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Functional forms of hardening internal state variables in modeling elasto-plastic behavior

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In this work use is made of functional forms of hardening state variables within a consistent thermodynamic formulation to model the elasto-plastic behavior of materials. The formulation is then numerically implemented using the developed plasticity model. In deriving the constitutive model, a local yield surface is used to determine the occurrence of plasticity. Isotropic hardening and kinematic hardening are incorporated as state variables to describe the change of the yield surface. The hardening conjugate forces (stress-like terms) are general nonlinear functions of their corresponding hardening state variables (strain-like terms) and can be defined basing on the desired material behavior. Various exponential and power law functional forms are studied in this formulation. The paper discusses the general concept of using such functional forms; however, it does not address the relevant appropriateness of certain forms to solve different problems. It is shown that, depending on the functions used, standard models known from the literature can be recovered. The use of this formulation in solving boundary value problems will be presented in future.

Key words: constitutive behavior, cyclic loading, finite elements.

1. Introduction

Plastic deformation of ductile materials can be explained in terms of the theory of dislocations as being independently introduced in 1934 by Orowan [12], Taylor [16], and Polanyi [13]. Although movement of the dislocations occurs due to any loading, this movement is insignificant until a critical threshold (the yield stress, σ_{yp}, in Fig. 1) occurs. At this point, loading causes dislocations to be generated, moved and stored. The ease with which dislocations are able to move determines the hardness of the material. With an increase in the dislocation density, there begins to appear more dislocation-dislocation interactions such that movement becomes more difficult and the stress required to produce additional plastic deformation increases, i.e. the material hardens.

Plastic material models are used to describe this behavior by defining the critical stress (the yield stress) through a yield criterion. Various plasticity models have been used throughout the literature to define the yield surface as well as the change in size, shape, and position of the yield surface. In this work,

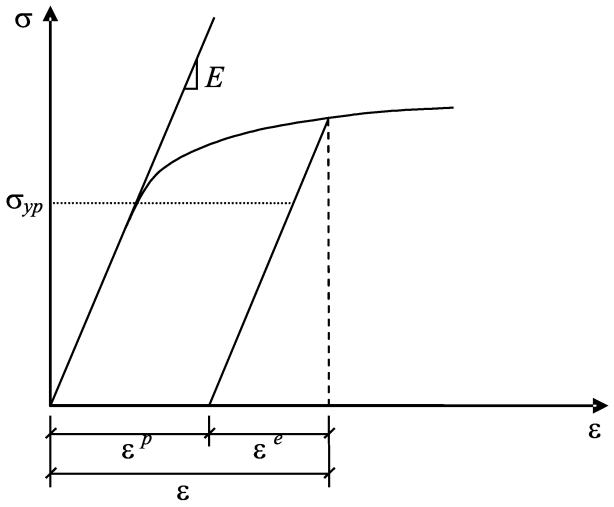

FIG. 1. Stress-strain curve for a plastically loaded material.

a von Mises-type yield criterion is used with both isotropic hardening (for example see HILL [8], CHABOCHE [3], etc.) corresponding to the change in the size of the yield surface and the kinematic hardening (for example see PRAGER [14], ARMSTRONG and FREDERICK [1], etc.) corresponding to the change in location of the yield surface.

The constitutive model is derived using consistent thermodynamics in the same fashion as the classical rate-independent continuum J2 plasticity model (e.g. DOGHRI [6], SIMO and HUGHES [15], BELYTSCHKO [2]). Based on the second law of thermodynamics, the Helmholtz free energy is introduced to describe the current state of energy in the material (MALVERN [11], COUSSY [4]), and is a function of the strain and the internal state variables under consideration.

In order to derive the model equations, the thermodynamics of irreversible processes is followed by introducing a local state consisting of state variables (MALVERN [11], LEMAITRE and CHABOCHE [10], COUSSY [4], DOGHRI [6]). A thermodynamic potential is used which allows the state laws to be defined basing on the state variables. The evolution of the thermodynamic conjugate forces are then obtained by assuming physical existence of the dissipation potential at the macroscale and owing to the use of the theory of functions of several variables with a Lagrange multiplier.

For convenience in developing the constitutive model and the finite element algorithm, tensorial notation will be used. Boldface terms indicate tensors of order one or greater, while italicized terms indicate scalars. Einstein's summation convention is used unless otherwise indicated.

2. Thermodynamic state variables

The local plasticity model is defined through the use of the method of material local state identification. In this method, a model is developed such that the thermodynamic state at a given point in space and time is completely determined by a given set of state variables at that point in space and time. The observable state variables used here are the total strain denoted by the second-order tensor $\boldsymbol{\varepsilon}$ and the temperature T. These variables can be measured and appear regardless of the material phenomena such as elasticity and plasticity. For pure elasticity, these observable state variables entirely define the point; however, for elasto-plasticity, the material has a history dependence which requires an additional set of internal state variables. These hardening internal state variables are unitless, strain-like quantities and are accumulated into a set, $\mathbf{V}^p$ of macroscopic measures of irreversible phenomena:

$$\mathbf{V}^p = [r, \boldsymbol{\alpha}], \tag{2.1}$$

where the internal state variables considered here are the plasticity-related variables representing the fluxes of the isotropic and kinematic hardening behaviors, denoted by the scalar r and the second-order tensor $\boldsymbol{\alpha}$, respectively. The isotropic hardening (HILL [8]), corresponds to the change in the size of the yield surface and the kinematic hardening (PRAGER [14]) corresponds to the change in location of the yield surface. Note also that the strain is assumed to be additively decomposed into two parts: a recoverable elastic strain, $\boldsymbol{\varepsilon}^e$, and an irreversible plastic strain, $\boldsymbol{\varepsilon}^p$ (Fig. 1). The reversible part is related to the stress through the usual linear elastic equations. Plasticity theory is concerned with characterizing the irreversible part which remains when external loads are removed.

3. Thermodynamic equations of state

In order to determine state laws which relate the internal state variable fluxes to their conjugate thermodynamic forces, a thermodynamic potential, ψ, defined as the Helmholtz free energy is used which is a state function of a thermodynamic system (MALVERN [11], LEMAITRE and CHABOCHE [10], COUSSY [4], DOGHRI [6]). This thermodynamic potential is used to describe the current state of energy in the material, is a function of the observable state variables and the internal state variables under consideration, and has been introduced through the the Clausius–Duhem inequality as follows:

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho\left(\dot{\psi} + s\dot{T}\right) - \mathbf{q} \cdot \frac{\nabla T}{T} \geq 0, \tag{3.1}$$

where $\boldsymbol{\sigma}$ is the second-order Cauchy stress tensor, ρ is the mass density, $\mathbf{q}$ is the heat flux vector, s is the entropy per unit mass representing the amount

of disorder or randomness in a system, ∇T is the temperature gradient, and $\dot{\psi}$ is the time derivative of ψ. Expanding $\dot{\psi}$ through the chain rule and utilizing the requirement that independent processes should satisfy the Clausius–Duhem inequality, the following thermo-elastic state laws can be written:

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}, \tag{3.2}$$

$$s = -\frac{\partial \psi}{\partial T}. \tag{3.3}$$

Thus, from the last two equations, the stress, $\boldsymbol{\sigma}$, and the enthalpy, s, are defined as the conjugate forces corresponding to the state variables $\boldsymbol{\varepsilon}^e$ and T, respectively. Similarly, a set of conjugate forces, $\mathbf{A}^p$, is defined which corresponds to the hardening internal state variables:

$$\mathbf{A}^p = [R, \mathbf{X}], \tag{3.4}$$

where the scalar R measures the expansion or contraction of the yield surface in the stress space while maintaining its shape and having a fixed center and the second-order tensor $\mathbf{X}$ measures the movement and distortion of the yield surface. Whereas the internal state variables are unitless, strain-like quantities, the thermodynamic conjugate forces are a set of stress-like quantities that are related to the state variables since the stress is related to the strain. These conjugate forces are defined in the Clausius–Duhem inequality by the following set of state laws:

$$\mathbf{A}^p = \rho \frac{\partial \psi}{\partial \mathbf{V}^p}. \tag{3.5}$$

4. Thermodynamic conjugate forces

Since the internal state variables are selected independently of each other, one can express the analytical form of the Helmholtz free energy in terms of its internal state variables as:

$$\rho\psi = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{C}^e : \boldsymbol{\varepsilon}^e + W(\mathbf{V}^p) - \rho T s, \tag{4.1}$$

where the fourth-order tensor $\mathbf{C}^e$ is the tangent elastic modulus. Using this form of the Helmholtz free energy and from the state law Eq. (2.21), the stress applied to extend or compress a body is defined in terms of the elastic strain through the standard Hookean relationship as follows:

$$\boldsymbol{\sigma} = \mathbf{C}^e : \boldsymbol{\varepsilon}^e. \tag{4.2}$$

The additional term, $W(\mathbf{V}^p)$, accounts for the energy introduced into the system by the hardening terms. In general, the hardening term for the energy may be introduced as fully coupled for the isotropic and kinematic hardening. However, in this paper it is assumed that the energy introduced by the hardening terms is uncoupled, so that the energy term is a sum of two terms:

$$W(\mathbf{V}^p) = W^r(r) + W^\alpha(\boldsymbol{\alpha}). \tag{4.3}$$

The energy term related to the plasticity "isotropic hardening – conjugate force" relationship can be, but not exclusively, in the form of a power or exponential relationship (DOGHRI [6]). Similarly, power and exponential relationships can be defined for the kinematic hardening energy terms (VOYIADJIS and DORGAN [17], DORGAN and VOYIADJIS [5]). Thus, the energy terms can either be selected from one of the following functional forms:

Power Laws:

$$W^r(r) = \frac{H_r}{m_r+1} r^{m_r+1}, \tag{4.4}$$

$$W^\alpha(\boldsymbol{\alpha}) = \frac{H_\alpha}{m_\alpha+1} \|\boldsymbol{\alpha}\|^{m_\alpha+1}. \tag{4.5}$$

Exponential Laws:

$$W^r(r) = R_\infty \left(r + \frac{1}{\gamma_r} e^{-\gamma_r r} - \frac{1}{\gamma_r} \right), \tag{4.6}$$

$$W^\alpha(\boldsymbol{\alpha}) = X_\infty \left(\|\boldsymbol{\alpha}\| + \frac{1}{\gamma_\alpha} e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|} - \frac{1}{\gamma_\alpha} \right). \tag{4.7}$$

In these relationships, H_i, m_i, R_∞, and X_∞ are material and geometrical parameters, where $i = r, \alpha$. For example, for the case of a composite, the geometrical properties may include size, shape and spacing of the fibers. Utilizing the energy terms in the Helmholtz free energy, the state laws, Eq. (3.5), result in definitions of the hardening – thermodynamic conjugate forces in the form of power and exponential relations of the corresponding state variables:

Power Laws:

$$R = H_r r^{m_r}, \tag{4.8}$$

$$\mathbf{X} = H_\alpha \|\boldsymbol{\alpha}\|^{m_\alpha - 1} \boldsymbol{\alpha}. \tag{4.9}$$

Exponential Laws:

$$R = R_\infty \left(1 - e^{-\gamma_r r}\right), \tag{4.10}$$

$$\mathbf{X} = X_\infty \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|} \left(1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}\right). \tag{4.11}$$

Note that these laws are subject to the constraint that $\mathbf{X} = \mathbf{0}$ when $\|\boldsymbol{\alpha}\| = 0$.

The internal "state variable – thermodynamic conjugate force" relationships are defined basing on the material being investigated, and different relationships can be selected for the isotropic hardening law and for the kinematic hardening law. For example, the isotropic hardening relationship can be assumed to be linear with an exponential law for the kinematic hardening. Though two typical models, the power and exponential laws, are used here to introduce the isotropic and kinematic hardening relations, more complex models can be incorporated in the same manner; however, the analysis of the material model is beyond the scope of this work. This work is focused on the development of a formulation based on a general functional form of the thermodynamic conjugate forces. This allows the constitutive model to be developed without making an assumption concerning the behavior of the material model such that the conjugate forces could be written as a general function of their corresponding internal state variable:

$$R = R(r); \qquad \mathbf{X} = \mathbf{X}(\boldsymbol{\alpha}). \tag{4.12}$$

For an example of how these relationships can be defined, consider the isotropic hardening conjugate force in plasticity, R. This conjugate force, which is a stress quantity, measures the expansion or contraction of the yield surface in the stress space, while maintaining its shape and having a fixed center. Thus, the radius of the yield surface, i.e. the current yield stress, is computed as the sum of the initial yield stress, σ_{ys}, and the isotropic hardening conjugate force. Some possible softening curves for the relationship between the current yield stress and the isotropic hardening are plotted in Fig. 2.

5. Dissipation potential and flow rules

The evolution of the thermodynamic conjugate forces can be obtained through the evolution relations of the internal state variables, which are obtained by assuming the physical existence of the dissipation potential at the macroscale, $\mathit{\Pi}^p$. The energy dissipation processes are set in conjunction with the Clausius–Duhem inequality with the thermodynamic state laws substituted and are thus given as the product of the thermodynamic conjugate forces with the respective

flux variables as follows:

$$\Pi^p = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \rho \mathbf{A}^p \cdot \dot{\mathbf{V}}^p - \mathbf{q} \cdot \frac{\nabla T}{T} \geq 0. \tag{5.1}$$

The theory of functions of several variables is used with the Lagrange multiplier $\dot{\lambda}^p$ to construct the objective function Ω in the following form:

$$\Omega = \Pi^p - F\dot{\lambda}^p, \tag{5.2}$$

where F is the plastic potential and will be defined later. In order to obtain the plastic strain rate, the objective function is extremized so that, for the case when $F \geq 0$, the evolution equations for the plastic strain and for the internal state variables are given as follows:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\partial F}{\partial \boldsymbol{\sigma}} \dot{\lambda}^p, \tag{5.3}$$

$$\dot{\mathbf{V}}^p = -\frac{\partial F}{\partial \mathbf{A}^p} \dot{\lambda}^p. \tag{5.4}$$

The following loading-unloading conditions known as the Kuhn–Tucker conditions (KUHN and TUCKER [9]) must also be enforced:

$$\dot{\lambda}^p \geq 0; \qquad f \leq 0; \qquad \dot{\lambda}^p f = 0. \tag{5.5}$$

The first relationship states that the plastic flow rate is always non-negative, while the second condition shows that the stress state is always within the yield surface or on the yield surface. The last condition can be met in two different

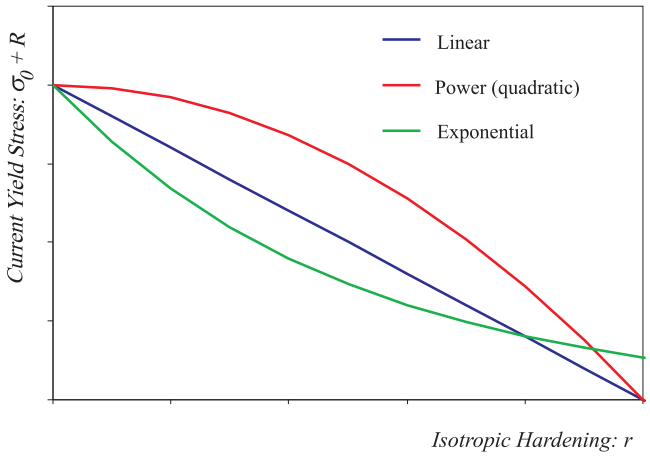

FIG. 2. Softening curves: possible relations between the isotropic hardening r and the corresponding conjugate force R.

loading cases. If the stress is in a state of elastic loading/unloading, then $\dot{\lambda}^p = 0$ and the condition is met. If, however, there is a plastic loading ($\dot{\lambda}^p > 0$), then the last condition makes the stress state to remain on the yield surface ($f = 0$). In order for this condition to be satisfied during loading, there can be no evolution of the yield surface ($\dot{f} = 0$). This last implication is the consistency condition for plasticity, and it will be used to determine the evolution of the plastic multiplier.

6. Plastic potential and yield condition

Associative plasticity can be used here to derive the evolution equations for the constitutive model such that the plastic potential, F, is set equal to the yield criterion, f:

$$F = f = \|\boldsymbol{\xi}\| - \sqrt{\frac{2}{3}}\,[\sigma_{yp} + R] \leq 0 \tag{6.1}$$

where $\|\boldsymbol{\xi}\|$ is the norm of the relative stress tensor, $\boldsymbol{\xi}$, and is defined in terms of the deviatoric stress, $\mathbf{s}$, and the backstress, $\mathbf{X}$, as follows:

$$\boldsymbol{\xi} = \mathbf{s} - \mathbf{X}. \tag{6.2}$$

With the plastic potential defined by Eq. (6.1), the normals to the plastic potential required in Eqs. (5.3) and (5.4) are derived to have the following forms:

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\xi}{\|\xi\|} = f_{,\boldsymbol{\sigma}}, \tag{6.3}$$

$$\frac{\partial F}{\partial R} = \frac{\partial f}{\partial R} = -\sqrt{\frac{2}{3}} = f_{,R}, \tag{6.4}$$

$$\frac{\partial F}{\partial \mathbf{X}} = \frac{\partial f}{\partial \mathbf{X}} = f_{,\mathbf{X}} = -f_{,\boldsymbol{\sigma}}. \tag{6.5}$$

Thus, the evolution equations for the local plasticity model have been derived and are written as follows:

$$\dot{\boldsymbol{\varepsilon}}^p = f_{,\boldsymbol{\sigma}}\dot{\lambda}^p; \qquad \dot{r} = -f_{,R}\dot{\lambda}^p = \dot{\varepsilon}^p_{eq}; \qquad \dot{\boldsymbol{\alpha}} = -f_{,\mathbf{X}}\dot{\lambda}^p = f_{,\boldsymbol{\sigma}}\dot{\lambda}^p. \tag{6.6}$$

In these equations, $\dot{\varepsilon}^p_{eq}$ is defined as the evolution of the equivalent plastic strain and takes the following form:

$$\dot{\varepsilon}^p_{eq} \equiv \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}^p : \boldsymbol{\varepsilon}^p} = \sqrt{\frac{2}{3}}\dot{\lambda}^p. \tag{6.7}$$

7. Plasticity consistency condition

At a plastic state when $f = 0$, the consistency condition $\dot{f} = 0$ results from the loading-unloading conditions of Eq. (5.5). Thus, since the yield criterion is a function of the effective Cauchy stress, the backstress and the isotropic hardening, the consistency condition can be expressed in terms of the conjugate forces:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial R}\dot{R} + \frac{\partial f}{\partial \mathbf{X}} : \dot{\mathbf{X}} \equiv 0. \tag{7.1}$$

Since the conjugate forces have been defined as general functions of the state flux variables as defined in Eqs. (4.12), the consistency condition can be rewritten in terms of the flux variables as follows:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial R}\frac{\partial R\,(r)}{\partial r}\dot{r} + \frac{\partial f}{\partial \mathbf{X}} : \frac{\partial \mathbf{X}\,(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} \equiv 0. \tag{7.2}$$

After substitution of the normals to the yield surface as defined in Eqs. (6.3) to (6.5) and by the evolution equations for the internal state variables as defined through Eqs. $(6.6)_1$ and $(6.6)_2$, the plastic multiplier can be obtained from this consistency condition and expressed in terms of the incremental stress as follows:

$$\dot{\lambda}^p = \frac{1}{H} f_{,\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}, \tag{7.3}$$

where H is defined here as:

$$H = f_{,R}^2 \frac{\partial R\,(r)}{\partial r} + f_{,\boldsymbol{\sigma}} : \frac{\partial \mathbf{X}\,(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} : f_{,\boldsymbol{\sigma}}. \tag{7.4}$$

An alternative form of the increment of the plastic multiplier can be found by substituting the incremental form of Eq. (4.9) and the evolution of the plastic strain, Eq. $(6.6)_1$, into the consistency condition, Eq. (7.2). The plastic multiplier is then expressed in terms of the incremental strain as follows:

$$\dot{\lambda}^p = \frac{1}{h} f_{,\boldsymbol{\sigma}} : \mathbf{C}^e : \dot{\boldsymbol{\varepsilon}}, \tag{7.5}$$

where h is defined here as:

$$h = H + f_{,\boldsymbol{\sigma}} : \mathbf{C}^e : f_{,\boldsymbol{\sigma}}. \tag{7.6}$$

Now again, using the rate form of the Hookean relationship and pre-multiplying by $f_{,\boldsymbol{\sigma}}$, the following relation is obtained:

$$f_{,\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = f_{,\boldsymbol{\sigma}} : \mathbf{C}^e : \dot{\boldsymbol{\varepsilon}} - f_{,\boldsymbol{\sigma}} : \mathbf{C}^e : f_{,\boldsymbol{\sigma}} \dot{\lambda}^p. \tag{7.7}$$

Using Eq. (7.5), the above equation can be expressed as follows:

$$\frac{f_{,\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}}{\dot{\lambda}^p} = \frac{\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\sigma}}}{\left(\dot{\lambda}^p\right)^2} = h - f_{,\boldsymbol{\sigma}} : \mathbf{C}^e : f_{,\boldsymbol{\sigma}} = H. \tag{7.8}$$

It can be seen from this equation, since the terms $\dot{\boldsymbol{\varepsilon}}^p$ and $\left(\dot{\lambda}^p\right)^2$ are always positive, that the hardening modulus, H, is positive when hardening occurs (i.e. positive increments of stress), and it is negative when softening occurs (i.e. negative increments of stress).

8. Elasto-plastic tangent modulus

In order to define the constitutive equation, the rate form of the Hookean stress must be derived by differentiating the stress-strain relation of Eq. (4.9):

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^e : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p)\,. \tag{8.1}$$

Using the evolution of the plastic strain and of the plastic multiplier defined by Eqs. $(6.6)_1$ and (7.5), the constitutive law can now be defined by the following expression:

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^{ep} : \dot{\boldsymbol{\varepsilon}}, \tag{8.2}$$

where the elastic-plastic continuum tangent modulus, $\mathbf{D}^{ep}$, is expressed as:

$$\mathbf{D}^{ep} = \begin{cases} \mathbf{C}^e & \text{if} \quad \dot{\lambda}^p = 0, \\ \mathbf{C}^e - \dfrac{1}{h}\mathbf{C}^e : f_{,\boldsymbol{\sigma}} \otimes f_{,\boldsymbol{\sigma}} : \mathbf{C}^e & \text{if} \quad \dot{\lambda}^p > 0. \end{cases} \tag{8.3}$$

Alternatively, if the plastic multiplier defined in terms of the incremental stress, Eqs. (7.3), is used, then the inverse of the elastic-plastic continuum tangent modulus takes the following form:

$$\mathbf{D}^{-ep} = \begin{cases} \mathbf{C}^{-e} & \text{if} \quad \dot{\lambda}^p = 0, \\ \mathbf{C}^{-e} + \dfrac{1}{H} f_{,\boldsymbol{\sigma}} \otimes f_{,\boldsymbol{\sigma}} & \text{if} \quad \dot{\lambda}^p > 0. \end{cases} \tag{8.4}$$

9. Conjugate force definitions

Now when the constitutive model has been derived, the definitions of the conjugate forces are revisited here in order to demonstrate the applicability of the material models used. In order to understand how these relationships can be defined, various models of defining the isotropic hardening conjugate force and the kinematic hardening conjugate force will be considered, and it will be shown that the hardening models take the form of standard models known from the literature.

9.1. Linear isotropic hardening model

The simplest case in plasticity hardening is the assumption of a linear hardening model such that the isotropic hardening conjugate force is a linear function of the equivalent plastic strain:

$$\dot{R} = a\dot{\varepsilon}^p_{eq}, \tag{9.1}$$

where the coefficient a is a constant derived from a simple monotonic uniaxial curve. In order to obtain this linear model using the model presented in this paper, a linear state law is utilized for the isotropic hardening as defined by setting $m_r = 1$ in Eq. (4.8). By taking the time derivative of this equation and utilizing the evolution equation Eq. $(6.6)_2$, the evolution equation of the isotropic hardening conjugate force is obtained in the desired form:

$$\dot{R} = H_r\dot{\varepsilon}^p_{eq}, \tag{9.2}$$

where the previously defined coefficient, a, is defined to be equal to the linear coefficient, H_r.

The derivative of the linear state law with respect to the isotropic hardening state variable to be used in Eq. (7.4) is defined as follows:

$$\frac{\partial R}{\partial r} = H_r. \tag{9.3}$$

9.2. Chaboche isotropic hardening model

CHABOCHE [3] introduced a nonlinear relationship between the isotropic hardening and the equivalent plastic strain such that:

$$\dot{R} = a\,(b - R)\,\dot{\varepsilon}^p_{eq}. \tag{9.4}$$

In order to obtain this Chaboche model using the model presented in this work, the exponential state law is utilized for the isotropic hardening as defined by Eq. (4.10). Differentiating this equation, we obtain the following evolution of the isotropic hardening conjugate force:

$$\dot{R} = R_\infty\gamma_r e^{-\gamma_r r}\dot{r}. \tag{9.5}$$

Use of the original state law, Eq. (4.10), and the evolution equation, Eq. $(6.6)_2$, results in the desired form analogous to Eq. (9.4):

$$\dot{R} = \gamma_r\,(R_\infty - R)\,\dot{\varepsilon}^p_{eq}. \tag{9.6}$$

Thus, the Chaboche model has been derived when the previously defined coefficient, a, is defined to be equal to γ_r and the coefficient b is defined to be

equal to R_∞. These coefficients can be determined from a plot of R versus r such that the coefficient R_∞ is the saturation stress at large values of r and the combination of coefficients $R_\infty \gamma_r$ is the initial slope of the curve near $\varepsilon^p_{eq} = 0$.

The derivative of the linear state law with respect to the isotropic hardening state variable to be used in Eq. (7.4) is defined as follows:

$$\frac{\partial R}{\partial r} = R_\infty \gamma_r e^{-\gamma_r r}. \tag{9.7}$$

9.3. Prager kinematic hardening model

The PRAGER model [14] was introduced to describe the motion of the yield surface such that the yield surface translates linearly with the plastic strain. Thus, the evolution of the back-stress can be defined by the following relationship:

$$\dot{\mathbf{X}} = a \dot{\boldsymbol{\varepsilon}}^p, \tag{9.8}$$

where the coefficient a is a constant derived from a simple monotonic uniaxial curve. In order to obtain this Prager model using the model presented in this work, a linear state law is utilized for the kinematic hardening as defined by setting $m_\alpha = 1$ in Eq. (4.9). By taking the time derivative of this equation and utilizing the evolution equation, Eq. $(6.6)_3$, the evolution equation of the kinematic hardening conjugate force is obtained in the desired form:

$$\dot{\mathbf{X}} = H_\alpha \dot{\boldsymbol{\varepsilon}}^p, \tag{9.9}$$

where the previously defined coefficient, a, is defined to be equal to the presented model's linear coefficient, H_α.

The derivative of the linear state law with respect to the isotropic hardening state variable to be used in Eq. (7.4) is defined as follows:

$$\frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}} = H_\alpha. \tag{9.10}$$

9.4. Armstrong–Frederick kinematic hardening model

The ARMSTRONG and FREDERICK model [1] was introduced to describe the motion of the yield surface, to simulate the multiaxial Bauschinger effect and to introduce nonlinear hardening. Its kinematic hardening rule was predicted by the following expression:

$$\dot{\mathbf{X}} = a \dot{\boldsymbol{\varepsilon}}^p - b \dot{\varepsilon}^p_{eq} \mathbf{X} \tag{9.11}$$

where the constants a and b are determined from uniaxial tests. In order to obtain this Armstrong–Frederick model using the model presented in this work,

either a power law or an exponential state law is utilized for the kinematic hardening as defined by Eqs. (4.9) and (4.11), respectively. By taking the time derivative of these equations, the following evolution equations are obtained:

Power:

$$\dot{\mathbf{X}} = H_{\boldsymbol{\alpha}} \left\|\boldsymbol{\alpha}\right\|^{m_{\boldsymbol{\alpha}}-1} \dot{\boldsymbol{\alpha}} - H_{\boldsymbol{\alpha}} \left(1 - m_{\boldsymbol{\alpha}}\right) \left\|\boldsymbol{\alpha}\right\|^{m_{\boldsymbol{\alpha}}-3} \left(\boldsymbol{\alpha} : \dot{\boldsymbol{\alpha}}\right) \boldsymbol{\alpha}. \tag{9.12}$$

Exponential:

$$\dot{\mathbf{X}} = X_\infty \left\{ \frac{\dot{\boldsymbol{\alpha}}}{\left\|\boldsymbol{\alpha}\right\|} - \frac{\boldsymbol{\alpha}}{\left\|\boldsymbol{\alpha}\right\|^3} \left(\boldsymbol{\alpha} : \dot{\boldsymbol{\alpha}}\right) \right\} \left(1 - e^{-\gamma_{\boldsymbol{\alpha}} \left\|\boldsymbol{\alpha}\right\|}\right) + X_\infty \gamma_{\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}}{\left\|\boldsymbol{\alpha}\right\|} \left(e^{-\gamma_{\boldsymbol{\alpha}} \left\|\boldsymbol{\alpha}\right\|}\right) \frac{\left(\boldsymbol{\alpha} : \dot{\boldsymbol{\alpha}}\right)}{\left\|\boldsymbol{\alpha}\right\|} \tag{9.13}$$

subject to the constraint that $\dot{\mathbf{X}} = \mathbf{0}$ when $\left\|\boldsymbol{\alpha}\right\| = 0$.

Use of the original state laws, Eqs. (4.9) and (4.11), and the evolution equation, Eq. (6.6_3, results in the desired forms of evolution of the kinematic hardening:

Power:

$$\dot{\mathbf{X}} = \left[H_\alpha \left\|\boldsymbol{\alpha}\right\|^{m_\alpha - 1}\right] \dot{\boldsymbol{\varepsilon}}^p - \left[\left(1 - m_\alpha\right) \left(\frac{\boldsymbol{\alpha}}{\left\|\boldsymbol{\alpha}\right\|} : \frac{\dot{\boldsymbol{\varepsilon}}^p}{\dot{\varepsilon}^p_{eq}}\right)\right] \dot{\varepsilon}^p_{eq} \mathbf{X}. \tag{9.14}$$

Exponential:

$$\dot{\mathbf{X}} = \left[X_\infty \frac{\left(1 - e^{-\gamma_\alpha \left\|\boldsymbol{\alpha}\right\|}\right)}{\left\|\boldsymbol{\alpha}\right\|}\right] \dot{\boldsymbol{\varepsilon}}^p - \left[\left(\frac{1 - e^{-\gamma_\alpha \left\|\boldsymbol{\alpha}\right\|} \left(1 + \left\|\boldsymbol{\alpha}\right\| \gamma_\alpha\right)}{\left\|\boldsymbol{\alpha}\right\| \left(1 - e^{-\gamma_\alpha \left\|\boldsymbol{\alpha}\right\|}\right)}\right) \left(\frac{\boldsymbol{\alpha}}{\left\|\boldsymbol{\alpha}\right\|} : \frac{\dot{\boldsymbol{\varepsilon}}^p}{\dot{\varepsilon}^p_{eq}}\right)\right] \dot{\varepsilon}^p_{eq} \mathbf{X}. \tag{9.15}$$

Thus, a modified form of the Armstrong – Frederick rule given by Eq. (9.11) has been derived. In this form, the coefficients are no longer constant. The coefficient a is now a function of the norm of the kinematic hardening flux variable, $\left\|\boldsymbol{\alpha}\right\|$, such that:

Power:

$$a = H_\alpha \left\|\boldsymbol{\alpha}\right\|^{m_\alpha - 1}. \tag{9.16}$$

Exponential:

$$a = X_\infty \frac{\left(1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}\right)}{\|\boldsymbol{\alpha}\|} \tag{9.17}$$

and the coefficient b is now a function of the kinematic hardening flux variable $\boldsymbol{\alpha}$; its norm $\|\boldsymbol{\alpha}\|$; and the evolutions of the plastic strain $\dot{\boldsymbol{\varepsilon}}^p$ and accumulated plastic strain, $\dot{\varepsilon}^p_{eq}$, such that:

Power:

$$b = (1 - m_\alpha) \left(\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|} : \frac{\dot{\boldsymbol{\varepsilon}}^p}{\dot{\varepsilon}^p_{eq}} \right). \tag{9.18}$$

Exponential:

$$b = \left(\frac{1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|} \left(1 + \|\boldsymbol{\alpha}\| \, \gamma_\alpha\right)}{\|\boldsymbol{\alpha}\| \left(1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}\right)} \right) \left(\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|} : \frac{\dot{\boldsymbol{\varepsilon}}^p}{\dot{\varepsilon}^p_{eq}} \right). \tag{9.19}$$

The derivatives of the exponential and power state laws with respect to the kinematic hardening state variable to be used in Eq. (7.4) are defined as follows:

Power:

$$\frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}} = H_\alpha \, \|\boldsymbol{\alpha}\|^{m_\alpha - 1} \, \mathbf{1} \otimes \mathbf{1} + H_\alpha \left(m_\alpha - 1\right) \|\boldsymbol{\alpha}\|^{m_\alpha - 3} \, \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}. \tag{9.20}$$

Exponential:

$$\begin{aligned} \frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}} = X_\infty \left\{ \frac{\mathbf{1} \otimes \mathbf{1}}{\|\boldsymbol{\alpha}\|} - \frac{\mathbf{1} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{1}}{\|\boldsymbol{\alpha}\|^3} \right\} \left(1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}\right) \\ + X_\infty \gamma_\alpha \left(\frac{\mathbf{1} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{1}}{\|\boldsymbol{\alpha}\|^2} \right) \left(e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}\right) \end{aligned} \tag{9.21}$$

subject to the constraint that $\dot{\mathbf{X}} = \mathbf{0}$ when $\|\boldsymbol{\alpha}\| = 0$.

Of further interest is the case of a uniaxial tension/compression loading. As the Poisson's ratio equals 0.5 for incompressible plasticity, the plastic strain tensor can be reduced as follows:

$$[\varepsilon^p] = \begin{bmatrix} \varepsilon^p_{11} & 0 & 0 \\ 0 & -\frac{1}{2}\,\varepsilon^p_{11} & 0 \\ 0 & 0 & -\frac{1}{2}\,\varepsilon^p_{11} \end{bmatrix} \tag{9.22}$$

which, from Eq. $(6.6)_3$, allows the backstress flux tensor to be reduced to the form:

$$\left\{\begin{array}{c}\alpha_{11}\\ \alpha_{22}\\ \alpha_{33}\end{array}\right\} = \left\{\begin{array}{c}\alpha_{11}\\ -\dfrac{1}{2}\,\alpha_{11}\\ -\dfrac{1}{2}\,\alpha_{11}\end{array}\right\},$$

$$[\alpha] = \left[\begin{array}{ccc}\alpha_{11} & 0 & 0\\ 0 & -\dfrac{1}{2}\,\alpha_{11} & 0\\ 0 & 0 & -\dfrac{1}{2}\,\alpha_{11}\end{array}\right]. \tag{9.23}$$

For this reduced form of the backstress flux tensor, the following relationships can be used:

$$\|\boldsymbol{\alpha}\| = \sqrt{\alpha_{11}^2 + \alpha_{22}^2 + \alpha_{33}^2} = \sqrt{\frac{3}{2}}\,\alpha_{11}, \tag{9.24}$$

$$\boldsymbol{\alpha} : \dot{\boldsymbol{\alpha}} = \alpha_{11}\dot{\alpha}_{11} + \alpha_{22}\dot{\alpha}_{22} + \alpha_{33}\dot{\alpha}_{33} = \frac{3}{2}\,\alpha_{11}\dot{\alpha}_{11}. \tag{9.25}$$

These relations allow the evolution equations for the kinematic hardening in a 1D state to be written in terms of the uniaxial components as follows:

Power:

$$\dot{X}_{11} = H_\alpha^{1/m_\alpha} m_\alpha \left(\sqrt{\frac{3}{2}}\,X_{11}\right)^{(m_\alpha - 1)/m_\alpha} \dot{\varepsilon}_{eq}^p. \tag{9.26}$$

Exponential:

$$\dot{X}_{11} = \gamma_\alpha \left(X_\infty - \sqrt{\frac{3}{2}}\,X_{11}\right) \dot{\varepsilon}_{eq}^p. \tag{9.27}$$

It is important to note that Eq. (9.27), the nonlinear kinematic hardening evolution rule for a 1D state, is identical to the standard 1D form of the Armstrong – Frederick equation.

The uniaxial forms of the derivatives of the exponential and power state laws with respect to the kinematic hardening state variable to be used in Eq. (7.4) are defined as follows:

Power:

$$\frac{\partial X_{11}}{\partial \alpha_{11}} = m_\alpha H_\alpha \left(\sqrt{\frac{3}{2}}\, \alpha_{11} \right)^{m_\alpha - 1}. \tag{9.28}$$

Exponential:

$$\frac{\partial X_{11}}{\partial \alpha_{11}} = X_\infty \gamma_\alpha e^{-\gamma_\alpha \sqrt{\frac{3}{2}} \alpha_{11}}. \tag{9.29}$$

10. Integration algorithm

In the solution procedure, a linearized form of the equilibrium equation is solved within an incremental iterative Newton–Raphson solution procedure for the increment of strain during the time increment Δt_j such that:

$$\boldsymbol{\varepsilon}_j = \boldsymbol{\varepsilon}_0 + \Delta t_j d\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \Delta \boldsymbol{\varepsilon}_j, \tag{10.1}$$

where the subscripts j and 0 indicate that the variable is computed at iteration j and at the previously converged state, respectively; and the symbol Δ denotes a total increment from the previously converged state to the iteration j. The increment of the plastic multiplier $\Delta\lambda_j^p$ is then computed and the state of the material is updated so that:

$$\boldsymbol{\varepsilon}_j^p = \boldsymbol{\varepsilon}_0^p + \Delta \boldsymbol{\varepsilon}_j^p; \qquad r_j = r_0 + \Delta r_j; \qquad \boldsymbol{\alpha}_j = \boldsymbol{\alpha}_0 + \Delta \boldsymbol{\alpha}_j, \tag{10.2}$$

$$\boldsymbol{\sigma}_j = \mathbf{C}^e : \left(\boldsymbol{\varepsilon}_j - \boldsymbol{\varepsilon}_j^p \right) = \boldsymbol{\sigma}_0 + \mathbf{C}^e : \left(\Delta \boldsymbol{\varepsilon}_j - \Delta \boldsymbol{\varepsilon}_j^p \right). \tag{10.3}$$

Early computational work would update the state using the plastic multiplier from Eq. (7.3), $\dot{\lambda}_0^p$, such that the increment of the plastic multiplier over the time increment Δt_j would be computed as:

$$\Delta\lambda_j^p = \dot{\lambda}_0^p \Delta t_j = \frac{\Delta t_j}{h_0} f_{,\boldsymbol{\sigma}_0} : \mathbf{C}^e : d\boldsymbol{\varepsilon} \tag{10.4}$$

where h_0 is written as:

$$h_0 = f_{,\boldsymbol{\sigma}_0} : \mathbf{C}^e : f_{,\boldsymbol{\sigma}_0} + f_{,R_0}^2 \frac{\partial R\left(r_0\right)}{\partial r} + f_{,\boldsymbol{\sigma}_0} : \frac{\partial \mathbf{X}\left(\boldsymbol{\alpha}_0\right)}{\partial \boldsymbol{\alpha}} : f_{,\boldsymbol{\sigma}_0} \tag{10.5}$$

and the unit normal, $f_{,\boldsymbol{\sigma}_0}$, is computed basing on the state at the beginning of the time step. Based on this increment, the state would be updated for a plastic

state using a simple forward Euler integration scheme, where the increments in Eqs. (10.2) are defined as follows:

$$\Delta\boldsymbol{\varepsilon}_j^p = f_{,\boldsymbol{\sigma}_0}\Delta\lambda_j^p; \qquad \Delta r_j = -f_{,R_0}\Delta\lambda_j^p; \qquad \Delta\boldsymbol{\alpha}_j = f_{,\boldsymbol{\sigma}_0}\Delta\lambda_j^p. \tag{10.6}$$

In this scheme, it is not secured that the yield condition at the end of the time step (time j) should be zero so that the solution will tend to drift from the yield surface producing inaccurate solutions. In order to obtain more accurate solutions, integration schemes must be used which secure that $f_j = 0$ at the end of the time step:

$$F_j = f_j = \left\|\boldsymbol{\xi}_j\right\| - \sqrt{\frac{2}{3}}\left[\sigma_{yp} + R_j\right] = 0 \tag{10.7}$$

where the relative stress is defined as follows:

$$\boldsymbol{\xi}_j = \mathbf{s}_j - \mathbf{X}_j. \tag{10.8}$$

The conjugate forces are defined as functions of the state variables such that:

$$R_j = R\left(r_j\right); \qquad \mathbf{X}_j = \mathbf{X}\left(\boldsymbol{\alpha}_j\right). \tag{10.9}$$

In order to address this type of problem, a return mapping algorithm is used. This algorithm has an initial elastic-predictor step, followed by a plastic-corrector step. In the elastic-predictor step, the incremental strains are assumed to be elastic so that an initial trial stress can be computed as:

$$\boldsymbol{\sigma}_j^{\text{trial}} = \boldsymbol{\sigma}_0 + \mathbf{C}^e : \Delta\boldsymbol{\varepsilon}_j. \tag{10.10}$$

The trial state $\left(\boldsymbol{\sigma}_j^{\text{trial}}, \boldsymbol{\varepsilon}_0^p, r_0, \boldsymbol{\alpha}_0\right)$ is then used to decide whether an elastic point enters the plastic regime or whether a plastic point elastically unloads through a trial yield criterion. For the case when $f_{\text{trial}} \leq 0$, the integration point is assumed to be elastic and the current state $\left(\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}_j^p, r_j, \boldsymbol{\alpha}_j\right)$ is set to the trial state $\left(\boldsymbol{\sigma}_j^{\text{trial}}, \boldsymbol{\varepsilon}_0^p, r_0, \boldsymbol{\alpha}_0\right)$. Alternatively, when $f_{\text{trial}} > 0$, the current state resulting from this trial state lies outside of the yield surface. Plasticity has occurred and the state must return to the yield surface. Using the definition of the Cauchy stress from Eq. (10.3) along with the definition of the trial stress, Eq. (10.10), the Cauchy stress is corrected as follows:

$$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j^{\text{trial}} - \mathbf{C}^e : \Delta\boldsymbol{\varepsilon}_j^p. \tag{10.11}$$

Thus, the correction of the stress during the plastic-corrector phase is defined as:

$$\Delta\boldsymbol{\sigma}_j = -\mathbf{C}^e : \Delta\boldsymbol{\varepsilon}_j^p. \tag{10.12}$$

While the trial stress is computed basing upon the increment of the total strain, this plastic corrector is computed basing upon the increment of the plastic multiplier which is computed basing on the integration scheme used. In this scheme, the increment of the plastic multiplier is initially assumed to be zero $\left(\Delta\lambda_j^{p(0)} = 0\right)$. At each iteration k, the plastic multiplier is then increased by $d\lambda_j^{p(k)}$ so that:

$$\Delta\lambda_j^{p(k+1)} = \Delta\lambda_j^{p(k)} + d\lambda_j^{p(k)}. \tag{10.13}$$

This increment is computed by using a linearized form of the nonlinear equation, $f\left(\Delta\lambda_j^p\right)$, such that:

$$f^{(k)} + \frac{df^{(k)}}{d\Delta\lambda_j^{p(k)}} d\lambda_j^{p(k)}. \tag{10.14}$$

This iterative procedure is followed until the state computed from the plastic multiplier converges. This occurs when the stress has returned to the yield surface.

11. Fully implicit backward Euler scheme

An implicit backward Euler scheme, as presented in BELYTSCHKO *et al.* [2] is used for the integration of the constitutive model. This type of integration scheme is implicit (computed at time j) in the plasticity multiplier, the plastic strain, the hardening variables, and the plastic flow direction. The integration scheme is defined by Eqs. (10.1) to (10.3) and Eq. (10.7), where the increments of the state variables are written as follows:

$$\Delta\boldsymbol{\varepsilon}_j^p = f_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^p; \qquad \Delta r_j = -f_{,R_j}\Delta\lambda_j^p; \qquad \Delta\boldsymbol{\alpha}_j = f_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^p. \tag{11.1}$$

It can be seen that the problem defined by this model can be entirely defined by solving for two unknowns, $\Delta\boldsymbol{\sigma}_j$ and $\Delta\lambda_j^p$, using the following two nonlinear equations obtained from Eqs. $(11.1)_1$ and (10.7):

$$\mathbf{a}_j = -\boldsymbol{\varepsilon}_j^p + \boldsymbol{\varepsilon}_0^p + f_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^p = 0, \tag{11.2}$$

$$f_j = \|\boldsymbol{\xi}_j\| - \sqrt{\frac{2}{3}}\left[\sigma_{yp} + R_j\right] = 0. \tag{11.3}$$

Making use of Eq. (10.12), these two equations can be linearized as in Eq. (10.14) so that, for each iteration k, the following equations hold:

$$\mathbf{a}_j^{(k)} + \mathbf{C}^{-e} : d\boldsymbol{\sigma}_j^{(k)} + df_{,\boldsymbol{\sigma}_j}^{(k)}\Delta\lambda_j^{p(k)} + f_{,\boldsymbol{\sigma}_j}^{(k)} d\lambda_j^{p(k)} = 0, \tag{11.4}$$

$$f_j^{(k)} + f_{,\boldsymbol{\sigma}_j}^{(k)} : d\boldsymbol{\sigma}_j^{(k)} + f_{,\Delta\lambda_j^p}^{(k)} d\lambda_j^{p(k)} = 0, \tag{11.5}$$

where the normals to the yield surface are defined as:

$$f_{,\boldsymbol{\sigma}_j}^{(k)} = \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_j^{(k)} = \frac{\boldsymbol{\xi}_j^{(k)}}{\left\|\boldsymbol{\xi}_j^{(k)}\right\|}, \tag{11.6}$$

$$f_{,\Delta\lambda_j^p}^{(k)} = \left(\frac{\partial f}{\partial \Delta\lambda^p}\right)_j^{(k)} = -f_{,R_j}^{(k)2}\frac{\partial R\left(r_j^{(k)}\right)}{\partial r} - f_{,\boldsymbol{\sigma}_j}^{(k)} : \frac{\partial \mathbf{X}\left(\boldsymbol{\alpha}_j^{(k)}\right)}{\partial \boldsymbol{\alpha}} : f_{,\boldsymbol{\sigma}_j}^{(k)}. \tag{11.7}$$

The increment of the stress normal can be expressed in terms of the increments of the unknowns such that:

$$df_{,\boldsymbol{\sigma}_j}^{(k)} = \left(\frac{\partial f_{,\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}}\right)_j^{(k)} : d\boldsymbol{\sigma}_j^{(k)} + \left(\frac{\partial f_{,\boldsymbol{\sigma}}}{\partial \Delta\lambda^p}\right)_j^{(k)} d\lambda_j^{p(k)}, \tag{11.8}$$

where:

$$\left(\frac{\partial f_{,\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}}\right)_j^{(k)} = \left(\frac{\partial f_{,\boldsymbol{\sigma}}}{\partial \boldsymbol{\xi}}\right)_j^{(k)} = \frac{\mathbf{I} - f_{,\boldsymbol{\sigma}_j}^{(k)} \otimes f_{,\boldsymbol{\sigma}_j}^{(k)}}{\left\|\boldsymbol{\xi}_j^{(k)}\right\|} = f_{,\boldsymbol{\sigma}_j\boldsymbol{\sigma}_j}^{(k)}, \tag{11.9}$$

$$\left(\frac{\partial f_{,\boldsymbol{\sigma}}}{\partial \Delta\lambda^p}\right)_j^{(k)} = -f_{,\boldsymbol{\sigma}_j\boldsymbol{\sigma}_j}^{(k)} : \frac{\partial \mathbf{X}\left(\boldsymbol{\alpha}_j^{(k)}\right)}{\partial \boldsymbol{\alpha}} : f_{,\boldsymbol{\sigma}_j}^{(k)} = f_{,\boldsymbol{\sigma}_j\Delta\lambda_j^p}^{(k)}. \tag{11.10}$$

After substituting Eq. (11.8) into Eq. (11.4), the increment of the stress can be expressed as follows:

$$d\boldsymbol{\sigma}_j^{(k)} = -\mathbf{A}_j^{(k)} : \mathbf{a}_j^{(k)} - \mathbf{A}_j^{(k)} : \mathbf{A}_j^{p(k)} d\lambda_j^{p(k)} \tag{11.11}$$

where:

$$\mathbf{A}_j^{(k)} = \left[\mathbf{C}^{-e} + f_{,\boldsymbol{\sigma}_j\boldsymbol{\sigma}_j}^{(k)} \Delta\lambda_j^{p(k)}\right]^{-1}, \tag{11.12}$$

$$\mathbf{A}_j^{p(k)} = f_{,\boldsymbol{\sigma}_j}^{(k)} + f_{,\boldsymbol{\sigma}_j\Delta\lambda_j^p}^{(k)} \Delta\lambda_j^{p(k)}. \tag{11.13}$$

This increment of stress can now be substituted into the linearized yield condition, Eq. (11.5). The resulting equation is then solved for $d\lambda_j^{p(k)}$ to yield:

$$d\lambda_j^{p(k)} = \frac{f_j^{(k)} - f_{,\boldsymbol{\sigma}_j}^{(k)} : \mathbf{A}_j^{(k)} : \mathbf{a}_j^{(k)}}{f_{,\boldsymbol{\sigma}_j}^{(k)} : \mathbf{A}_j^{(k)} : \left(f_{,\boldsymbol{\sigma}_j}^{(k)} + f_{,\boldsymbol{\sigma}_j\Delta\lambda_j^p}^{(k)} \Delta\lambda_j^{p(k)}\right) - f_{,\Delta\lambda_j^p}^{(k)}}. \tag{11.14}$$

Thus, the increments of the unknown stress and the unknown plastic multiplier have been derived at iteration . Using these increments from Eqs. (11.11) and (11.14), the unknowns are updated as follows:

$$\Delta\lambda_j^{p(k+1)} = \Delta\lambda_j^{p(k)} + d\lambda_j^{p(k)}, \tag{11.15}$$

$$\boldsymbol{\sigma}_j^{(k+1)} = \boldsymbol{\sigma}_j^{(k)} + d\boldsymbol{\sigma}_j^{(k)}. \tag{11.16}$$

and the state variables are updated as follows:

$$\boldsymbol{\varepsilon}_j^{p(k+1)} = \boldsymbol{\varepsilon}_j^{p(k)} - \mathbf{C}^{-e} : d\boldsymbol{\sigma}_j^{(k)}, \tag{11.17}$$

$$r_j^{(k+1)} = r_j^{(k)} + f_{,R_j}^{(k)} d\lambda_j^{p(k)}, \tag{11.18}$$

$$\boldsymbol{\alpha}_j^{(k+1)} = \boldsymbol{\alpha}_j^{(k)} - \mathbf{C}^{-e} : d\boldsymbol{\sigma}_j^{(k)}. \tag{11.19}$$

The Newton iteration procedure is repeated until convergence is obtained by checking $\mathbf{a}_j$ and f_j from Eqs. (11.2) and (11.3). An algorithm for this solution procedure is given in Table 1.

Table 1. Fully implicit backward Euler return algorithm.

1. Initialize and compute trial elastic state for iteration $k = 0$
$\Delta\lambda_j^{p(0)} = 0$
$\boldsymbol{\sigma}_j^{(0)} = \boldsymbol{\sigma}_j^{\text{trial}} = \boldsymbol{\sigma}_0 + \mathbf{C}^e : \Delta\boldsymbol{\varepsilon}_j$
$\boldsymbol{\varepsilon}_j^{p(0)} = \boldsymbol{\varepsilon}_0^p; \quad r_j^{(k)} = r_0; \quad \boldsymbol{\alpha}_j^{(k)} = \boldsymbol{\alpha}_0$
2. Check convergence for iteration iteration k
$f_j^{(k)} = \left\|\boldsymbol{\xi}_j^{(k)}\right\| - \sqrt{\frac{2}{3}}\left[\sigma_{yp} + R_j^{(k)}\right]$
$\mathbf{a}_j^{(k)} = -\boldsymbol{\varepsilon}_j^{p(k)} + \boldsymbol{\varepsilon}_0^p + f_{,\boldsymbol{\sigma}_j}^{(k)} \Delta\lambda_j^{p(k)}$
if $f_j^{(k)} < TOL_1$ and $\left\|\mathbf{a}_j^{(k)}\right\| < TOL_2$ then:
$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j^{(k)}$
$\boldsymbol{\varepsilon}_j^p = \boldsymbol{\varepsilon}_j^{p(k)}; \quad r_j = r_j^{(k)}; \quad \boldsymbol{\alpha}_j = \boldsymbol{\alpha}_j^{(k)}$
exit
end if

3. Compute increments of unknowns

$$d\lambda_j^{p(k)} = \frac{f_j^{(k)} - f_{,\boldsymbol{\sigma}_j}^{(k)} : \mathbf{A}_j^{(k)} : \mathbf{a}_j^{(k)}}{f_{,\boldsymbol{\sigma}_j}^{(k)} : \mathbf{A}_j^{(k)} : \left(f_{,\boldsymbol{\sigma}_j}^{(k)} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p}^{(k)} \Delta\lambda_j^{p(k)} \right) - f_{,\Delta\lambda_j^p}^{(k)}}$$

$$d\boldsymbol{\sigma}_j^{(k)} = -\mathbf{A}_j^{(k)} : \mathbf{a}_j^{(k)} - \mathbf{A}_j^{(k)} : \left(\mathbf{n}_{\boldsymbol{\sigma}_j}^{(k)} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p}^{(k)} \Delta\lambda_j^{p(k)} \right) d\lambda_j^{p(k)}$$

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4. Update state

$$\Delta\lambda_j^{p(k+1)} = \Delta\lambda_j^{p(k)} + d\lambda_j^{p(k)}$$

$$\boldsymbol{\sigma}_j^{(k+1)} = \boldsymbol{\sigma}_j^{(k)} + d\boldsymbol{\sigma}_j^{(k)}$$

$$\varepsilon_j^{p(k+1)} = \varepsilon_j^{p(k)} - \mathbf{C}^{-e} : d\boldsymbol{\sigma}_j^{(k)}$$

$$r_j^{(k+1)} = r_j^{(k)} + f_{,R_j}^{(k)} d\lambda_j^{p(k)}$$

$$\boldsymbol{\alpha}_j^{(k+1)} = \boldsymbol{\alpha}_j^{(k)} - \mathbf{C}^{-e} : d\boldsymbol{\sigma}_j^{(k)}$$

$k \leftarrow k+1$ go to **2**

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12. Consistent tangent operator

The trial stress can be used to predict whether the integration point has entered the plastic regime, and the internal state variables can then be updated using the integration scheme. In order to obtain proper quadratic convergence, the choice of a tangent operator must be consistent with the integration scheme. The consistent tangent operator is defined as follows (e.g. SIMO and TAYLOR [15]):

$$\mathbf{D}_j^{\text{alg}} = \left(\frac{d\boldsymbol{\sigma}}{d\boldsymbol{\varepsilon}} \right)_j . \tag{12.1}$$

Following the procedure given in BELYTSCHKO *et al.* [2], the following set of equations is used which corresponds to the integration scheme of the previous section:

$$d\boldsymbol{\sigma}_j = \mathbf{C}^e : \left(d\boldsymbol{\varepsilon}_j - d\boldsymbol{\varepsilon}_j^p \right), \tag{12.2}$$

$$d\boldsymbol{\varepsilon}_j^p = df_{,\boldsymbol{\sigma}_j} \Delta\lambda_j^p + f_{,\boldsymbol{\sigma}_j} d\lambda_j^p, \tag{12.3}$$

$$df_j = f_{,\boldsymbol{\sigma}_j} : d\boldsymbol{\sigma}_j + f_{,\Delta\lambda_j^p} d\lambda_j^p = 0, \tag{12.4}$$

where the normals to the yield surface and the increment of the stress normal are defined by evaluating Eqs. (11.6) to (11.8) at time step j. After substituting the increment of the plastic strain, Eq. (12.3), and the increment of the stress normal, Eq. (11.8), at time step j, into Eq. (12.2), the increment of the stress is determined

$$d\boldsymbol{\sigma}_j = \mathbf{A}_j : d\boldsymbol{\varepsilon}_j - \mathbf{A}_j : \left(f_{,\boldsymbol{\sigma}_j} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p} \Delta\lambda_j^p \right) d\lambda_j^p \tag{12.5}$$

where:

$$\mathbf{A}_j = \left[\mathbf{C}^{-e} + f_{,\boldsymbol{\sigma}_j \boldsymbol{\sigma}_j} \Delta\lambda_j^p \right]^{-1}. \tag{12.6}$$

This increment of stress can now be substituted into the incremental consistency condition, Eq. (12.4). The resulting equation is then solved for $d\lambda_j^p$ leading to:

$$d\lambda_j^p = \frac{f_{,\boldsymbol{\sigma}_j} : \mathbf{A}_j : d\boldsymbol{\varepsilon}_j}{f_{,\boldsymbol{\sigma}_j} : \mathbf{A}_j : \left(f_{,\boldsymbol{\sigma}_j} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p} \Delta\lambda_j^p \right) - f_{,\Delta\lambda_j^p}}. \tag{12.7}$$

Substitution of the above relation in Eq. (12.5) gives the algorithmic relation between the increment of the stress and the elastic strain as follows:

$$d\boldsymbol{\sigma}_j = \mathbf{D}_j^{\text{alg}} : d\boldsymbol{\varepsilon}_j \tag{12.8}$$

where $\mathbf{D}_j^{\text{alg}}$, the algorithmic elastic stiffness operator which is defined as follows:

$$\mathbf{D}_j^{\text{alg}} = \mathbf{A}_j - \left(\frac{\mathbf{A}_j : \left(f_{,\boldsymbol{\sigma}_j} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p} \Delta\lambda_j^p \right) \otimes f_{,\boldsymbol{\sigma}_j} : \mathbf{A}_j}{f_{,\boldsymbol{\sigma}_j} : \mathbf{A}_j : \left(f_{,\boldsymbol{\sigma}_j} + f_{,\boldsymbol{\sigma}_j \Delta\lambda_j^p} \Delta\lambda_j^p \right) - f_{,\Delta\lambda_j^p}} \right). \tag{12.9}$$

In order to solve a boundary value problem, the finite element approach can be adopted such that the displacement field is discretized. The algorithm requires a weak satisfaction of the equilibrium condition such that:

$$\int_{\mathbf{V}} \delta\boldsymbol{\varepsilon} : \mathbf{D}_j^{\text{alg}} : d\boldsymbol{\varepsilon}_j d\mathbf{V} = \int_{\mathbf{V}} \delta\mathbf{u} : \mathbf{b}_j d\mathbf{V} + \int_{\boldsymbol{\Gamma}_t} \delta\mathbf{u} : \hat{\mathbf{t}}_j d\boldsymbol{\Gamma} - \int_{\mathbf{V}} \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}_j d\mathbf{V}. \tag{12.10}$$

Note that this equation is enforced within the entire body, including both the plastic domain and the elastic domain. The governing equation can be linearized consistently and a finite element procedure is then followed to solve the equations. The problem defined by this set of equations is nonlinear since the stiffness and the load residuals are a functions of the body deformation. The

degrees of freedom thus cannot automatically be determined from the system of equations. An iterative procedure is thus necessary to obtain the degrees of freedom such that the left-hand side of the governing equations is in equilibrium with the load vector residuals of the right-hand side. A typical procedure is the Newton–Raphson method; however, other methods such as the modified Newton–Raphson may also be used.

13. Conclusions

In this work, thermodynamically consistent theoretical formulations and the numerical implementation of a classical continuum plasticity model have been presented. Following standard thermodynamics and using local state variables, a complete set of constitutive equations were derived where a local yield surface was used to determine the occurrence of plasticity. This elasto-plastic model for materials is introduced here within a framework that uses functional forms of the isotropic hardening and the kinematic hardening internal state variables. The hardening conjugate forces (stress-like terms) were defined as general functions of their corresponding hardening state variables (strain-like terms). Various exponential and power-law functional forms are studied in this formulation, and it was shown that, depending on the functions used, standard models from the literature can be recovered.

The use of this formulation in solving boundary value problems will be presented in future work where the finite element approach must be adopted, so that the displacement field will be discretized. The fully implicit backward Euler scheme will be used in a Newton–Raphson solution procedure.

References

1. P.J. ARMSTRONG and C.O. FREDERICK, *A mathematical representation of the multiaxial Bauschinger effect,* CEGB Report RD/B/N/731, Berkeley Laboratories, R and D Department, CA. 1966.

2. T. BELYTSCHKO, W.K. LIU, and B. MORAN, *Nonlinear finite elements for continua and structures*, John Wiley and Sons Ltd, New York 2000.

3. J.-L. CHABOCHE, *Constitutive equations for cyclic plasticity and cyclic viscoplasticity,* International Journal of Plasticity, **5**, 247–302, 1989.

4. O. COUSSY, *Mechanics of porous continua*, John Wiley and Sons Ltd, New York 1995.

5. R.J. DORGAN and G.Z. VOYIADJIS, *A mixed finite element implementation of a gradient enhanced coupled damage-plasticity model,* International Journal of Damage Mechanics, **15**, 201–235, 2006.

6. I. DOGHRI, *Fully implicit integration and consistent tangent modulus in elasto-plasticity,* International Journal for Numerical Methods in Engineering, **36**, 3915–3932, 1993.

7. I. DOGHRI, *Mechanics of deformable solids: linear and nonlinear*, Analytical and Computational Aspects, Springer, Germany 2000.

8. R. HILL, *The mathematical theory of plasticity*, Oxford 1950.

9. H. KUHN and A. TUCKER, *Nonlinear programming,* Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probabilistics, University of California Press, 481–492, 1951.

10. J. LEMAITRE and J.-L. CHABOCHE, *Mechanics of solid materials*, Cambridge University Press, London 1994.

11. L.E. MALVERN, *Introduction to the mechanics of a continuous medium*, Prentice-Hall, Inc., New Jersey 1969.

12. E. OROWAN, *Zur Kristallplastizität III: Über die Mechanismus des Gleitvorganges,* Zeitschrift für Physik, **89**, 634–659, 1934.

13. M. POLANYI, *Über eine Art Gitterstörung, die einem Kristall plastisch machen könnte,* Zeitschrift für Physik, **89**, 660–664, 1934.

14. W. PRAGER, *A new method of analyzing stresses and strains in work-hardening plastic solids,* Journal of Applied Mechanics, ASME, **23**, 493–496, 1956.

15. J.C. SIMO and T.J.R. HUGHES, *Computational Inelasticity*, Springer-Verlag, New York 1998.

16. G.I. TAYLOR, *The mechanism of plastic deformation of crystals, I. theoretical,* Proceedings of the Royal Society A, **145**, 362–387, 1934.

17. G.Z. VOYIADJIS and R.J. DORGAN, *A gradient enhanced, generalized plasticity/damage model: Rigorous mathematical formulation and finite element implementation,* Journal of the Mechanical Behavior of Materials, **15**, 4–5, 309–340, 2004.

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