Arch. Mech., **59**, 6, pp. 519–539, Warszawa 2007

Weak solutions to anti-plane boundary value problems in a linear theory of elasticity with microstructure

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In this paper we formulate the interior and exterior Dirichlet and Neumann boundary value problems of anti-plane micropolar elasticity in a weak (Sobolev) space setting, we show that these problems are well-posed and the corresponding weak solutions depend continuously on the data. We show that the problem of torsion of a micropolar beam of (non-smooth) *arbitrary* cross-section can be reduced to an interior Neumann boundary value problem in antiplane micropolar elasticity and consider an example which demonstrates the significance of material microstructure.

Mathematics Subject Classification (2000). 74B99, 74A99, 45P05.

Key words: anti-plane micropolar elasticity, Sobolev spaces, boundary integral equation method.

1. Introduction

The governing equations and fundamental boundary value problems of a theory describing the three-dimensional deformations of a linearly elastic, homogeneous and isotropic solid which incorporates the effects of material microstructure [1] (also known as the linear theory of *micropolar*, *asymmetric or Cosserat elasticity*) were established by Eringen in [2] (see [3] for a review of works in this area and an extensive bibliography). The theory was intended to eliminate discrepancies between the classical elasticity and experiments, since the classical elasticity failed to present acceptable results when the effects of material microstructure were known to contribute significantly to the body's overall deformation, for example, in the case of granular bodies with large molecules (e.g. polymers), graphite or human bones (see, for example, [4]). These cases are becoming increasingly important in the design and manufacture of modern day advanced materials, as small-scale effects become paramount in the prediction of the overall mechanical behavior of these materials.

In [5], the boundary value problems of three-dimensional Cosserat elasticity were shown to be well-posed and subsequently solved in a rigorous manner using the boundary integral equation method. The corresponding problems in plane and anti-plane deformations (see [6], for example, which provides a comprehensive overview of the results obtained for classical anti-plane theory and their importance for applications) and in the bending of plates [7], are not accommodated by the results in [5]. This can be attributed to the fact that, in each case, the matrix of fundamental solutions has a highly unsuitable asymptotic behavior. In a series of recent papers, (see [8–12]), the boundary integral equation method has been developed and extended to allow for a rigorous analysis and solution of these problems which, as shown in [13] and [14], are of the great importance for applications since, for example, the problem of torsion of a micropolar beam of *arbitrary* cross-section can be reduced to an interior Neumann boundary value problem in anti-plane micropolar elasticity.

However, since the boundary value problems considered in [8–14] were formulated in a L^2 space, the corresponding solutions can be found only if the boundary is sufficiently smooth and cannot be obtained in the case of the reduced boundary smoothness or if the domain contains cracks. To obtain solutions for the domain with an irregular boundary, Chudinovich and Constanda [15] suggested that the boundary value problems arising in the bending of plates should be formulated in a weak (Sobolev) space setting. Such formulation allows to employ the boundary integral equation method to derive weak solutions in the form of integral potentials. In addition, this approach facilitates the close monitoring of the performance of numerical schemes in domains with relatively low degree of smoothness. Recently, Shmoylova *et al.* [16, 17] has demonstrated that the method introduced in [15] may be successfully applied to the investigation of the boundary value problems in the theory of plane micropolar elasticity. Meanwhile, consideration of the corresponding boundary value problems for domains with irregular boundaries in the case of anti-plane micropolar elasticity is, to the authors' knowledge, absent from the literature.

In this paper we formulate interior and exterior Dirichlet and Neumann boundary value problems of anti-plane micropolar elasticity in a weak (Sobolev) space setting, and we show that these problems are well-posed and the corresponding weak solutions depend continuously on the data. Further we show that the problem of torsion of a micropolar beam of non-smooth arbitrary cross-section can be reduced to the interior Neumann boundary-value problem in anti-plane Cosserat elasticity and the corresponding (weak) solution for a warping function may be found in terms of the modified integral potential. Using the computational procedure presented in [18], we find the solution to the interior Neumann problem in terms of generalized Fourier series and consider an example related to torsion of a graphite micropolar beam of square cross-section, which demonstrates that material microstructure has a significant effect on the warping function of a beam.

2. Preliminaries

In what follows Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$-matrices, E_n is the identity element in $\mathcal{M}_{m\times n}$, a superscript T indicates matrix transposition and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. Also, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Let S be a domain in $\mathbb{R}^2$ bounded by a closed C^2-curve ∂S and occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants $\lambda, \mu, \alpha, \beta$, γ and κ. The state of micropolar anti-plane shear is characterized by a displacement field $u\,(x') = (u_1\,(x')\,, u_2\,(x')\,, u_3\,(x'))^T$ and a microrotation field $\Phi\,(x') = (\phi_1\,(x')\,, \phi_2\,(x')\,, \phi_3\,(x'))^T$ of the form:

$$
\begin{aligned}
&u_\alpha\left(x'\right) = 0, \quad u_3(x') = u_3\left(x\right), \\
&\phi_3\left(x'\right) = 0, \quad \phi_\alpha(x') = \phi_\alpha\left(x\right),
\end{aligned} \tag{2.1}
$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. From (2.1) we find that the equilibrium equations of micropolar anti-plane shear written in terms of displacements and microrotations are given by [3]:

$$
L(\partial_x)u(x) + q\,(x) = 0, \qquad x \in S, \tag{2.2}
$$

in which now, denoting ϕ_α by u_α, we have $u(x) = (u_1, u_2, u_3)^T$, the matrix of the partial differential operator $L(\partial x) = L(\partial/\partial x_\alpha)$ is defined by

$$
L\,(\xi) = L\,(\xi_\alpha)
$$

$$
= \begin{pmatrix} (\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_1^2 & (\beta+\gamma-\kappa)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\beta+\gamma-\kappa)\xi_1\xi_2 & (\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\mu+\alpha)\Delta \end{pmatrix},
$$

where $\Delta = \xi_\alpha\xi_\alpha$ and $F = (F_1, F_2, F_3)^T$ represent body forces and couples.

Together with L we consider the boundary stress operator $T(\partial x) = T\,(\partial/\partial x_\alpha)$ defined by

$$T\,(\xi) = T\,(\xi_\alpha)$$

$$= \begin{pmatrix} (2\gamma+\beta)\,\xi_1 n_1 + (\gamma+\kappa)\,\xi_2 n_2 & (\gamma-\kappa)\xi_2 n_1 + \beta\xi_1 n_2 & -2\alpha n_2 \\ (\gamma-\kappa)\xi_1 n_2 + \beta\xi_2 n_1 & (\gamma+\kappa)\,\xi_1 n_1 + (2\gamma+\beta)\,\xi_2 n_2 & 2\alpha n_1 \\ 0 & 0 & (\mu+\alpha)\xi_\alpha n_\alpha \end{pmatrix}^T,$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S.

The internal energy density is given by

$$E\,(u,v) = E_1\,(u,v) + E_2\,(u,v) + \frac{\mu}{2}(u_{3,1}v_{3,1} + u_{3,2}v_{3,2}) + \frac{\alpha}{2}\Big[\,(2v_1 - u_{3,2})\,(2u_1 - v_{3,2}) + (2u_2 + u_{3,1})\,(2v_2 + v_{3,1})\Big], \tag{2.3}$$

where

$$E_1\,(u,v) \;=\; \frac{\gamma+\kappa}{2}\,(u_{1,2}v_{1,2} + u_{2,1}v_{2,1}) + \frac{(\gamma-\kappa)}{2}\,(u_{1,2}v_{2,1} + u_{2,1}v_{1,2})\,,$$

$$E_2\,(u,u) \;=\; \left(\gamma + \frac{\beta}{2}\right)(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) + \frac{\beta}{2}(u_{1,1}v_{2,2} + u_{2,2}v_{1,1}).$$

Throughout what follows we assume that

$$2\gamma + \beta > 0, \quad \kappa, \alpha, \gamma, \mu > 0.$$

Noting that the matrix $L_0\,(\xi)$ corresponding to the second order derivatives in the system (2.2) is invertible for all $\xi \neq 0$, since

$$\det L_0\,(\xi) = (\mu+\alpha)\,(\gamma+\kappa)\,(2\gamma+\beta)\,(\xi_1^2 + \xi_2^2)^3,$$

it is clear that (2.2) is an elliptic system and that $E\,(u,u)$ is a positive quadratic form. In fact, $E(u,u) = 0$ if and only if

$$u(x) = (0,0,c)^T, \tag{2.4}$$

where c is an arbitrary constant. This is the most general rigid displacement and microrotation associated with (2.1). Clearly, the space of such rigid displacements and microrotations $\mathcal{F}$ is spanned by the single vector $(0,0,1)$. Accordingly, we denote by $\mathbb{F}$ the matrix

$$\mathbb{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

from which it can be seen that $L\mathbb{F} = 0$ in $\mathbb{R}^2$, $T\mathbb{F} = 0$ on ∂S and a generic vector of the form (2.4) can be written as $\mathbb{F}k$, where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary.

Let S^+ be a domain in $\mathbb{R}^2$ bounded by a closed curve ∂S, and $S^- = \mathbb{R}^2 \backslash \overline{S^+}$. Using the same technique as in the derivation of the Betti formula [9], it is easy to show that if u is a solution of (2.2) in S^+, then for any $v \in C^2(S^+) \cap C^1(\overline{S}^+)$

$$\int_{S^+} v^T q dx = -\int_{S^+} v^T L u dx = 2\int_{S^+} E(u,v)\, dx - \int_{\partial S} v^T T u\, ds. \tag{2.5}$$

The analogue of the Betti formula in the exterior domain S^- requires that we should restrict the behavior of u at infinity. To this end, we consider the class $\mathcal{A}$ of vectors $u \in \mathcal{M}_{3\times 1}$ whose components, in terms of polar coordinates, admit an asymptotic expansion (as $r = |x| \to \infty$) of the form [9]

$$\begin{aligned} u_1(r,\theta) &= r^{-2}\left[m_0 \sin 2\theta + m_1(1-\cos 2\theta) + m_2\right] + O(r^{-3}), \\ u_2(r,\theta) &= r^{-2}\left[-m_0 \sin 2\theta - m_1(1-\cos 2\theta) + m_3\right] + O(r^{-3}), \\ u_3(r,\theta) &= r^{-1}\left[(m_3 - m_0)\cos\theta - (m_2 - m_1)\sin\theta\right] + O(r^{-2}), \end{aligned} \tag{2.6}$$

where $m_0, ..., m_3$ are arbitrary constants.

We introduce also the set

$$\mathcal{A}^* = \left\{u : u = \mathcal{F}k + s^{\mathcal{A}}\right\},$$

where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary and $s^{\mathcal{A}} \in \mathcal{M}_{3\times 1} \cap \mathcal{A}$. In view of (2.3), $\mathcal{A}$ and $\mathcal{A}^*$ are classes of finite energy functions.

For the exterior domain the Betti formula [9] is as follows: if u is a solution of (2.2) in S^-, then for any $v \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$

$$\int_{S^-} v^T q dx = -\int_{S^-} v^T L u dx = 2\int_{S^-} E(u,v)\, dx + \int_{\partial S} v^T T u\, ds. \tag{2.7}$$

3. Basic definitions of Sobolev spaces

For any $m \in \mathbb{R}$, let $H_m(\mathbb{R}^2)$ be the standard real Sobolev space of three-component distributions, equipped with the norm

$$\| u \|_m^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |\widetilde{u}(\xi)|^2 d\xi,$$

where $\widetilde{u}$ is the Fourier transform of u. In what follows we do not distinguish between equivalent norms and denote them by the same symbol; thus, the norm in $H_1(\mathbb{R}^2)$ can be defined by

$$\| u \|_1^2 = \| u \|_0^2 + \sum_{i=1}^{3} \| \nabla u_i \|_0^2 .$$

The spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to duality induced by $\langle \cdot, \cdot \rangle_0$.

We introduce the space $L^2_\omega(\mathbb{R}^2)$ of $(3{\times}1)$-vector functions $u = (\overline{u}, u_3)^T$, where $\overline{u} = (u_1, u_2)^T$, such that

$$\| u \|_{0,\omega}^2 = \int_{\mathbb{R}^2} \frac{|\overline{u}(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx + \int_{\mathbb{R}^2} \frac{|u_3(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx < \infty.$$

We consider the bilinear form $b(u,v) = 2\int_{\mathbb{R}^2} E(u,v)\ dx$. Let $H_{1,\omega}(\mathbb{R}^2)$ be the space of three-component distributions on $\mathbb{R}^2$ for which

$$\| u \|_{1,\omega}^2 = \| u \|_{0,\omega}^2 + b(u,u) < \infty,$$

$H_{-1,\omega}(\mathbb{R}^2)$ is dual to $H_{1,\omega}(\mathbb{R}^2)$ with respect to the duality generated by $\langle \cdot, \cdot \rangle_0$. The norm in $H_{-1,\omega}(\mathbb{R}^2)$ is denoted by $\| \cdot \|_{-1,\omega}$.

Let $\overset{\circ}{H}_m(S^+)$ be the subspace of $H_m(\mathbb{R}^2)$ consisting of all u which have a compact support in S^+. $H_m(S^+)$ is the space of the restrictions to S^+ of all $u \in H_m(\mathbb{R}^2)$. Denoting by $\pi^\pm$ the operators of restrictions from $\mathbb{R}^2$ to $S^\pm$, respectively, we introduce the norm of $u \in H_m(S^+)$ by $\| u \|_{m;S^+} = \inf_{v \in H_m(\mathbb{R}^2):\pi^+ v = u} \| v \|_m$. If $m = 1$, then the norms of $u \in \overset{\circ}{H}_1(S^+)$ and $u \in H_1(S^+)$ are equivalent to

$$\left\{ \| u \|_{0;S^+}^2 + \sum_{i=1}^{3} \int_{S^+} |\nabla u_i(x)|^2 dx \right\}^{1/2} .$$

The spaces $\overset{\circ}{H}_m(S^+)$ and $H_{-m}(S^+)$ are dual with respect to the duality induced by $\langle \cdot, \cdot \rangle_{0;S^+}$.

Let $\overset{\circ}{H}_{1,\omega}(S^-)$ be the subspace of $H_{1,\omega}(\mathbb{R}^2)$ consisting of all u which have compact support in S^-. $H_{1,\omega}(S^-)$ is the space of the restrictions to S^- of all $u \in H_{1,\omega}(\mathbb{R}^2)$. The norm in $H_{1,\omega}(S^-)$ is defined by

$$\| u \|_{1,\omega;S^-} = \inf_{v \in H_{1,\omega}(\mathbb{R}^2):\pi^- v = u} \| v \|_{1,\omega} \,.$$

From the definition it follows that $H_{1,\omega}(S^-)$ is isometric to $H_{1,\omega}(\mathbb{R}^2)\backslash \overset{\circ}{H}_1(S^+)$. It can be shown that the norm of $u \in H_{1,\omega}(S^-)$ is equivalent to

$$\left\{ \| u \|^2_{0,\omega;S^-} + b_-(u,u) \right\}^{1/2},$$

where

$$\| u \|^2_{0,\omega;S^-} = \int\limits_{S^-} \frac{|\overline{u}(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx + \int\limits_{S^-} \frac{|u_3(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx$$

and $b_\pm(u,v) = 2\int_{S^\pm} E(u,v)\ dx$. This norm is compatible with asymptotic class $\mathcal{A}$.

The dual of $\overset{\circ}{H}_{1,\omega}(S^-)$ with respect to the duality generated by $\langle \cdot,\cdot \rangle_{0;S^-}$ is the space $H_{-1,\omega}(S^-)$, with norm $\| \cdot \|_{-1,\omega;S^-}$; the dual of $H_{1,\omega}(S^-)$ is $\overset{\circ}{H}_{-1,\omega}(S^-)$, which can be identified with a subspace of $H_{-1,\omega}(\mathbb{R}^2)$.

Let $H_m(\partial S)$ be the standard Sobolev space of distributions on ∂S, with norm $\| \cdot \|_{m;\partial S}$ $H_m(\partial S)$ and $H_{-m}(\partial S)$ are dual with respect to the duality generated by the inner product $\langle \cdot,\cdot \rangle_{0;\partial S}$ in $L^2(\partial S)$.

We denote by γ^+ and γ^- the trace operators defined first on $C_0^\infty(S^\pm)$ and then extended by continuity to the surjections $\gamma^+ : H_1(S^+) \longrightarrow H_{1/2}(\partial S)$, $\gamma^- : H_{1,\omega}(S^-) \longrightarrow H_{1/2}(\partial S)$. This is possible because of the local equivalence of $H_{1,\omega}(S^-)$ and $H_1(S^-)$. We also consider a continuous extension operators $l^+ : H_{1/2}(\partial S) \longrightarrow H_1(S^+)$, $l^- : H_{1/2}(\partial S) \longrightarrow H_1(S^-)$; the latter, since the norm in $H_1(S^-)$ is stronger than that in $H_{1,\omega}(S^-)$, can also be regarded as a continuous operator from $H_{1/2}(\partial S)$ into $H_{1,\omega}(S^-)$.

To proceed further we will need the following well-known fact from the functional analysis.

Theorem 1. *(Lax–Milgram Lemma). Let H be a Hilbert space and $b(u,v)$ be a bilinear functional defined for every ordinate pair $u,v \in H$, for which there exist two constants h and k such that:*

$$|b(u,v)| \leqslant h\,\|u\|\,\|v\|\,, \quad \|u\|^2 \leqslant k\,|b(u,u)| \quad \forall u,v \in H.$$

In this case we say that $b(u,v)$ is coercive. Then however we assign the bounded linear functional $\mathcal{L}(v)$ on H, there exists one and only one u such that

$$b(u,v) = \mathcal{L}(v), \quad \forall v \in H, \quad \|u\| \leqslant c\,\|\mathcal{L}\|_*\,,$$

where $\|\cdot\|_*$ is the norm on the dual H' of H.

The proof of this lemma can be found in [19].

4. Interior boundary value problems

We consider the Dirichlet and Neumann interior boundary value problems.

The (interior) Dirichlet problem is formulated as follows.

(D$^+$) Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (2.2) such that $u\,|_{\partial S} = f,.$

where f is prescribed on ∂S.

Let (D$_0^+$) be the interior Dirichlet problem with $f = 0$. From (2.5) we see that a solution u of (D$_0^+$) satisfies

$$b_+(u,v) = \langle q, v\rangle_{0,S^+} \quad \forall v \in C_0^\infty(S^+). \tag{4.1}$$

Since $C_0^\infty(S^+)$ is dense in $\mathring{H}_1(S^+)$, it is clear that (4.1) holds for any $v \in \mathring{H}_1(S^+)$. Obviously, any $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (4.1) for any $v \in \mathring{H}_1(S^+)$ and $u|_{\partial S} = 0$ is a classical (regular) solution of (D$_0^+$). Hence, the variational formulation of (D$_0^+$) is as follows.

Find $u \in \mathring{H}_1(S^+)$ such that

$$b_+(u,v) = \langle q, v\rangle_{0,S^+} \quad \forall v \in \mathring{H}_1(S^+). \tag{4.2}$$

THEOREM 2. *There exists a constant $c = c(S^+) > 0$ such that*

$$b_+(u,u) + \| u \|^2_{0;S^+} \geqslant c \| u \|^2_{1;S^+} \ \forall u \in H_1(S^+). \tag{4.3}$$

P r o o f. In view of the condition on α, β, γ, κ and μ, $E(u,u)$ is a positive quadratic form. Consequently, we may introduce the space G of all (3×1)-vector functions u on S^+ with norm

$$\| u \|^2_G = b_+(u,u) + \| u \|^2_{0;S^+}\ .$$

Let $\{u^{(n)}\}$ be a Cauchy sequence in G. From the definition of $b_+(u,v)$ it follows that there are ρ_{11}, ρ_{22}, ρ_{12}, ρ_{21}, ρ_{31}, ρ_{32}, ρ_{132}, ρ_{231}, $\beta \in L^2(S^+)$ such that

$$\begin{aligned} u^{(n)}_{1,1} &\to \rho_{11},\ u^{(n)}_{2,2} \to \rho_{22},\ u^{(n)}_{1,2} \to \rho_{12},\ u^{(n)}_{2,1} \to \rho_{21}, \\ u^{(n)}_{3,1} &\to \rho_{31},\ u^{(n)}_{3,2} \to \rho_{32}, \\ 2u^{(n)}_1 - u^{(n)}_{3,2} &\to \rho_{132},\ 2u^{(n)}_2 + u^{(n)}_{3,1} \to \rho_{231},\ u^{(n)} \to \rho \end{aligned}$$

in $L^2(S^+)$. Then

$$\begin{aligned} u_{1,1}^{(n)} &\to \rho_{11} = \rho_{1,1},\ u_{2,2}^{(n)} \to \rho_{22} = \rho_{2,2}, \\ u_{1,2}^{(n)} &\to \rho_{12} = \rho_{1,2},\ u_{2,1}^{(n)} \to \rho_{21} = \rho_{2,1}, \\ u_{3,1}^{(n)} &\to \rho_{31} = \rho_{3,1},\ u_{3,2}^{(n)} \to \rho_{32} = \rho_{3,2}, \\ 2u_1^{(n)} - u_{3,2}^{(n)} &\to \rho_{132} = 2\rho_1 - \rho_{3,2}, \\ 2u_2^{(n)} + u_{3,1}^{(n)} &\to \rho_{231} = 2\rho_2 + \rho_{3,1}, \end{aligned}$$

in the sense of distributions, hence, also in $L^2(S^+)$. This implies that ρ, $\rho_{1,1}$, $\rho_{2,2}$, $\rho_{1,2}, \rho_{2,1}$, $\rho_{3,1}$, $\rho_{3,2}$, $2\rho_1 - \rho_{3,2}$, $2\rho_2 + \rho_{3,1} \in L^2(S^+)$ and $\| u^{(n)} - \rho \|_G \to 0$, what means that G is complete. For any $u \in G$ we have $u_1 \in L^2(S^+)$, $u_{1,1} \in L^2(S^+)$, $u_{1,2} \in L^2(S^+)$, consequently, $u_1 \in H_1(S^+)$. The facts that $u_2 \in H_1(S^+)$ and $u_3 \in H_1(S^+)$ are shown similarly. This indicates that G is a subset of $H_1(S^+)$. The converse statement being obvious, we conclude that G and $H_1(S^+)$ coincide as sets. The imbedding operator $\mathcal{I} : H_1(S^+) \to G$ is bijective and continuous, therefore, by Banach's theorem [19] on the inverse operator $\mathcal{I}^{-1}$ is continuous; in other words, $\| u \|_G^2 \geqslant c \| u \|_{1,S^+}^2$, which is the same as (4.3). □

Theorem 3. *There exists a constant $c = c(S^+) > 0$ such that*

$$b_+(u,u) \geqslant c \| u \|_1^2 \ \forall u \in \mathring{H}_1(S^+). \tag{4.4}$$

P r o o f. We claim that there is a $c = c(S^+) > 0$ such that

$$b_+(u,u) \geqslant c \| u \|_{0,S^+}^2 \ \forall u \in \mathring{H}_1(S^+). \tag{4.5}$$

Indeed, if the opposite is true, then we can construct a sequence $\{u^{(n)}\}$ in $\mathring{H}_1(S^+)$ such that

$$b_+(u^{(n)}, u^{(n)}) \to 0, \qquad \| u^{(n)} \|_{0,S^+} = 1 \text{ for all } n. \tag{4.6}$$

By (4.3), $\{u^{(n)}\}$ is bounded in $H_1(S^+)$ so, by Rellich's lemma [19], it contains a convergent subsequence (again denoted by $\{u^{(n)}\}$, for simplicity); that is, there is a $u \in L^2(S^+)$ such that $u^{(n)} \to u$ in $L^2(S^+)$. This means that, in view of (4.6),

$$u_{1,1}^{(n)} \to 0 = u_{1,1}, \quad u_{2,2}^{(n)} \to 0 = u_{2,2},$$

$$u_{1,2}^{(n)} \to 0 = u_{1,2}, \quad u_{2,1}^{(n)} \to 0 = u_{2,1},$$

$$u_{3,1}^{(n)} \to 0 = u_{3,1}, \quad u_{3,2}^{(n)} \to 0 = u_{3,2},$$

$$2u_1^{(n)} - u_{3,2}^{(n)} \to 0 = 2u_1 - u_{3,2},$$

$$2u_2^{(n)} + u_{3,1}^{(n)} \to 0 = 2u_2 + u_{3,1}$$

in $L^2(S^+)$. These equalities imply that u is a rigid displacement. Since $u = 0$ on ∂S, it follows that $u = 0$ in $\overline{S^+}$ which contradicts the corollary $\| u \|_{0,S^+} = 1$ of (4.6). Hence, (4.5) holds, and the statement of the theorem is now obtained from (4.5) and (4.3). □

Theorem 4. *Problem (4.2) has a unique solution $u \in \overset{\circ}{H}_1(S^+)$ for every $q \in H_{-1}(S^+)$, and this solution satisfies the estimate*

$$\| u \|_1 \leqslant c \| q \|_{-1,S^+} . \tag{4.7}$$

P r o o f. Since $H_{-1}(S^+)$ is the dual of $\overset{\circ}{H}_1(S^+)$ with respect to duality induced by $\langle \cdot, \cdot \rangle_{0,S^+}$, it follows that $\langle q, v \rangle_{0,S^+}$ is continuous linear functional on $\overset{\circ}{H}_1(S^+)$ for every $q \in H_{-1}(S^+)$. By Theorem 2, $b_+(u, v)$ is a continuous bilinear form on $\overset{\circ}{H}_1(S^+) \times \overset{\circ}{H}_1(S^+)$. By Theorem 3, $b_+(u, u)$ is coercive on $\overset{\circ}{H}_1(S^+)$. We now apply the Lax-Milgram lemma to complete the proof. □

The variational formulation of (D$^+$) is as follows.

Find $u \in H_1(S^+)$ such that

$$b_+(u, v) = \langle q, v \rangle_{0,S^+} \; \forall v \in \overset{\circ}{H}_1(S^+) \tag{4.8}$$

and

$$\gamma^+ u = f. \tag{4.9}$$

Theorem 5. *Problem (4.8)–(4.9) has a unique solution $u \in H_1(S^+)$ for any $q \in H_{-1}(S^+)$ and any $f \in H_{1/2}(\partial S)$, and this solution satisfies the estimate*

$$\| u \|_{1,S^+} \leqslant c \left(\| q \|_{-1,S^+} + \| f \|_{1/2,\partial S} \right) . \tag{4.10}$$

P r o o f. The substitution $u = u_0 + l^+ f$ reduces (4.8)-(4.9) to a new variational problem, consisting in finding $u_0 \in \overset{\circ}{H}_1(S^+)$ such that

$$b_+(u_0, v) = \langle q, v \rangle_{0,S^+} - b_+(l^+ f, v) \quad \forall v \in \overset{\circ}{H}_1(S^+). \tag{4.11}$$

Clearly, $b_+(u,v)$ is continuous on $H_1(S^+) \times H_1(S^+)$, which implies that

$$\langle q, v \rangle_{0,S^+} - b_+(l^+ f, v)$$

is a continuous linear functional on $\overset{\circ}{H}_1(S^+)$. Also,

$$\begin{aligned} | \langle q, v \rangle_{0,S^+} - b_+(l^+ f, v) \ | &\leqslant \| q \|_{-1,S^+} \| v \|_1 + c \| l^+ f \|_{1,S^+} \| v \|_1 \\ &\leqslant c \left(\| q \|_{-1,S^+} + \| l^+ f \|_{1,S^+} \right) \| v \|_1 \ . \end{aligned} \tag{4.12}$$

The statement now follows from Theorem 4, with (4.10) obtained from (4.12) and the continuity of l^+. □

The (interior) Neumann problem is formulated as follows.

(N^+) Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (2.2) and $Tu = g$ on ∂S,

where g is prescribed on ∂S.

In this case (2.5) leads to the following variational formulation.

Find $u \in H_1(S^+)$ such that

$$b_+(u,v) = \langle q, v \rangle_{0,S^+} + \left\langle g, \gamma^+ v \right\rangle_{0,\partial S} \quad \forall v \in H_1(S^+). \tag{4.13}$$

It is clear that, in view of the properties of rigid displacements,

$$\left\langle q, \mathbb{F}^{(i)} \right\rangle_{0,S^+} + \left\langle g, \gamma^+ \mathbb{F}^{(i)} \right\rangle_{0,\partial S} = 0 \tag{4.14}$$

is a necessary solvability condition for (N^+). In what follows we assume that (4.14) holds.

THEOREM 6. *There is a $c = c(S^+) > 0$ such that for any $u \in H_1(S^+)$*

$$b_+(u,u) + \sum_{i=1}^{3} \left\langle u, \mathbb{F}^{(i)} \right\rangle_{0,S^+}^2 \geqslant c \| u \|_{1,S^+}^2, \tag{4.15}$$

$$b_+(u,u) + \sum_{i=1}^{3} \left\langle \gamma^+ u, \gamma^+ \mathbb{F}^{(i)} \right\rangle_{0,\partial S}^2 \geqslant c \| u \|_{1,S^+}^2 \ . \tag{4.16}$$

P r o o f. If either (4.15) or (4.16) does not hold, then, by repeating the argument in the proof of Theorem 3, we find that there is a $u \in \mathcal{F}$ such that $\langle u, \mathbb{F}^{(i)} \rangle_{0,S^+} = 0$ in the case of (4.15) or $\langle \gamma^+ u, \gamma^+ \mathbb{F}^{(i)} \rangle_{0,\partial S} = 0$ in the case of (4.16), while $\| u \|_{1,S^+} = 1$, which is an obvious contradiction. Inequalities (4.15) and (4.16) hold. □

THEOREM 7. *Problem (4.13) is solvable for any* $q \in \overset{\circ}{H}_{-1}(S^+)$ *and* $g \in H_{-1/2}(\partial S)$. *Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,S^+} \leqslant c \left(\| q \|_{-1,S^+} + \| g \|_{-1/2,\partial S} \right). \tag{4.17}$$

P r o o f. We introduce the factor space $\mathcal{H}_1(S^+) = H_1(S^+) \backslash \mathcal{F}$, the bilinear form

$$\mathcal{B}_+(U,V) = b_+(u,v) \text{ on } \mathcal{H}_1(S^+) \times \mathcal{H}_1(S^+),$$

and the linear functional

$$\mathcal{L}(V) = \langle q, v \rangle_{0,S^+} + \left\langle g, \gamma^+ v \right\rangle_{0,\partial S} \text{ on } \mathcal{H}_1(S^+),$$

where u and v are arbitrary representatives of the classes $U, V \in \mathcal{H}_1(S^+)$. We define the norm in $\mathcal{H}_1(S^+)$ by

$$\| U \|_{\mathcal{H}_1(S^+)} = \inf_{\substack{u \in H_1(S^+) \\ u \in U}} \| u \|_{1,S^+} .$$

Instead of (4.13) we now consider the new variational problem of finding $U \in \mathcal{H}_1(S^+)$ such that

$$\mathcal{B}_+(U,V) = \mathcal{L}(V) \quad \forall V \in \mathcal{H}_1(S^+). \tag{4.18}$$

In view of the definition of $\mathcal{B}_+(U,V)$, we see that for any U, $V \in \mathcal{H}_1(S^+)$ and any $u \in U$, $v \in V$,

$$|\mathcal{B}_+(U,V)| = |b_+(u,v)| \leqslant c \| u \|_{1,S^+} \| v \|_{1,S^+},$$

therefore

$$|\mathcal{B}_+(U,V)| \leqslant c \inf_{\substack{u \in H_1(S^+) \\ u \in U}} \| u \|_{1,S^+} \inf_{\substack{v \in H_1(S^+) \\ v \in U}} \| v \|_{1,S^+} = c \| U \|_{\mathcal{H}_1(S^+)} \| V \|_{\mathcal{H}_1(S^+)},$$

which shows that $\mathcal{B}_+(U,V)$ is continuous on $\mathcal{H}_1(S^+) \times \mathcal{H}_1(S^+)$.

Next, we can choose $\widetilde{u} \in U$ such that $\left\langle \widetilde{u}, \mathbb{F}^{(i)} \right\rangle_{0,S^+} = 0$. Then, by Theorem 6,

$$\mathcal{B}_+(U,U) = b_+(\widetilde{u},\widetilde{u}) \geqslant c \| \widetilde{u} \|^2_{1,S^+} \geqslant c \inf_{\substack{u \in H_1(S^+) \\ u \in U}} \| u \|_{1,S^+} = c \| U \|_{\mathcal{H}_1(S^+)},$$

so $\mathcal{B}_+(U,U)$ is coercive on $\mathcal{H}_1(S^+)$.

Finally, since γ^+ is continuous on $H_1(\partial S)$, for any $V \in \mathcal{H}_1(S^+)$

$$\begin{aligned}\mathcal{L}(V) &\leqslant \| q \|_{-1,S^+} \| v \|_{1,S^+} + \| g \|_{-\frac{1}{2},\partial S} \| \gamma^+ v \|_{\frac{1}{2},\partial S} \\ &\leqslant c \left(\| q \|_{-1,S^+} + \| g \|_{-\frac{1}{2},\partial S} \right) \| v \|_{1,S^+},\end{aligned}$$

which shows that $\mathcal{L}$ is continuous linear functional on $\mathcal{H}_1(S^+)$.

By the Lax–Milgram lemma, problem (4.18) has a unique solution $U \in \mathcal{H}_1(S^+)$, and this solution satisfies the estimate

$$\| U \|_{\mathcal{H}_1(S^+)} \leqslant c \left(\| q \|_{-1,S^+} + \| g \|_{-\frac{1}{2},\partial S} \right).$$

Clearly, any $u \in U$ is a solution of (4.13), and $u_0 \in U$ such that

$$\| u_0 \|_{1,S^+} = \| U \|_{\mathcal{H}_1(S^+)}$$

satisfies (4.17). □

5. Exterior boundary value problems

We consider the Dirichlet and Neumann exterior boundary value problems.

The (exterior) Dirichlet problem is formulated as follows:

(D^-) Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (2.2) such that $u|_{\partial S} = f$,

where f is prescribed on ∂S.

Let (D_0^-) be the exterior Dirichlet problem with $f = 0$. From (2.6) we see that a solution u of (D_0^-) satisfies

$$b_-(u, v) = \langle q, v \rangle_{0,S^-} \qquad \forall v \in C_0^\infty(S^-). \tag{5.1}$$

Since $C_0^\infty(S^-)$ is dense in $\overset{\circ}{H}_{1,\omega}(S^-)$, it is clear that (4.19) holds for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$. Obviously, any $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (4.19) for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$ and $u|_{\partial S} = 0$ is a classical (regular) solution of (D_0^-). Hence, the variational formulation of (D_0^-) is as follows.

Find $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ such that

$$b_-(u, v) = \langle q, v \rangle_{0,S^-} \qquad \forall v \in \overset{\circ}{H}_{1,\omega}(S^-). \tag{5.2}$$

Let $K_R^- = \left\{ x \in \mathbb{R}^2 : |x| > R \right\}$, $R > 1$, and $\partial K_R = \left\{ x \in \mathbb{R}^2 : |x| = R \right\}$.

THEOREM 8. *There are* $c_i(R) = \text{const} > 0$ *such that*

$$\| u \|^2_{0,\omega;K_R^-} \leqslant c_1 b_{K_R^-}(u,u) + c_2 \| u \|^2_{0,\partial K_R}, \quad \forall u \in H_{1,\omega}(K_R^-), \tag{5.3}$$

where $\| \cdot \|_{0,\partial K_R}$ *and* $\| \cdot \|_{1/2,\partial K_R}$ *are the norms in* $L^2(\partial K_R)$ *and* $H_{1/2}(\partial K_R)$, *respectively.*

The proof of this theorem follows the procedure described in [20].

THEOREM 9. *There is a* $c = c(S^-) = const > 0$ *such that any* $u \in H_{1,\omega}(S^-)$ *satisfies the estimates*

$$\| u \|^2_{1,\omega;S^-} \leqslant c \left[b_-(u,u) + \left| \int_{\Gamma_0} u \; ds \right|^2 \right], \tag{5.4}$$

$$\| u \|^2_{1,\omega;S^-} \leqslant c \left[b_-(u,u) + \sum_{i=1}^{3} \left\langle u, \gamma^- \mathbb{F}^{(i)} \right\rangle^2_{0,\partial S} \right], \tag{5.5}$$

where $\Gamma_0 \subseteq \partial S$, *measure of* Γ_0 *is larger than zero.*

P r o o f. We claim that for any $u \in H_{1,\omega;S^-}$

$$\| u \|^2_{0,\omega;S^-} \leqslant c \left[b_-(u,u) + \left| \int_{\Gamma_0} u \; \mathrm{d}s \right|^2 \right], \tag{5.6}$$

$$\| u \|^2_{0,\omega;S^-} \leqslant c \left[b_-(u,u) + \sum_{i=1}^{3} \left\langle u, \gamma^- \mathbb{F}^{(i)} \right\rangle^2_{0,\partial S} \right]. \tag{5.7}$$

First suppose that the opposite of formula (5.6) is true. Then we can construct a sequence $\{u^{(n)}\} \subset H_{1,\omega}(S^-)$ such that

$$b_-(u^{(n)}, u^{(n)}) \to 0, \qquad \int_{\Gamma_0} u^{(n)} ds \to 0, \tag{5.8}$$

while

$$\| u \|^2_{0,\omega;S^-} = 1. \tag{5.9}$$

Let ∂K_R be a circle with the center at the origin and of radius $R > 1$ sufficiently large so that ∂S is contained inside ∂K_R. We write $S_R = S^- \cap K_R^-$. Since S_R

is bounded, we may repeat the proof of Theorem 3 to deduce that there is a $c_R = const > 0$ such that

$$\| u \|^2_{1;S_R} \leqslant c_R \left[b_{S_R}(u,u) + \left| \int_{\Gamma_0} u \, \mathrm{d}s \right|^2 \right] \quad \forall u \in H_1(S_R). \tag{5.10}$$

Then, by Theorem 8,

$$\| u^{(n)} \|^2_{0,\omega;S^-} = \| u^{(n)} \|^2_{0,\omega;S_R} + \| u^{(n)} \|^2_{0,\omega;K_R^-} \leqslant \| u^{(n)} \|^2_{0,S_R} + \| u^{(n)} \|^2_{0,\omega;K_R^-}$$

$$\leqslant c_R \left[b_{S_R}(u^{(n)}, u^{(n)}) + \left| \int_{\Gamma_0} u^{(n)} \, \mathrm{d}s \right|^2 \right] + c_1 b_{K_R^-}(u^{(n)}, u^{(n)})$$

$$+ c_2 \| \overset{\frown}{u}^{(n)} \|^2_{1/2,\partial K_R} + c_3 \| u_3^{(n)} \|^2_{0,\partial K_R} .$$

From (5.10) for $u^{(n)}$ we now conclude that $u^{(n)} \to 0$ in $H_1(S_R)$. Then $u^{(n)} \to 0$ in $H_{1/2}(\partial K_R)$, hence in $L^2(\partial K_R)$. Consequently, from the last inequality we find that $\lim_{n\to\infty} \| u^{(n)} \|^2_{0,\omega;S^-} = 0$, which contradicts (5.9). Formula (5.7) is proved similarly. □

Theorem 10. *The variational problem (5.2) has a unique solution* $u \in \overset{\circ}{H}_{1,\omega}(S^-)$ *for every* $q \in H_{-1,\omega}(S^-)$*, and this solution satisfies the estimate*

$$\| u \|_{1,\omega} \leqslant c \| q \|_{-1,\omega;S^-} .$$

P r o o f. By Theorem 9,

$$\| u \|^2_{1,\omega} \leqslant c b_-(u,u) \quad \forall u \in \overset{\circ}{H}_{1,\omega}(S^-),$$

which means that $b_-(u,u)$ is coercive on $\overset{\circ}{H}_{1,\omega}(S^-)$. Since $b_-(u,u)$ is clearly continuous on $\overset{\circ}{H}_{1,\omega}(S^-) \times \overset{\circ}{H}_{1,\omega}(S^-)$, we apply the Lax–Milgram lemma to complete the proof. □

The variational formulation of (D^-) is as follows.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u,v) = \langle q, v \rangle_{0,S^-} \ \forall v \in \overset{\circ}{H}_{1,\omega}(S^-) \tag{5.11}$$

and

$$\gamma^- u = f. \tag{5.12}$$

THEOREM 11. *Problem (5.11)–(5.12) has a unique solution $u \in H_{1,\omega}(S^-)$ for any $q \in H_{-1,\omega}(S^-)$ and any $f \in H_{\frac{1}{2}}(\partial S)$, and this solution satisfies the estimate*

$$\| u \|_{1,\omega;S^-} \leqslant c \left(\| q \|_{-1,\omega;S^-} + \| f \|_{\frac{1}{2},\partial S} \right).$$

P r o o f. The substitution $u = u_0 + l^- f$ reduces (5.11)–(5.12) to a new variational problem, consisting in finding $u_0 \in \overset{\circ}{H}_{1,\omega}(S^-)$ such that

$$b_-(u_0, v) = \langle q, v \rangle_{0,S^-} - b_-(l^- f, v) \quad \forall v \in \overset{\circ}{H}_{1,\omega}(S^-). \tag{5.13}$$

Since for any $v \in \overset{\circ}{H}_{1,\omega}(S^-)$

$$\begin{aligned} |\langle q, v \rangle_{0,S^-} - b_-(l^- f, v)| &\leqslant \| q \|_{-1,\omega;S^-} \| v \|_{1,\omega} + \left[b_-(l^- f, l^- f) \right]^{1/2} \left[b_-(v, v) \right]^{1/2} \\ &\leqslant \left(\| q \|_{-1,\omega;S^-} + \| l^- f \|_{1,\omega;S^-} \right) \| v \|_{1,\omega} \\ &\leqslant c \left(\| q \|_{-1,\omega;S^-} + \| f \|_{1/2,\partial S} \right) \| v \|_{1,\omega}, \end{aligned}$$

the linear form $\langle q, v \rangle_{0,S^-} - b_-(l^- f, v)$ is a continuous linear functional on $\overset{\circ}{H}_{1,\omega}(S^-)$. The statement of the theorem now follows from the Lax–Milgram lemma applied to the auxiliary problem (5.13) and the estimates

$$\| u_0 \|_{1,\omega} \leqslant c \left(\| q \|_{-1,\omega;S^-} + \| f \|_{1/2,\partial S} \right)$$

$$\| u \|_{1,\omega;S^-} \leqslant \| u_0 \|_{-1,\omega;S^-} + \| l^- f \|_{1,\omega;S^-} \leqslant c \left(\| q \|_{-1,\omega;S^-} + \| f \|_{\frac{1}{2},\partial S} \right).$$

□

The (exterior) Neumann problem is formulated as follows.

(N$^-$) Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$ satisfying (2.2) and $Tu = g$ on ∂S,

where g is prescribed on ∂S.

In this case (2.6) leads to the following variational formulation.

Find $u \in H_{1,\omega}(S^-)$ such that

$$b_-(u, v) = \langle q, v \rangle_{0,S^-} - \left\langle g, \gamma^- v \right\rangle_{0,\partial S} \quad \forall v \in H_{1,\omega}(S^-). \tag{5.14}$$

In view of the properties of rigid displacements, the condition

$$\left\langle q, \mathbb{F}^{(i)} \right\rangle_{0,S^-} - \left\langle g, \gamma^- \mathbb{F}^{(i)} \right\rangle_{0,\partial S} = 0 \tag{5.15}$$

is a necessary solvability condition for (5.14). In what follows we assume that (5.15) holds.

THEOREM 12. *Problem (5.14) is solvable for any* $q \in \overset{\circ}{H}_{-1,\omega}(S^-)$ *and* $g \in H_{-1/2}(\partial S)$. *Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,\omega;S^-} \leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-1/2,\partial S} \right). \tag{5.16}$$

P r o o f. We introduce the factor space $\mathcal{H}_{1,\omega}(S^-) = H_{1,\omega}(S^-) \backslash \mathcal{F}$, the bilinear form

$$\mathcal{B}_-(U,V) = b_-(u,v) \text{ on } \mathcal{H}_{1,\omega}(S^-) \times \mathcal{H}_{1,\omega}(S^-),$$

and the linear functional

$$\mathcal{L}(V) = \langle q, v \rangle_{0,S^-} - \left\langle g, \gamma^- v \right\rangle_{0,\partial S} \text{ on } \mathcal{H}_{1,\omega}(S^-),$$

where u and v are arbitrary representatives of the classes $U, V \in \mathcal{H}_{1,\omega}(S^-)$. We define the norm in $\mathcal{H}_{1,\omega}(S^-)$ by

$$\| U \|_{\mathcal{H}_{1,\omega}(S^-)} = \inf_{\substack{u \in H_{1,\omega}(S^-) \\ u \in U}} \| u \|_{1,\omega;S^-} .$$

Instead of (5.14) we now consider the new variational problem of finding $U \in \mathcal{H}_{1,\omega}(S^-)$ such that

$$\mathcal{B}_-(U,V) = \mathcal{L}(V) \quad \forall V \in \mathcal{H}_{1,\omega}(S^-). \tag{5.17}$$

In view of the definition of $\mathcal{B}_-(U,V)$, we see that for any U, $V \in \mathcal{H}_{1,\omega}(S^-)$ and any $u \in U$, $v \in V$

$$|\mathcal{B}_-(U,V)| = |b_-(u,v)| \leqslant c \| u \|_{1,\omega;S^-} \| v \|_{1,\omega;S^-},$$

therefore

$$\begin{aligned} |\mathcal{B}_-(U,V)| \; &\leqslant \; c \inf_{\substack{u \in H_{1,\omega}(S^-) \\ u \in U}} \| u \|_{1,\omega;S^-} \inf_{\substack{v \in H_{1,\omega}(S^-) \\ v \in U}} \| v \|_{1,\omega;S^-} \\ &= \; c \| U \|_{\mathcal{H}_{1,\omega}(S^-)} \| V \|_{\mathcal{H}_{1,\omega}(S^-)}, \end{aligned}$$

which shows that $\mathcal{B}_-(U,V)$ is continuous on $\mathcal{H}_{1,\omega}(S^-) \times \mathcal{H}_{1,\omega}(S^-)$.

Next, we can choose $\widetilde{u} \in U$ such that $\left\langle \gamma^- \widetilde{u}, \gamma^- \mathbb{F}^{(i)} \right\rangle_{0,\partial S} = 0$. Then, by (5.5),

$$\mathcal{B}_-(U,U) = b_-(\widetilde{u}, \widetilde{u}) \geqslant c \| \widetilde{u} \|^2_{1,\omega;S^-} \geqslant c \inf_{\substack{u \in H_{1,\omega}(S^-) \\ u \in U}} \| u \|^2_{1,\omega;S^-} = k \| U \|^2_{\mathcal{H}_{1,\omega}(S^-)},$$

so $\mathcal{B}_-(U,U)$ is coercive on $\mathcal{H}_{1,\omega}(S^-)$.

Finally, since γ^- is continuous on $H_{1,\omega}(S^-)$, for any $V \in \mathcal{H}_{1,\omega}(S^-)$

$$\begin{aligned}\mathcal{L}(V) &\leqslant \| q \|_{-1,\omega} \| v \|_{1,\omega;S^-} + \| g \|_{-\frac{1}{2},\partial S} \| \gamma^- v \|_{\frac{1}{2},\partial S} \\ &\leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-\frac{1}{2},\partial S} \right) \| v \|_{1,\omega;S^-},\end{aligned}$$

which shows that $\mathcal{L}$ is a continuous linear functional on $\mathcal{H}_{1,\omega}(S^-)$.

By the Lax–Milgram lemma, problem (5.17) has a unique solution $U \in \mathcal{H}_{1,\omega}(S^-)$, and this solution satisfies the estimate

$$\| U \|_{\mathcal{H}_{1,\omega}(S^-)} \leqslant c \left(\| q \|_{-1,\omega} + \| g \|_{-\frac{1}{2},\partial S} \right).$$

Clearly, any $u \in U$ is a solution of (5.14), and $u_0 \in U$ such that

$$\| u_0 \|_{1,\omega;S^-} = \| U \|_{\mathcal{H}_{1,\omega}(S^-)}$$

satisfies (5.16). □

6. Example: Torsion of a graphite micropolar beam of square cross-section

As an example, consider the torsion of a graphite micropolar beam of square cross-section in which the length of each side is equal to 40 mm. The elastic constants for graphite take the following values : $\alpha = 9050$ MPa, $\beta = 0$ N, $\gamma = 5434$ N, $\kappa = 61132$ N, and $\mu = 2123$ MPa [21]. Assume that the origin of the coordinate system is located in the center of the beam cross-section: consequently, the domain S is bounded by the square

$$-20 < x_1 < 20, \quad -20 < x_2 < 20.$$

As an auxiliary contour ∂S_* we take a circle of radius equal to 40 mm with the center at the origin

$$x_1 = 40 \cos t, \quad x_2 = 40 \sin t.$$

Using the Gauss quadrature formula with 16 ordinates to evaluate the integrals over ∂S and following the computational procedure discussed in [18], the approximate solution of the interior Neumann boundary-value problem (4.13) in terms of generalized Fourier series is found to converge to twelve decimal places accuracy for $n = 56$ terms of the series. Numerical values are presented below for a set of representative points inside the square cross-section:

Table 1. Approximate Solution of Micropolar Beam with Square Cross-section with $n = 56$ in (4.13). φ_1 – microrotation about x_1 axis, φ_2 – microrotation about x_2 axis, u – anti-plane displacement.

Point in Cross-Section	φ_1	φ_2	u
(0, 0)	0	0	0
(10, 10)	−0.005081325117	−0.036497251651	0.314159265343
(20, 20)	−0.000220837450	−0.062474342712	1.256302804033
(20, 10)	−0.021153286073	−0.029701921647	0.525061872292
(20, 0)	−0.024678497292	0	0
(20, −10)	−0.021153286076	0.029701921641	−0.525061872203
(20, −20)	−0.000220837450	0.062474342715	−1.256302804069
(10, −20)	0.001677453459	0.083988380805	−0.731533594421
(0, −20)	0	0.087536496618	0
(−10, −20)	−0.001677453459	0.083988380803	0.731533594488
(−20, −20)	0.000220837450	0.062474342718	1.256302804062
(−20, −10)	0.021153286079	0.029701921641	0.525061872208
(−20, 0)	0.024678497298	0	0
(−20, 10)	0.021153286071	−0.029701921643	−0.525061872272
(−20, 20)	0.000220837450	−0.062474342714	−1.256302804335
(−10, 20)	−0.001677453459	−0.083988380807	−0.731533594413
(0, 20)	0	−0.087536496614	0
(10, 20)	0.001677453459	−0.083988380806	0.731533594418

Note, that these results are in good agreement with the experimental data obtained for prismatic micropolar beams by PARK and LAKES in [4]. Also, the method used in our investigation is easily extended, with only minor changes in detail, to the analysis of torsion of micropolar beams of any (non-smooth) cross-section, where we again expect a significant contribution from the material microstructure.

7. Summary

In this paper we have formulated the interior and exterior Dirichlet and Neumann problems of anti-plane Cosserat elasticity in Sobolev spaces and established the existence, uniqueness and continuous dependence on the data of the results for these problems. We have shown that the problem of torsion of a micropolar beam of non-smooth arbitrary cross-section may be considered as the interior Neumann boundary value problem in anti-plane micropolar elasticity and provided an example, which demonstrates that material microstructure does indeed have a significant effect on the warping function.

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Received January 31, 2007; revised version May 7, 2007.