23}$$

Denoted by

$$\begin{aligned}\hat{s}^{(1)}_{lk} &= (\lambda_0 + \nu_0)E_{rr}\delta_{lk} + 2(\mu_0 + \zeta_0)E_{lk} + (\alpha_0 + \nu_0)G_{rr}\delta_{lk} \\ &\qquad + (2\beta_0 + \zeta_0)G_{lk} + (2\gamma_0 + \zeta_0)G_{kl}, \\ \hat{s}^{(2)}_{lk} &= \nu_0 E_{rr}\delta_{lk} + 2\zeta_0 E_{kl} + \alpha_0 G_{rr}\delta_{kl} + 2\beta_0 G_{kl} + 2\gamma_0 G_{lk},\end{aligned} \tag{3.24}$$

we consider the corresponding vector field of $\mathcal{V}$

$$\mathbf{s} \equiv \{s^{(1)}_{lk}, s^{(2)}_{lk}\}, \quad \text{with} \quad s^{(1)}_{lk} = s^{(1)}_{kl} = \frac{1}{2}\left(\hat{s}^{(1)}_{lk} + \hat{s}^{(1)}_{kl}\right), \quad s^{(2)}_{lk} = \hat{s}^{(1)}_{kl} + \hat{s}^{(2)}_{lk}. \tag{3.25}$$

From Eqs. (3.23)–(3.25) and using Schwarz's inequality, we have

$$\begin{aligned}\hat{s}^{(1)}_{lk}\hat{s}^{(1)}_{lk} + \hat{s}^{(2)}_{lk}\hat{s}^{(2)}_{lk} = 2\mathcal{F}[\mathbf{E},\mathbf{s}] \le [2\mathcal{F}[\mathbf{E},\mathbf{E}]]^{1/2}[2\mathcal{F}[\mathbf{s},\mathbf{s}]]^{1/2} \\ \le [2\mathcal{F}[\mathbf{E},\mathbf{E}]]^{1/2}[3\varsigma_M(\hat{s}^{(1)}_{lk}\hat{s}^{(1)}_{lk} + \hat{s}^{(2)}_{lk}\hat{s}^{(2)}_{lk})]^{1/2},\end{aligned}$$

so that

$$\hat{s}^{(1)}_{lk}\hat{s}^{(1)}_{lk} + \hat{s}^{(2)}_{lk}\hat{s}^{(2)}_{lk} \le 6\varsigma_M\mathcal{F}[\mathbf{E},\mathbf{E}]. \tag{3.26}$$

For $\omega_{lk}^{(1)} = u_{l,k}$, $\omega_{lk}^{(2)} = w_{l,k}$, $E_{lk} = e_{lk}$, $G_{lk} = g_{lk}$, $\mathcal{F}[\mathbf{E},\mathbf{E}]$ reduces to $\mathcal{E}_1$ and relations (3.24) become $(2.4)_{1,2}$. Consequently, Eq. (3.26) implies

$$\pi_{lk}^{(1)}\pi_{lk}^{(1)} + \pi_{lk}^{(2)}\pi_{lk}^{(2)} \leq 6\varsigma_M \mathcal{E}_1 \leq 6\varsigma_M(\mathcal{E}_0 + \mathcal{E}_1). \tag{3.27}$$

Furthermore, Eqs. (2.6), (3.20) and the obvious relation $1+\dfrac{1}{\epsilon_0} > 1$ lead, for any $\epsilon > 0$, to

$$\tilde{\pi}_{lk}^{(1)}\tilde{\pi}_{lk}^{(1)} + \tilde{\pi}_{lk}^{(2)}\tilde{\pi}_{lk}^{(2)} \leq (1+\epsilon)(\pi_{lk}^{(1)}\pi_{lk}^{(1)} + \pi_{lk}^{(2)}\pi_{lk}^{(2)}) \tag{3.28}$$
$$+\left(1+\frac{1}{\epsilon}\right)(1+\epsilon_0)\pi_{lk}^{*}\pi_{lk}^{*} + 3\left(1+\frac{1}{\epsilon}\right)\left(1+\frac{1}{\epsilon_0}\right)\left[(\varkappa_1^0+\varkappa_2^0)^2 + (\varkappa_2^0)^2\right]\vartheta^2.$$

Using Schwarz's and arithmetic–geometric mean inequalities in conjunction with Eqs. (3.5), (3.18), (3.27), (3.28), we arrive, for any $\epsilon_1 > 0$, at

$$\left|\tilde{\pi}_{lk}^{(1)}\dot{u}_k n_l + \tilde{\pi}_{lk}^{(2)}\dot{w}_k n_l\right| \leq \epsilon_1 \mathcal{T} + \frac{3\varsigma_M\left[\mathcal{E}_0+\mathcal{E}_1\right]}{\epsilon_1 \min\{1,\rho\}}(1+\epsilon) \tag{3.29}$$
$$+\frac{3\left[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2\right]}{\epsilon_1\min\{1,\rho\}\chi}\left(1+\frac{1}{\epsilon}\right)\left(1+\frac{1}{\epsilon_0}\right)\frac{\chi}{2}\vartheta^2$$
$$+\frac{\sigma_M^*\mathcal{H}_0}{2\epsilon_1\min\{1,\rho\}}\left(1+\frac{1}{\epsilon}\right)(1+\epsilon_0).$$

On the other hand, by means of Eqs. $(2.6)_5$, $(3.2)_2$ and of relation

$$(L_k+F_k)(L_k+F_k) \leq (1+\epsilon)L_kL_k + \left(1+\frac{1}{\epsilon}\right)F_kF_k \tag{3.30}$$

valid for any $\epsilon > 0$ and for each couple of vectors $\{L_k, F_k\}$, we get

$$\tilde{Q}_k\tilde{Q}_k \leq 2\left(f_0^{*2}\dot{d}_k\dot{d}_k + K_0^2\vartheta_{,k}\vartheta_{,k}\right) \leq \frac{2\max\left\{f_0^{*2},K_0^2\right\}\mathcal{H}_1}{\varsigma_m^*} \tag{3.31}$$

and

$$\left|\chi\vartheta\tilde{Q}_l n_l\right| \leq \epsilon_2\frac{\chi}{2}\vartheta^2 + \frac{\chi\max\left\{f_0^{*2},K_0^2\right\}\mathcal{H}_1}{\epsilon_2\varsigma_m^*}, \qquad \forall\epsilon_2 > 0. \tag{3.32}$$

Collecting Eqs. (3.3), (3.29), (3.32), we have, for any $\epsilon > 0$, $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$,

$$|\Gamma(r,t)| \leq \int_0^t \int_{S_r} e^{-\omega s} \Big\{ \epsilon_1 \mathcal{T}(s) + \frac{3\varsigma_M \left[\mathcal{E}_0(s) + \mathcal{E}_1(s)\right]}{\epsilon_1 \min\{1,\rho\}}(1+\epsilon)$$

$$+ \left[\frac{3\left[(\varkappa_1^0 + \varkappa_2^0)^2 + (\varkappa_2^0)^2\right]}{\epsilon_1 \min\{1,\rho\}\chi} \left(1 + \frac{1}{\epsilon_0}\right)\left(1 + \frac{1}{\epsilon}\right) + \epsilon_2 \right] \frac{\chi}{2}\vartheta^2(s)$$

$$+ \frac{\sigma_M^* \mathcal{H}_0(s)}{2\epsilon_1 \min\{1,\rho\}}\left(1 + \frac{1}{\epsilon}\right)(1+\epsilon_0) + \frac{\chi \max\left\{f_0^{*2}, K_0^2\right\}\mathcal{H}_1(s)}{\epsilon_2 \varsigma_m^*} \Big\} da ds.$$

If we choose the constants such that

$$\epsilon_1 = \frac{3\left[(\varkappa_1^0 + \varkappa_2^0)^2 + (\varkappa_2^0)^2\right]}{\epsilon_1 \min\{1,\rho\}\chi}\left(1 + \frac{1}{\epsilon_0}\right)\left(1 + \frac{1}{\epsilon}\right) + \epsilon_2 = \frac{3\varsigma_M}{\epsilon_1 \min\{1,\rho\}}(1+\epsilon)$$

$$= \frac{\omega}{2\epsilon_1 \min\{1,\rho\}}\left(1 + \frac{1}{\epsilon}\right)(1+\epsilon_0)\sigma_M^* = \frac{\chi\omega \max\left\{f_0^{*2}, K_0^2\right\}}{\epsilon_2 \varsigma_m^*},$$

the following relations are valid

$$\epsilon = \frac{\sigma_M^* \omega}{6\varsigma_M}(1+\epsilon_0), \qquad \epsilon_1 = \sqrt{\frac{3\varsigma_M}{\min\{1,\rho\}}(1+\epsilon)},$$

$$\epsilon_2 = \frac{\chi\omega \max\left\{f_0^{*2}, K_0^2\right\}}{\varsigma_m^*}\sqrt{\frac{\min\{1,\rho\}}{3\varsigma_M(1+\epsilon)}};$$

moreover, ϵ_0 is the positive root of the algebraic equation (3.15) and $\epsilon_1 = \upsilon_1$, as defined in (3.14). Consequently, we conclude that

$$|\Gamma(r,t)| \leq \frac{\upsilon_1}{\omega} \int_0^t \int_{S_r} e^{-\omega s} \Big\{ \omega \left[\mathcal{T}(s) + \mathcal{E}_0(s) + \mathcal{E}_1(s) + \frac{\chi}{2}\vartheta^2(s)\right] + \mathcal{H}_0(s) + \mathcal{H}_1(s) \Big\} da ds. \tag{3.33}$$

Therefore, Eqs. (3.16), (3.33) lead to Eq. (3.13) and the proof is complete. ■

For a bounded body B, Eqs. (3.11), (3.13) imply

$$\frac{\partial}{\partial r}\left[\Gamma(r,t) e^{\frac{\omega}{\upsilon_1} r}\right] \leq 0, \qquad \text{for all} \quad r \in [0,L], \quad t \in [0,\tau_0], \tag{3.34}$$

while, for an unbounded body B, Eqs. (3.12), (3.13) imply, for a fixed $t \in [0, \tau_0]$:

$$
\text{(3.35)} \qquad
\begin{aligned}
&\text{if for any } r \geq 0, \quad \varGamma(r,t) \geq 0, \Rightarrow && \frac{\partial}{\partial r}\left[\varGamma(r,t) e^{\frac{\omega}{\upsilon_1} r}\right] \leq 0;\\
&\text{if for any } r \geq \bar{r}_t, \quad \varGamma(r,t) < 0 \Rightarrow && \frac{\partial}{\partial r}\left[\varGamma(r,t)\, e^{-\frac{\omega}{\upsilon_1} r}\right] \leq 0.
\end{aligned}
$$

Using Eqs. (3.13), (3.34), we can prove the next theorem, valid for a bounded body. This theorem provides a spatial decay estimate of Saint–Venant type through a time-independent decay rate.

Theorem 1. *(Spatial behaviour for a bounded body). Let B be a bounded body and let the hypotheses of Lemma 1 be valid. It follows*

$$
\text{(3.36)} \qquad \varGamma(r,t) \leq \varGamma(0,t) e^{-\frac{\omega}{\upsilon_1} r}, \qquad \textit{for all} \quad r \in [0, L], \; t \in [0, \tau_0]. \qquad \blacksquare
$$

Now, we can formulate, with the aid of Eqs. (3.13), (3.35), a theorem concerning the spatial behaviour for an unbounded body. The obtained decay estimate is of Phragmén–Lindelöf type.

Theorem 2. *(Spatial behaviour for an unbounded body). Let B be an unbounded body and let the hypotheses of Lemma 1 hold true. It results, for any fixed $t \in [0, \tau_0]$:*

$$
\text{(3.37)} \qquad
\begin{aligned}
&\textit{if for any } r \geq 0 \quad \varGamma(r,t) \geq 0 \Rightarrow && \varGamma(r,t) \leq \varGamma(0,t) e^{-\frac{\omega}{\upsilon_1} r};\\
&\textit{if for any} r \geq \bar{r}_t \quad \varGamma(r,t) < 0 \Rightarrow && -\varGamma(r,t) \geq -\varGamma(\bar{r}_t, t) e^{\frac{\omega}{\upsilon_1}(r-\bar{r}_t)}. \qquad \blacksquare
\end{aligned}
$$

4. Another surface power function and related estimates

In this section, we will investigate another cross-section function, in order to obtain different exponential decay estimates suitable for short values of time. We introduce the following function

$$
\text{(4.1)} \qquad \varGamma_1(r,t) = \int_0^t \varGamma(r,s)ds
$$

$$
= -\int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\left[\tilde{\pi}_{lk}^{(1)}(\tau)\dot{u}_k(\tau) + \tilde{\pi}_{lk}^{(2)}(\tau)\dot{w}_k(\tau) + \chi\vartheta(\tau)\tilde{Q}_l(\tau)\right] n_l da d\tau ds.
$$

As a consequence of Eqs. (3.16), (4.1), it is

$$\frac{\partial \Gamma_1}{\partial r}(r,t) = -\int_0^t \int_{S_r} e^{-\omega s}\left[\mathcal{T}(s)+\mathcal{E}_0(s)+\mathcal{E}_1(s)+\frac{\chi}{2}\vartheta^2(s)\right] da ds - \int_0^t \int_0^s \int_{S_r} e^{-\omega\tau}\left\{\omega\left[\mathcal{T}(\tau)+\mathcal{E}_0(\tau)+\mathcal{E}_1(\tau)+\frac{\chi}{2}\vartheta^2(\tau)\right] + \mathcal{H}_0(\tau)+\mathcal{H}_1(\tau)\right\} da d\tau ds. \tag{4.2}$$

Under hypotheses (2.7), (2.10), (2.11), for any fixed $t \in [0,\tau_0]$, Eq. (4.2) implies that

$$\frac{\partial \Gamma_1}{\partial r}(r,t) \le 0, \qquad \forall r \in [0,L]$$

and that $\Gamma_1(r,t)$ is non-increasing with respect to r.

As for $\Gamma(r,t)$, also for $\Gamma_1(r,t)$, when B is a bounded body, it follows

$$\Gamma_1(r,t) \ge 0 \quad \text{for all} \quad r \in [0,L], \quad t \in [0,\tau_0],$$

while, if B is an unbounded body, then it is

either $\Gamma_1(r,t) \ge 0 \quad \forall r \ge 0,$ or $\exists\, \bar r_t \ge 0$ such that $\Gamma_1(r,t) < 0 \quad \forall r \ge \bar r_t.$

Lemma 2. *Let the hypotheses of Lemma 1 be true. For any fixed $t \in (0,\tau_0]$, $\Gamma_1(r,t)$ satisfies the inequality*

$$\frac{1}{\upsilon_2(t)\sqrt{t}}\Big|\Gamma_1(r,t)\Big| + \frac{\partial}{\partial r}\Gamma_1(r,t) \le 0, \tag{4.3}$$

where

$$\upsilon_2(t) = \sqrt{\frac{6t\varsigma_M + \sigma_M^*\left[1+\varepsilon_0(t)\right]}{2\min\{1,\rho\}}} \tag{4.4}$$

and $\varepsilon_0(t)$ is the positive root of the algebraic equation

$$\varepsilon_0^2 + \left(1 + \frac{6t\varsigma_M}{\sigma_M^*} - \frac{6t[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2]}{\chi\sigma_M^*} - \frac{2\chi\max\{f_0^{*2},K_0^2\}\min\{1,\rho\}}{\varsigma_m^*\sigma_M^*}\right)\varepsilon_0 - \left(1+\frac{6t\varsigma_M}{\sigma_M^*}\right)\frac{6t[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2]}{\chi\sigma_M^*} = 0. \tag{4.5}$$

P r o o f. Remembering Eq. (3.29) and using the well-known relations

$$\int_0^t \int_0^s a(z)dzds = \int_0^t (t-z)a(z)dz \le t\int_0^t a(z)dz, \tag{4.6}$$

where $a(z)$ is any non-negative function, we get, for any $\varepsilon > 0$, $\varepsilon_0 > 0$, $\varepsilon_1 > 0$,

$$\left|\int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\left[\tilde{\pi}_{lk}^{(1)}(\tau)\dot{u}_k(\tau) + \tilde{\pi}_{lk}^{(2)}(\tau)\dot{w}_k(\tau)\right] n_l da d\tau ds\right| \tag{4.7}$$

$$\le \sqrt{t}\int_0^t \int_{S_\alpha} e^{-\omega\tau}\left[\sqrt{t}\varepsilon_1 \mathcal{T}(\tau) + \frac{3\sqrt{t}\varsigma_M\left[\mathcal{E}_0(\tau)+\mathcal{E}_1(\tau)\right]}{\varepsilon_1 \min\{1,\rho\}}(1+\varepsilon)\right.$$

$$\left. + \frac{3\sqrt{t}\left[(\varkappa_1^0+\varkappa_2^0)^2 + (\varkappa_2^0)^2\right]}{\varepsilon_1 \min\{1,\rho\}\chi}\left(1+\frac{1}{\varepsilon_0}\right)\left(1+\frac{1}{\varepsilon}\right)\frac{\chi}{2}\vartheta^2(\tau)\right] da d\tau$$

$$+ \int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\frac{\sigma_M^* \mathcal{H}_0(\tau)}{2\varepsilon_1 \min\{1,\rho\}}\left(1+\frac{1}{\varepsilon}\right)(1+\varepsilon_0) da d\tau ds.$$

Moreover, using arithmetic-geometric mean inequality and by means of Eqs. (3.31), (4.6), it results, for any $\varepsilon_2 > 0$,

$$\left|\int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\chi\vartheta(\tau)\tilde{Q}_l(\tau) n_l da d\tau ds\right| \tag{4.8}$$

$$\le \left(\sqrt{t}\int_0^t \int_{S_\alpha} e^{-\omega\tau}\chi\vartheta^2(\tau) da d\tau\right)^{1/2}\left(\sqrt{t}\int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\chi\tilde{Q}_l(\tau)\tilde{Q}_l(\tau) da d\tau ds\right)^{1/2}$$

$$\le \sqrt{t}\int_0^t \int_{S_\alpha} e^{-\omega\tau}\varepsilon_2\frac{\chi\vartheta^2(\tau)}{2} da d\tau + \sqrt{t}\int_0^t \int_0^s \int_{S_\alpha} e^{-\omega\tau}\frac{\chi \max\left\{f_0^{*2}, K_0^2\right\}\mathcal{H}_1(\tau)}{\varsigma_m^* \varepsilon_2} da d\tau ds.$$

Collecting Eqs. (4.1), (4.7), (4.8), we obtain

$$(4.9)\qquad |\Gamma_1(r,t)| \leq \sqrt{t}\int_0^t\int_{S_\alpha} e^{-\omega\tau}\Bigg\{\varepsilon_1\sqrt{t}\mathcal{T}(\tau)+\frac{3\sqrt{t}\varsigma_M\left[\mathcal{E}_0(\tau)+\mathcal{E}_1(\tau)\right]}{\varepsilon_1\min\{1,\rho\}}(1+\varepsilon)$$
$$+\left[\frac{3\sqrt{t}\left[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2\right]}{\varepsilon_1\min\{1,\rho\}\chi}\left(1+\frac{1}{\varepsilon_0}\right)\left(1+\frac{1}{\varepsilon}\right)+\varepsilon_2\right]\frac{\chi}{2}\vartheta^2(\tau)\Bigg\}\,da\,d\tau$$
$$+\sqrt{t}\int_0^t\int_0^s\int_{S_\alpha} e^{-\omega\tau}\left[\frac{\sigma_M^*\mathcal{H}_0(\tau)}{2\varepsilon_1\min\{1,\rho\}\sqrt{t}}\left(1+\frac{1}{\varepsilon}\right)(1+\varepsilon_0)\right.$$
$$\left.+\frac{\chi\max\left\{f_0^{*2},K_0^2\right\}\mathcal{H}_1(\tau)}{\varepsilon_2\varsigma_m^*}\right]da\,d\tau\,ds.$$

If ε, ε_0, ε_1, ε_2 satisfy the following relations

$$\varepsilon_1=\frac{3\varsigma_M}{\varepsilon_1\min\{1,\rho\}}(1+\varepsilon)=\frac{3\left[(\varkappa_1^0+\varkappa_2^0)^2+(\varkappa_2^0)^2\right]}{\varepsilon_1\min\{1,\rho\}\chi}\left(1+\frac{1}{\varepsilon_0}\right)\left(1+\frac{1}{\varepsilon}\right)+\frac{\varepsilon_2}{\sqrt{t}}$$
$$=\frac{\sigma_M^*}{2\varepsilon_1\min\{1,\rho\}t}\left(1+\frac{1}{\varepsilon}\right)(1+\varepsilon_0)=\frac{\chi\max\left\{f_0^{*2},K_0^2\right\}}{\varepsilon_2\varsigma_m^*\sqrt{t}},$$

we arrive at

$$\varepsilon=\frac{\sigma_M^*}{6t\varsigma_M}(1+\varepsilon_0),\qquad \varepsilon_1=\sqrt{\frac{3\varsigma_M}{\min\{1,\rho\}}(1+\varepsilon)},$$
$$\varepsilon_2=\frac{\chi\max\left\{f_0^{*2},K_0^2\right\}}{\varsigma_m^*}\sqrt{\frac{\min\{1,\rho\}}{3t\varsigma_M(1+\varepsilon)}},$$

where ε_0 is the positive root of Eq. (4.5) and $\varepsilon_1\sqrt{t}=\upsilon_2(t)$, as defined in (4.4). Then, Eq. (4.9) can be rewritten as

$$(4.10)\qquad |\Gamma_1(r,t)|\leq\sqrt{t}\upsilon_2(t)\Bigg\{\int_0^t\int_{S_\alpha}e^{-\omega\tau}\Big[\mathcal{T}(\tau)+\mathcal{E}_0(\tau)+\mathcal{E}_1(\tau)+\frac{\chi}{2}\vartheta^2(\tau)\Big]\,da\,d\tau$$
$$+\int_0^t\int_0^s\int_{S_\alpha}e^{-\omega\tau}\left[\mathcal{H}_0(\tau)+\mathcal{H}_1(\tau)\right]da\,d\tau\,ds\Bigg\}.$$

Finally, comparing Eq. (4.2) with Eq. (4.10), we obtain Eq. (4.3). ∎

Using properties of Γ_1 and proceeding in the same way as for Theorems 1-2, we establish an exponential spatial decay estimate of Saint–Venant type for a bounded body and we also get an alternative evaluation of Phragmén–Lindelöf type, for an unbounded body.

Theorem 3. *(Spatial behaviour for a bounded body). Under the same hypotheses of Theorem 1, it follows*

$$\Gamma_1(r,t) \le \Gamma_1(0,t)e^{-\frac{r}{v_2(t)\sqrt{t}}}, \qquad \textit{for all} \quad r \in [0,L], \quad t \in (0,\tau_0]. \tag{4.11}$$

Theorem 4. *(Spatial behaviour for an unbounded body). Under the same hypotheses of Theorem 2, it follows, for any fixed $t \in (0,\tau_0]$:*

(4.12)

$$\textit{if for any } r \ge 0 \quad \Gamma_1(r,t) \ge 0 \Rightarrow \quad \Gamma_1(r,t) \le \Gamma_1(0,t)e^{-\frac{r}{v_2(t)\sqrt{t}}};$$

$$\textit{if for any } r \ge \bar{r}_t \quad \Gamma_1(r,t) < 0 \Rightarrow \quad -\Gamma_1(r,t) \ge -\Gamma_1(\bar{r}_t,t)e^{\frac{1}{v_2(t)\sqrt{t}}(r-\bar{r}_t)}.$$

■

5. Conclusions

In the present paper, we study the classical Kelvin–Voigt viscoelastic model, using a Lagrangian description and considering a mixture made of an elastic solid and a viscous fluid. In the context of linear theory used to describe the previous model, we establish spatial behaviours of different types, defining appropriate time-weighted surface power functions (3.3), (4.1) and studying their properties. The estimates of first kind (3.36), (3.37) are characterized by a time-independent decay rate, with results indicated for large values of time. The relations (4.11), (4.12) show time-dependent decay rates suitable for short values of time. We are able to establish spatial decay estimates of Saint–Venant type (3.36), (4.11), for bounded bodies, and of Phragmén–Lindelöf type (3.37), (4.12), for unbounded bodies.

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