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The method of finite cells for the analysis of terrain subsidence caused by tectonic movements or by subterrain exploitation

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A simple numerical procedure for determining two- and three-dimensional field of displacements in the terrain subsidence caused by tectonic movements is proposed. The procedure is based on the concept of J. Litwiniszyn, who treated the gravity flow of a solid medium as a stochastic process. In this procedure, an imagined two-dimensional system of cells analogous to the Galton's board is assumed. Theoretical considerations are supported by simple experimental simulations.

Key words: terrain subsidence, kinematics, stochastic approach.

1. Introduction

The subsidence of the terrain, often of catastrophic consequences, may be caused by natural phenomena such as sudden tectonic translation of the bedrock or by human activity connected with the industrial exploitation of natural resources. The mechanics of such movements is complex. In the case of subsidence caused by underground exploitation there are proposed specialised methods based mainly on observations of real deformations of the upper surface of the terrain in the vicinity of coal mines – cf. e. g. the early works [1, 2] of Polish researchers, or the standard finite elements procedures – see e. g. [9].

As stated by J. Litwiniszyn [5], during the gravity flow of granular media we cannot expect that the condition of continuity is fulfilled. The contacts between the grains are changing. The grains being in contact at a given movement may after a while lose the contact. Thus the standard methods of mechanics of granular media, in which the continuity condition is assumed, are inadequate when problems of gravity flow are analysed. It seems to be reasonable to use a different model, in which the medium is regarded as a collection of discrete elements. Motion is characterised by random changes of contacts and random displacements of grains.

Such an approach may be supported by an experimental simulation on simple two-dimensional models. The simplest of them, based on the standard theory of limit states of granular media treating them as a continuum, consists in rigid blocks movement as shown in Fig. 1. On the left side is shown the initial configuration. A thick layer of soil rests on the bedrock with the potential fault $B-D$. On the right side is shown the configuration after the right-hand segment of the bedrock has been displaced downwards. Displacements of particular blocks of soil shifted along certain sliplines $A-B$ and $B-C$ are shown in the figure.

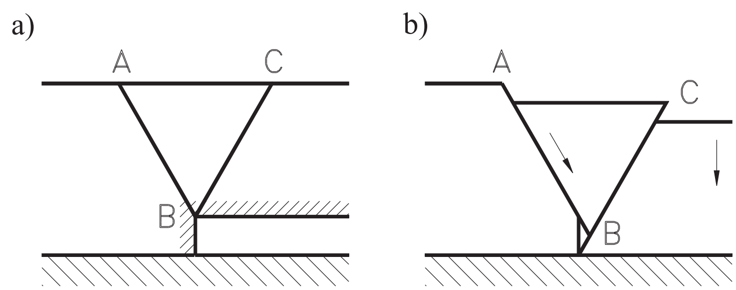

Fig. 1. Simple mechanism of terrain subsidence; a) initial configuration, b) configuration after the right-hand segment of the bedrock has been moved downwards.

This simple theoretical deformation mechanism may be illustrated when the soil is represented by an idealised regular array of discs of the same diameter – Fig. 2. This regular array is disturbed after the right-hand sector of the bedrock has been shifted downwards. The slip-lines $A-B$ and $B-C$ are visible. Let us note that the soil is in certain applications treated as a discrete medium which consists of individual elements (grains) – cf. e. g. [6].

If the subsidence of the terrain is caused by underground mining operations, the simplest deformation mechanism takes the form shown in Fig. 3.

In Fig. 4 is presented a simple experimental simulation of such a mechanism of deformation. The coins of the same diameter forming a regular array are located on a glass plate in the horizontal position as shown in the upper photograph. To simulate the initial configuration corresponding to that shown in Fig. 3a, two bottom rows on the right side have been left without coins. Then the plate was inclined with respect to its initial horizontal position and the coins slid downwards due to gravity forces only. The final configuration is shown in the lower photograph. The general layout of coins is close to the simple theoretical solution shown in Fig. 3.

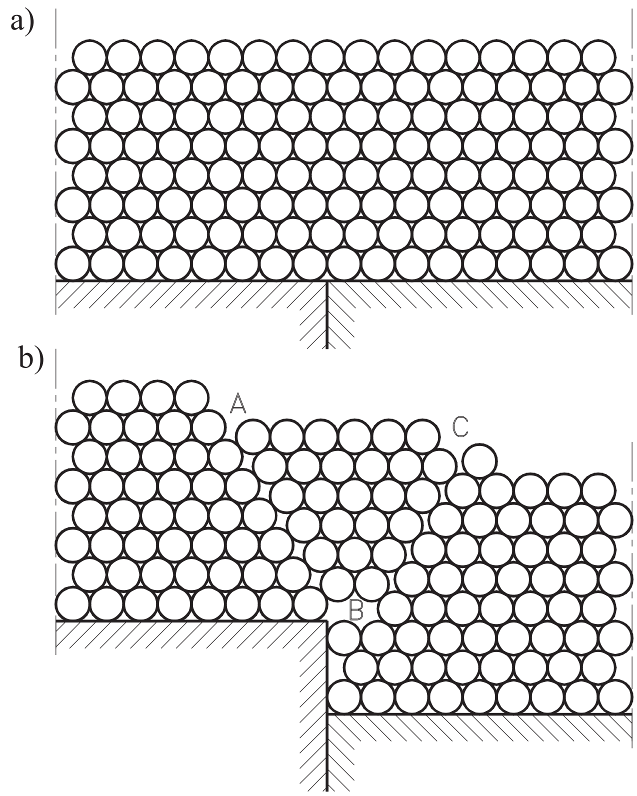

FIG. 2. Theoretical simulation of the simple terrain subsidence mechanism shown in Fig. 1 a) initial configuration, b) deformed configuration.

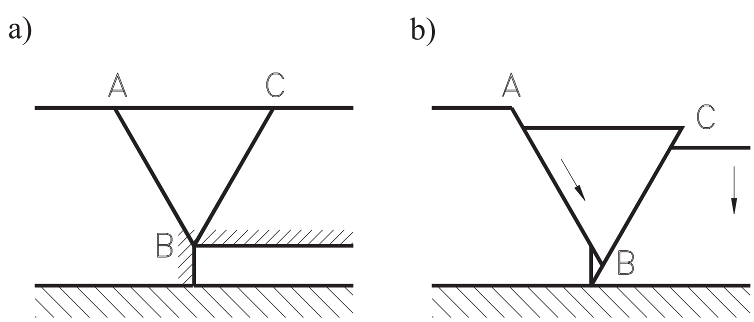

FIG. 3. Simple mechanism of terrain subsidence caused by underground exploitation; a) initial configuration, b) final configuration.

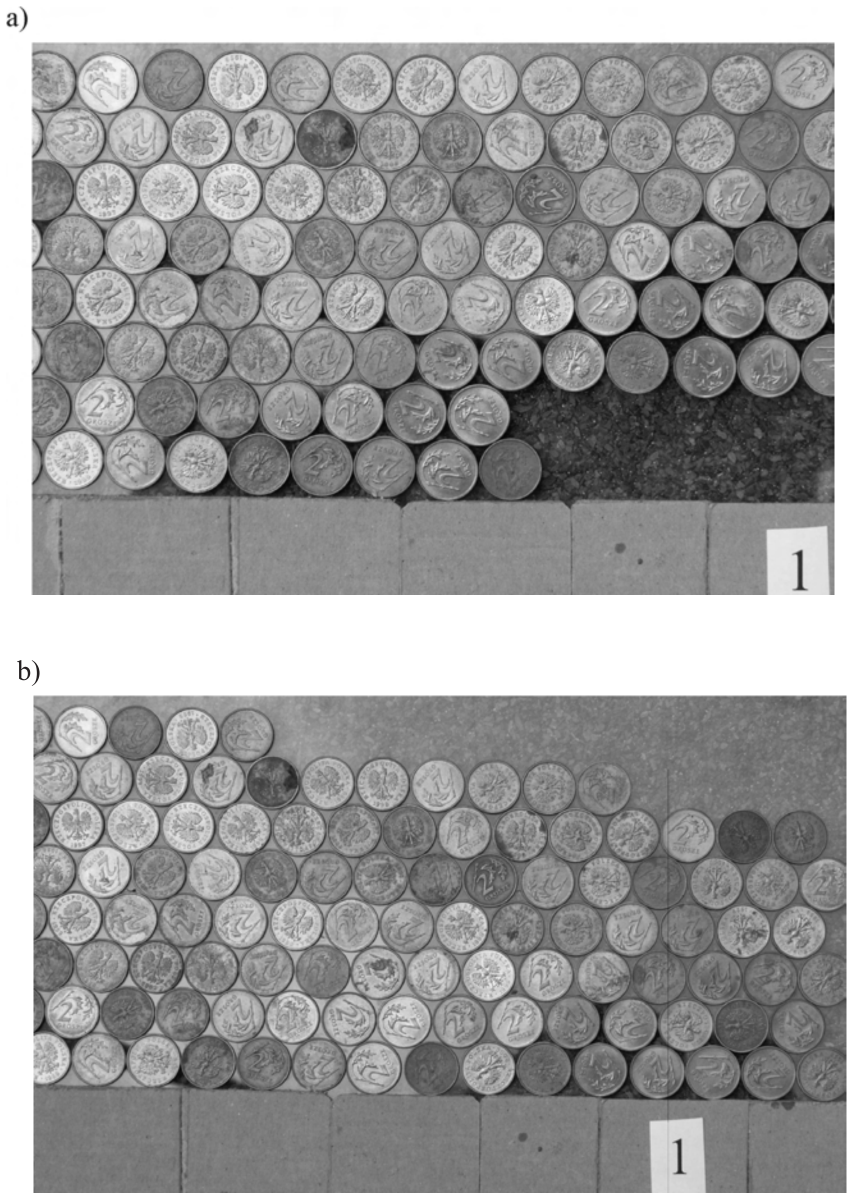

Fig. 4. Experimental simulation of the deformation mode shown in Fig. 3; a) initial configuration of coins located on the glass plate in horizontal position, b) final configuration after the plate was inclined with respect to horizontal position.

A similar displacement mode has been obtained when instead of coins, the hexagonal elements were used – Fig. 5.

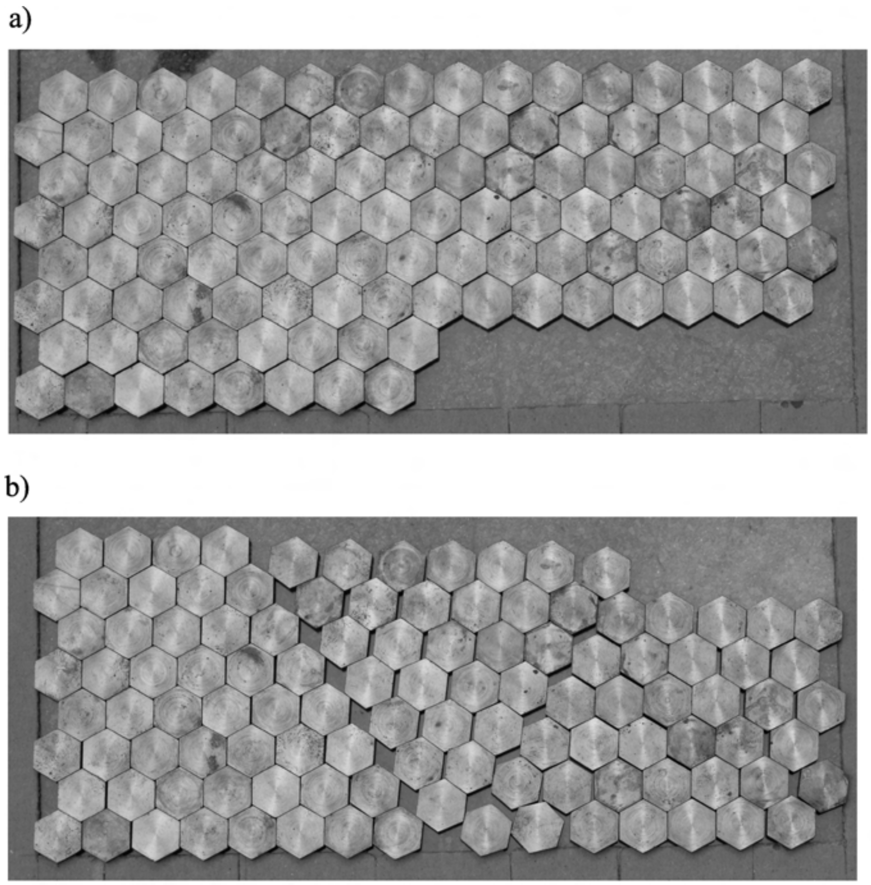

FIG. 5. Experimental simulation of the deformation mode shown in Fig. 3; a) – initial configuration of coins located on the glass plate in horizontal position, b) – final configuration after the plate was inclined with respect to horizontal position.

However, when the experimental simulation shown in Fig. 4 is repeated with the assembly of coins of different diameters, the obtained field of displacement is not so regular (Fig. 6). It is strongly influenced by random deviations from that presented in Fig. 4. There are no sudden jumps in the upper layer of coins. No slip-lines as those appearing in Figs. 2b, 4b and 5b are visible in Fig. 6b.

In Fig. 7 the displacements of central points of particular coins resulting from this experimental simulation are shown. Let us note that this displacement field is distinctly different from that in which coins of the same diameter were used. Now the displacements of coins of different diameters forming a random array are influenced by random changes of mutual contacts. Consequently, their displacements are random.

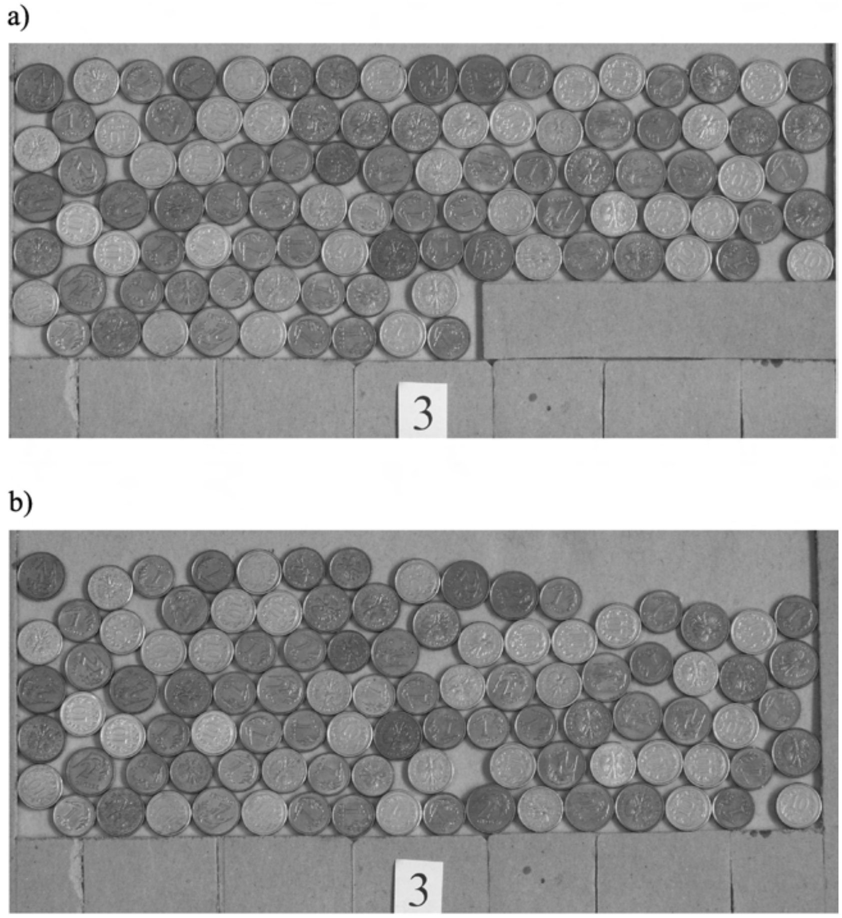

FIG. 6. Experimental simulation of the gravity walk of the particles of a granular medium; a) initial configuration of coins of different diameters located on a glass plate in its horizontal position, b) final configuration after the blocking strip at the bottom has been removed and the plate has been inclined with respect to the initial horizontal position.

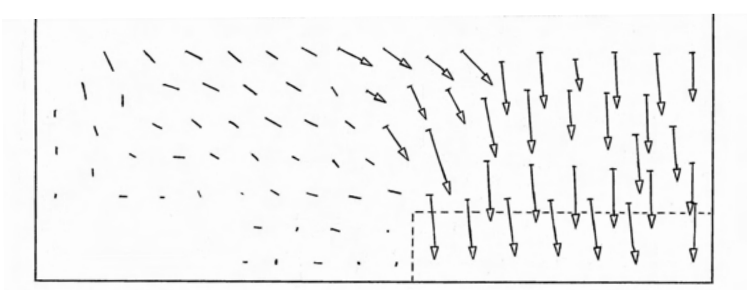

FIG. 7. Measured displacements of coins simulating particles of a granular medium in the test shown in Fig. 6.

Experimental simulations shown in Figs. 4, 5 and 6 indicate, that the movement of particles of a granular medium caused by gravity forces only, may be treated as a kinematical problem without any analysis of forces occurring between particular grains.

2. The method of finite cells

Such a walk influenced by random factors plays a distinct role in soil moving due to gravity forces only. This observation constitutes the basis of the method proposed by J. LITWINISZYN [3–5]. The idea of this method is presented in Fig. 8. In the layer of the soil resting on a bedrock we assume certain hypothetical system of rectangular "cells". Particular layers of the system are shifted with respect to the neighbouring layers by one half of the division. Let us now assume that a single cell of the lowest layer is empty. The volume of it will be treated as a unit cavity. The process of subsidence of the soil is treated as migration of that cavity upwards. It is assumed that each time the cavity moves upwards with the same probability equal to 1/2 into the left or into the right-hand cell lying above it. Proceeding according to this rule we calculate how the initial unit cavity will finally be distributed after reaching the upper free surface. The numbers in particular cells indicate how large is the portion of the initial unit cavity which has passed through the cell during the migration process. The stepwise lines in Fig. 9 corresponding to these numbers indicate how much the soil has been shifted downwards in particular cells. Note that these lines represent a normal distribution.

The ratio a/b of the dimensions of cells depends on the properties of the soil. It should be determined experimentally – cf. [7].

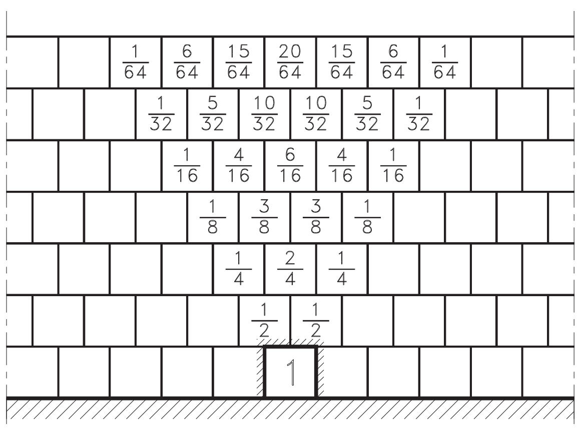

FIG. 8. The assumed system of finite cells for the analysis of migration of a unit cavity at the bottom according to an assumed, simple stochastic rule. Numbers in cells indicate which fraction of the cavity passed through the cell.

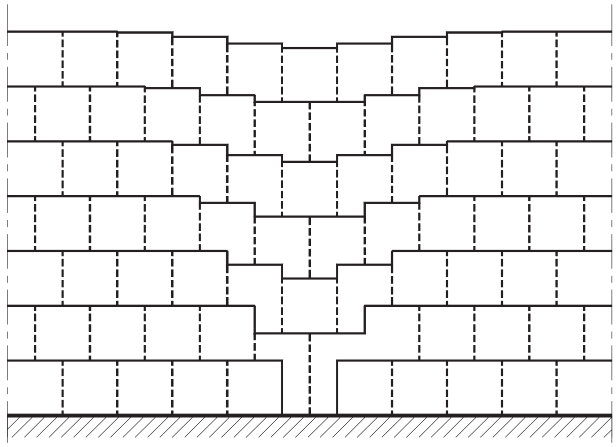

FIG. 9. Stepwise approximation of the deformation mode, caused by migration upwards of the unit cavity at the bottom.

3. Analysis of displacements

Let us now analyse any arbitrary set of three adjacent cells taken from Fig. 8. They are represented in Fig. 10. The numbers in them correspond to the fraction of the initial unit cavity which passed through the cell during migration towards the free surface of the bulk of the medium. According to the finite cells procedure only one half of these fractions migrate from each cell to the cell lying just above it. It is assumed that the migration takes place along the respective lines $A-C$ or $B-C$ joining central points of cells. Directions and sizes of these migrating portions of the unit cavity may be represented by vectors $\mathbf{w}_{BC}$ and $\mathbf{w}_{AC}$, as shown in Fig. 10b. They may be treated as components of the resulting vector $\mathbf{w}_c$, representing the direction and magnitude of the averaged momentary flux of the cavity into cell C during the migration process. The opposite vector $\mathbf{w}_m$ may be treated as representation of the flux of the mass of soil filling the space left by the cavities moving upwards. Its vertical component must be the same as the corresponding vertical displacement calculated before and shown in Fig. 9. The displacement vectors shown in Fig. 11 have been calculated using such a procedure.

Using this procedure one can calculate deformations of the soil for numerous two-dimensional problems of subsidence, caused by tectonic movements or by underground exploitation. In Sec. 5 a variant of the procedure for three-dimensional problems is presented.

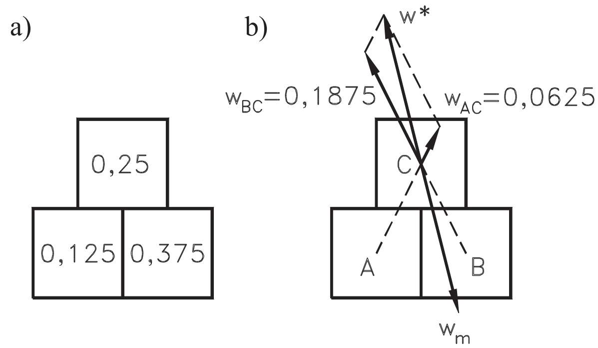

FIG. 10. Procedure of determining the displacement vector $\mathbf{w}_m$ of a granular medium caused by the migration of a cavity upwards.

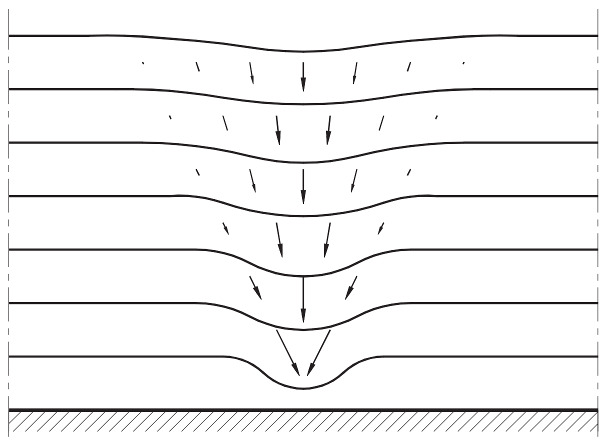

FIG. 11. Calculated displacements of granular medium after migration of a unit cavity, located initially as shown in Fig. 8.

4. Example of application

A simple example is shown in Fig. 12. In the lower part of the soil resting on a bedrock, the empty space $A-B-C-D$ has been left by underground exploitation. In the following process of subsidence this empty space will be filled by the soil migrating downwards. Let us divide this empty space into a number of cells, each of them being of unit volume. These unit cavities migrate upwards through the system of cells shown in the figure. It is assumed that each time a cavity in the particular cell migrates upwards, the probability that it moves to the left or to the right cell, just above it, is equal to 1/2. Numbers shown in

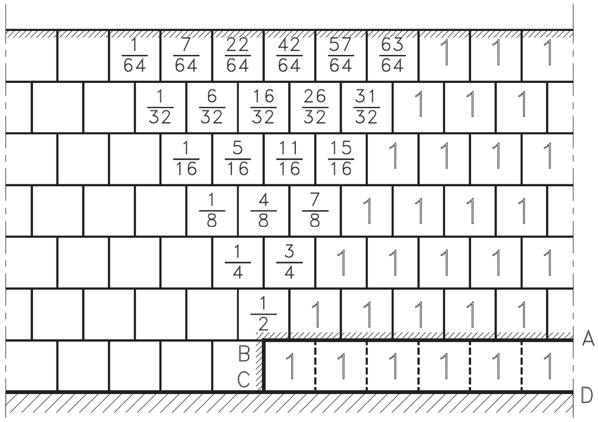

FIG. 12. The assumed system of finite cells for the analysis of the terrain subsidence. The initial long cavity $A-B-C-D$ has been divided into six cells, each of a unit volume. Numbers in particular cells indicate what portion of a unit cavity has passed through each cell in the migration process towards the upper surface.

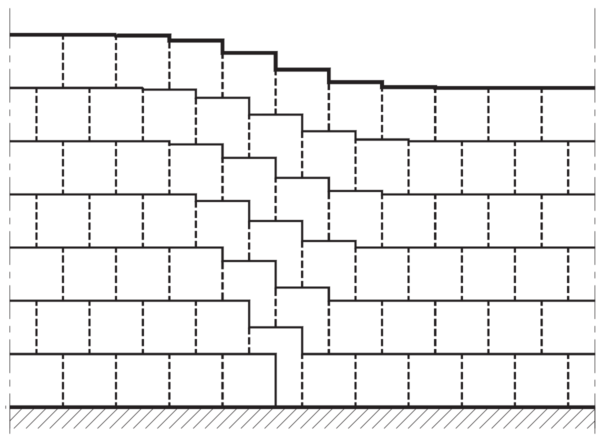

FIG. 13. Stepwise approximation of the terrain subsidence in the case shown in Fig. 12.

particular cells indicate how large was the portion of unit cavity which has passed through the cell during the migration process. On the basis of these numbers, the diagram representing a stepwise approximation of the final subsidence shown in Fig. 13 has been prepared.

The procedure described in Sec. 3 allows us to calculate the vectors of displacements in the entire deformation zone Fig. 14. Let us note that this theoretical displacements field is very similar to that resulting from experimental simulation cf. Fig. 7.

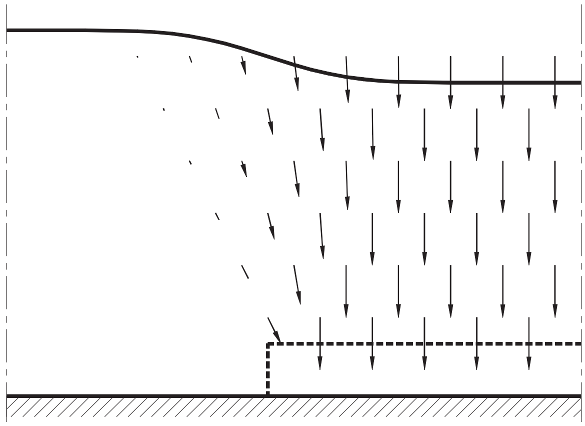

Fig. 14. Calculated displacements of the granular medium after migration of the cavity $A-B-C-D$ towards the upper surface of the medium.

5. Three-dimensional problems

As an introduction let us consider an elementary problem analogous to that shown in Fig. 8. In a bulk of soil with initially flat upper surface, we assume the existence of a cuboidal cavity A of unit volume Fig. 15. The frontal part of the bulk has not been shown in the figure. The process of subsidence will be analysed as migration of the cavity A towards the upper surface of the bulk. Let us assume now a certain system of plates, each containing a number of cells identical with the initial cell A. These cells in consecutive plates are arranged in such a manner that central axes of cells in each plate coincide with the common line of four corners of cells in the adjacent plates, just above and below.

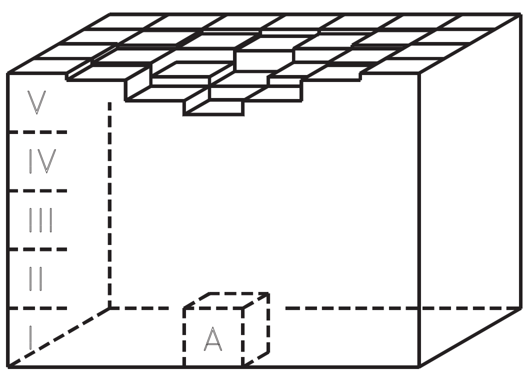

Fig. 15. Stepwise approximation of the deformation of the upper surface of a granular medium caused by migration of cavity A of unit volume. Vertical displacements have been enlarged five times. The frontal part of the bulk has not been shown.

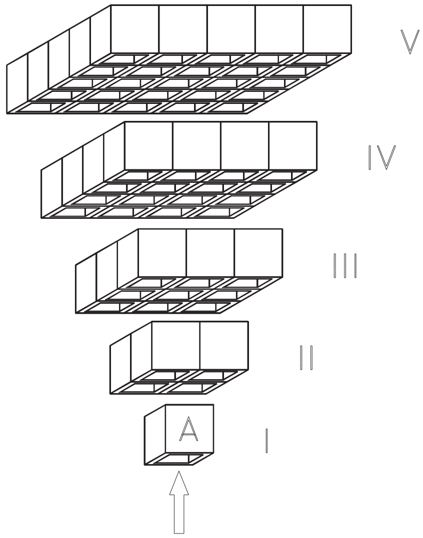

Fig. 16. Assumed system of finite cells for the analysis of migration of a single cavity A towards the upper surface of the bulk cf. Fig. 15.

A portion of this system of cells is shown in Fig. 16.

It is assumed that each time the cavity moves upwards with the same probability equal to 1/4into one of four cells just above it in the adjacent plate. The resulting probabilities in cells, located of the upper fifth plate are shown in Fig. 17. The numbers in particular cells indicate how large was the portion of the initial unit cavity at the bottom which has ended its migration in the cell. In other words, they indicate how much the surface of the bulk has been lowered in the cell.

$\frac{1}{256}$	$\frac{4}{256}$	$\frac{6}{256}$	$\frac{4}{256}$	$\frac{1}{256}$
$\frac{4}{256}$	$\frac{16}{256}$	$\frac{24}{256}$	$\frac{16}{256}$	$\frac{4}{256}$
$\frac{6}{256}$	$\frac{24}{256}$	$\frac{36}{256}$	$\frac{24}{256}$	$\frac{6}{256}$
$\frac{4}{256}$	$\frac{16}{256}$	$\frac{24}{256}$	$\frac{16}{256}$	$\frac{4}{256}$
$\frac{1}{256}$	$\frac{4}{256}$	$\frac{6}{256}$	$\frac{4}{256}$	$\frac{1}{256}$

Fig. 17. Cells in the upper layer V shown in Fig. 16. Numbers in particular cells indicate how large was the portion of the unit cavity A which has reached the upper surface of the bulk.

According to these numbers, the stepwise deformation of the upper surface of the bulk has been approximately found (see Fig. 15). The vertical displacements have been five times enlarged to make the deformation pattern more clearly visible. Note that the distribution of displacements in this basic solution is close to the circular normal distribution.

Using this procedure we can solve numerous complex three-dimensional problems of terrain subsidence. An example is shown in Fig. 18. Let a long and narrow part of the bedrock, with a layer of soil resting on it, be moved downwards due to a tectonic displacement. The essential (for our considerations) portion of the initially empty space near the end of displaced part of the bedrock has been divided into eighteen cells, each of the assumed unit volume. These empty cells constitute the lowest layer of the assumed system of finite cells. Other layers of the system of finite cells are formed in the manner presented in Fig. 16. Using the procedure of migration of the cavities upwards, the portions of cavity of unit volume, which reached the cells of the fifth upper layer of cells, have been calculated. These portions are shown in Fig. 19. Using these numbers a stepwise approximation of the deformed surface of the terrain shown in Fig. 18 has been found.

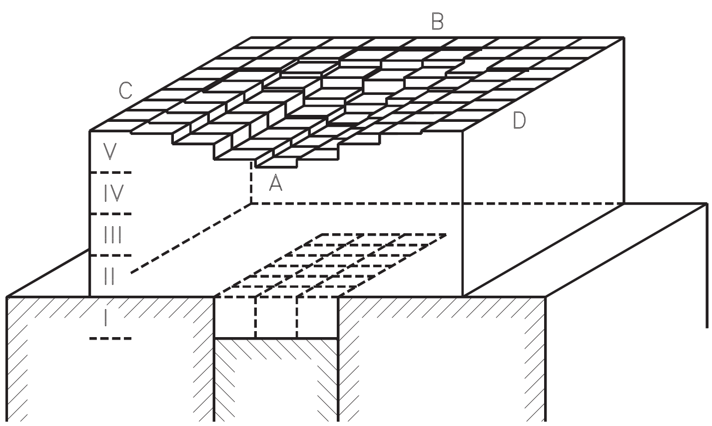

FIG. 18. Calculated subsidence of the upper surface of the bulk caused by tectonic displacement of a long narrow portion of the bedrock. The empty space formed by the end part of the displaced bedrock has been divided into 18 cells, each of a unit volume.

Displacement vectors can be calculated by the generalized procedure described in Sec. 3 for two-dimensional problems. When three-dimensional problems cases are investigated, we must analyse at each place a set of five cells: four adjacent cells taken from a layer in question and one cell taken from the layer just above these four cells below (Fig. 20).

	$\frac{1}{256}$	$\frac{5}{256}$	$\frac{11}{256}$	$\frac{15}{256}$	$\frac{16}{256}$	$\frac{16}{256}$	$\frac{16}{256}$	$\frac{16}{256}$
	$\frac{5}{256}$	$\frac{25}{256}$	$\frac{55}{256}$	$\frac{75}{256}$	$\frac{80}{256}$	$\frac{80}{256}$	$\frac{80}{256}$	$\frac{80}{256}$
	$\frac{11}{256}$	$\frac{55}{256}$	$\frac{121}{256}$	$\frac{165}{256}$	$\frac{176}{256}$	$\frac{176}{256}$	$\frac{176}{256}$	$\frac{176}{256}$
	$\frac{14}{256}$	$\frac{70}{256}$	$\frac{154}{256}$	$\frac{210}{256}$	$\frac{224}{256}$	$\frac{224}{256}$	$\frac{224}{256}$	$\frac{224}{256}$
	$\frac{11}{256}$	$\frac{55}{256}$	$\frac{121}{256}$	$\frac{165}{256}$	$\frac{176}{256}$	$\frac{176}{256}$	$\frac{176}{256}$	$\frac{176}{256}$
	$\frac{5}{256}$	$\frac{25}{256}$	$\frac{55}{256}$	$\frac{75}{256}$	$\frac{80}{256}$	$\frac{80}{256}$	$\frac{80}{256}$	$\frac{80}{256}$
	$\frac{1}{256}$	$\frac{5}{256}$	$\frac{11}{256}$	$\frac{15}{256}$	$\frac{16}{256}$	$\frac{16}{256}$	$\frac{16}{256}$	$\frac{16}{256}$

Fig. 19. Cells in the upper layer V shown in Fig. 18. Numbers in particular cells indicate how large was the portion of the cavity of unit volume which has reached the upper surface of the bulk.

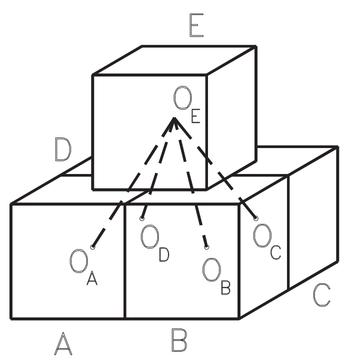

Fig. 20. Local configuration of five cells taken from the assumed system of finite cells for calculation of the displacement vector.

In the previous analysis of migration of the cavities upwards, the probabilities p_A, p_B, p_C and p_D that a unit cavity passed through the lower four cells A, B, C, D respectively, have been calculated. According to the finite cells technique described above, only one quarter of these portions of the unit cavity migrates from each cell A, B, C and D to the upper cell E. It is assumed that this migration takes place along the respective lines O_{AOE}, O_{BOE}, O_{COE}, O_{DOE}, joining central point O_E of the upper cell with the respective central points of each of the lower cells O_A, O_B, O_C and O_D (Fig. 21). Directions and magnitudes of these portions of unit cavity, which passed from respective lower cells to the cell E just above them, are represented by vectors $\mathbf{w}_{EA}$, $\mathbf{w}_{EB}$, $\mathbf{w}_{EC}$ and $\mathbf{w}_{ED}$. The resultant vector $\mathbf{w}_{\text{cav}}$ of those four vectors, represents the averaged flux of cavities migrating through the upper cell. The opposite vector $\mathbf{w}_m$ may be treated as representation of the average flux of the mass of the medium through the cell E.

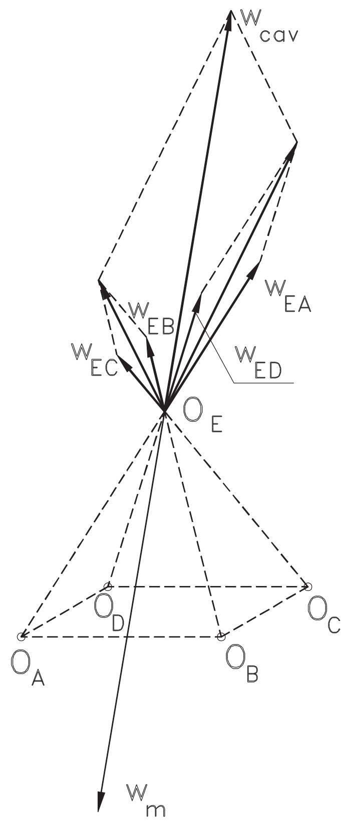

FIG. 21. Procedure of determining the displacement vector $\mathbf{w}_m$. of granular medium, caused by three-dimensional migration of cavities.

Vector $\mathbf{u}$ of the averaged displacement of medium particles in cell E has has the same orientation and direction as vector $\mathbf{w}_m$. It is assumed that vertical component of the displacement vector is the same as the previously calculated vertical displacement of the medium in the cell. Using this procedure, the vectors of surface displacements along two lines BA and CD (cf. Fig. 18) have been calculated – Fig. 22.

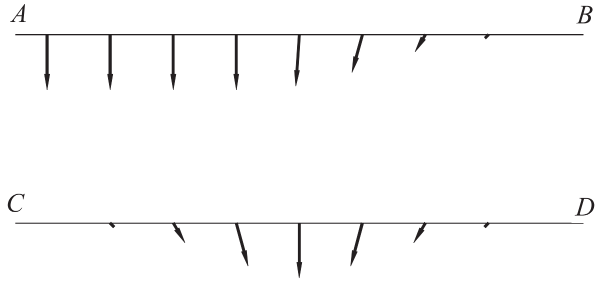

Fig. 22. Surface displacement vectors along two lines BA and CD (cf. Fig. 18).

6. Final remarks

The numerical procedure based on an assumed system of finite cells described above and supported by experimental simulation, may be used for the analysis of numerous problems of practical significance. The gravity flow of a granular medium is treated as a kinematical problem without the analysis of stresses. This numerical method constitutes a continuation of the ingenious idea of J. Litwiniszyn [5], who proposed to treat such problems in terms of stochastic analysis see also [10, 11]. In a simple way it allows to estimate the general mode and extent of terrain subsidence.

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